

Three-axis Rigid Body under Action of Gyroscopic Forces for Classical Problems of Celestial Mechanics

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Abstract. To determine the stable stationary rotations of a triaxial satellite relative to its center of mass located at a libration point of the restricted circular three-body problem, within the satellite approximation model, the problem of the motion of a rigid body with a fixed point under the action of gyroscopic forces with a constant intensity vector is used.

Introduction

We discuss a generalization of the restricted circular three-body problem — instead of a small point mass, we consider the rotation of an extended, small, triaxial rigid body around its center of mass, located at the libration point. In an approximate satellite model for this problem, we determine the moments of all forces acting on the rigid body, relative to the center of mass, namely, the moments of gravitational, centrifugal, and Coriolis forces. We prove that in a linear approximation in the parameter of the vector of angular velocity of the orbital motion of the Primaries, only the moment of the Coriolis (gyroscopic) force exerts the dominant influence.

Thus, it becomes necessary to consider the problem of a triaxial rigid body with a fixed point, where the body is subject to gyroscopic forces of constant intensity. An arbitrary constant vector of the gyroscopic force intensity is in some sense analogous to the gravity acceleration vector in the classical problem for a heavy rigid body with a fixed point. In the restricted three-body problem, it corresponds to twice the angular velocity vector of the circular orbital motion of the two primary bodies.

1. Gyroscopic forces with constant intensity vector, equations of motion and first integrals

Let us consider the problem of the motion of a rigid body with a fixed point under the action of gyroscopic forces with a constant intensity vector [1]. A similar problem arises when analyzing the motion of a rigid body relative to its center of mass in the extended circular three-body problem.

A point mass dm with radius vector $\bar{r}$ is subject to a gyroscopic force $d\bar{F} = \bar{c} \times \bar{v}dm$. The constant vector in absolute space $\bar{c}$ is the intensity vector of the gyroscopic forces. We introduce a system $Ox_1x_2x_3$ of principal axes of inertia of a rigid body, and denote by $\bar{r} = (x_1, x_2, x_3)$ the radius vector of the rigid body's points in this system, and by $\bar{\omega} = (\omega_1, \omega_2, \omega_3)$ angular velocity of rigid body.

The total moment of gyroscopic forces acting on a rigid body

$$\bar{M}_O = \int_V \bar{r} \times d\bar{F} = \int_V \bar{r} \times (\bar{c} \times (\bar{\omega} \times \bar{r}))dm,$$

$$\bar{M}_O = -\bar{\omega} \times J^* \bar{c}, \quad J^* = \text{diag}\{I_1^*, I_2^*, I_3^*\} = \frac{(I_1 + I_2 + I_3)}{2} E - J.$$

The equations of motion of a rigid body in Euler-Poisson form:

$$J\dot{\bar{\omega}} + \bar{\omega} \times (J\bar{\omega} + J^*\bar{c}) = 0, \quad \dot{\bar{c}} + \bar{\omega} \times \bar{c} = 0, \quad (1)$$

where $J = \text{diag}\{I_1, I_2, I_3\}$ is the matrix of the rigid body's inertia tensor in the principal axes of inertia.

Applying to a rigid body with the center of mass at the libration point, we have gyroscopic force intensity vector $\bar{c} = (0, 0, -2\Omega)$, where Ω is the orbital angular velocity of the main bodies,

In the satellite approximation model, when calculating the moments of two gravitational forces, the centrifugal force and the Coriolis force relative to the libration point, it was found that in the linear approximation, the Coriolis force moment has a dominant effect — the moments of centrifugal and gravitational forces are of the order of $\approx \Omega^2$, and the moment of gyroscopic forces is of the order of $\approx \Omega$, both in the vicinity of triangular and collinear libration points.

So, the body is acted upon only by gyroscopic forces, their power is zero, so the energy integral exists: $H = (J\bar{\omega}, \bar{\omega})/2 = h = \text{const}$. As usual, the Euler-Poisson equations also admit the trivial integral $F = (\bar{c}, \bar{c}) = c_0^2 = \text{const}$. By direct differentiation, by virtue of equations (1), we can verify that the function $K = (J(\bar{\omega} - \bar{c}/2), \bar{c}) = k$ is a first integral. If we move to a system rotating with angular velocity $\bar{c}/2$, then K is the projection onto the vector $\bar{c}$ of the rigid body's angular momentum.

Note that if the body is dynamically symmetric, then the system also contains a fourth integral [1]. Since equations (1) admit an invariant measure with constant density, in this case they are integrated by quadratures.

