

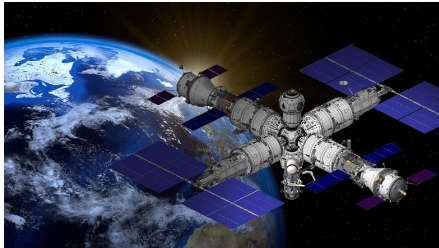
"Analytical Methods of Celestial Mechanics"
Euler International Mathematical Institute, St. Petersburg, RUSSIA

Three-axis Rigid Body under Action of Gyroscopic Forces for Classical Problems of Celestial Mechanics