2. Stability of the stationary rotations

Substituting the condition $\bar{\omega}(t) = \text{const}$ into (1), we obtain the conditions for the body's motion to be stationary: $\bar{\omega} \times (J\bar{\omega} + J^*\bar{c}) = 0$, and $\dot{\bar{c}} + \bar{\omega} \times \bar{c} = 0$. Differentiating the first equation with respect to time and substituting $\dot{\bar{c}}$ from the second, we obtain that the vector $\bar{c}$ is collinear with the vector $\bar{\omega}$, i.e. $\bar{c} = \mu\bar{\omega}$, where μ is some number. In this case, $\bar{\omega} \times \tilde{J}\bar{\omega} = 0$, where $\tilde{J} = J + \mu J^*$. Since J and $\tilde{J}$ are diagonal matrices and $\det J \neq 0$, then the eigenvectors of matrix J are also eigenvectors of matrix $\tilde{J}$. Consequently, rotations around an arbitrary principal axis of inertia with an arbitrary angular velocity are stationary rotations of a rigid body.

If μ is such that the matrix $\tilde{J}$ is non-singular (the non-critical case), then the set of its eigenvectors coincides with the set of eigenvectors of the matrix J . Then there are no other stationary motions of the rigid body.

Without loss of generality, we assume that the absolute coordinate system is chosen such that in it $\bar{c} = (c_0, 0, 0)$, $c_0 > 0$. Consider the stationary solution

$$\bar{\omega}(t) = (\omega_0, 0, 0), \quad \bar{c}(t) = (c_0, 0, 0), \quad c_0 > 0 \quad (2)$$

projected onto the principal axes of inertia of the body. Then $h = I_1\omega_0^2/2$, $k = I_1c_0\omega_0 - I_1c_0^2/2$

Assuming that the quantities $\omega_2, \omega_3, c_2, c_3$ are infinitesimal, we can express the quantities ω_1 and c_1 from the integrals H and F to third order. Substituting them into the integral K , we obtain to third order:

$$K = f_1(\omega_2, c_2) + f_2(\omega_3, c_3) + O(\omega_2^4 + \omega_3^4 + c_2^4 + c_3^4),$$

where

$$f_1(\omega_2, c_2) = -\left(\frac{I_2c_0}{2\omega_0}\right)\omega_2^2 + I_2c_2\omega_2 + \left(-\frac{I_1\omega_0}{2c_0} + \frac{I_1 - I_2}{2}\right)c_2^2$$

$$f_2(\omega_3, c_3) = -\left(\frac{I_3c_0}{2\omega_0}\right)\omega_3^2 + I_3c_3\omega_3 + \left(-\frac{I_1\omega_0}{2c_0} + \frac{I_1 - I_3}{2}\right)c_3^2.$$

According to Routh's theory, for the stationary motion (2) to be Lyapunov stable, it is sufficient that at the point $(\omega_2, \omega_3, c_2, c_3) = (0, 0, 0, 0)$ the quadratic part of K reaches a strict minimum or strict maximum. But the sign of the coefficient of ω_2^2 in the form f_1 is the same as the sign of the coefficient of ω_3^2 in the form f_2 . This means that the type of extremum of these forms (maximum or minimum) is the same. Therefore, a sufficient condition for stability is that both forms have definite signs. Using the fact that $I_2 > 0$ and $I_3 > 0$, we find that this condition consists of the simultaneous satisfaction of two inequalities:

$$(I_1 - I_2)\left(\frac{c_0}{\omega_0} - 1\right) < 0, \quad (I_1 - I_3)\left(\frac{c_0}{\omega_0} - 1\right) < 0 \quad (3)$$

Assuming that $I_2 \leq I_3$ is certain, we obtain three cases:

1. Stationary rotation around the axis with the smallest moment of inertia ($I_1 < I_2 \leq I_3$) is stable for $0 < \omega_0 < c_0$.

2. Stationary rotation about an axis with the intermediate moment of inertia ($I_2 < I_1 < I_3$) is unstable since the conditions (3) cannot be performed simultaneously.

3. Stationary rotation about an axis with the largest moment of inertia ($I_2 \leq I_3 < I_1$) is stable in two cases. If $\omega_0 > 0$, then it is sufficient that $c_0 < \omega_0 < +\infty$. If $\omega_0 < 0$, then it is sufficient that $-\infty < \omega_0 < 0$.

Unlike Euler's problem for a triaxial rigid body, in which stable rotations around the largest or smallest axis of the ellipsoid of inertia occur at any values of angular velocity, in this formulation, the intervals of admissible values of angular velocity for stationary states are not arbitrary.

Note that the obtained conditions for the stability of stationary rotation are sufficient conditions.

Conclusion

The above study has important application to the analysis of the evolution of a satellite with the center of mass at the libration point. Then gyroscopic force intensity vector $\bar{c} = (0, 0, -2\Omega)$ specifies one of the directions of the absolute coordinate system.

The results of an analytical study within the satellite approximation are compared with the results of numerical modeling for parameters of the Earth-Moon-Satellite system for various libration points (and also taking into account solar perturbation). Analytically obtained steady states are justified by numerical simulations.

References

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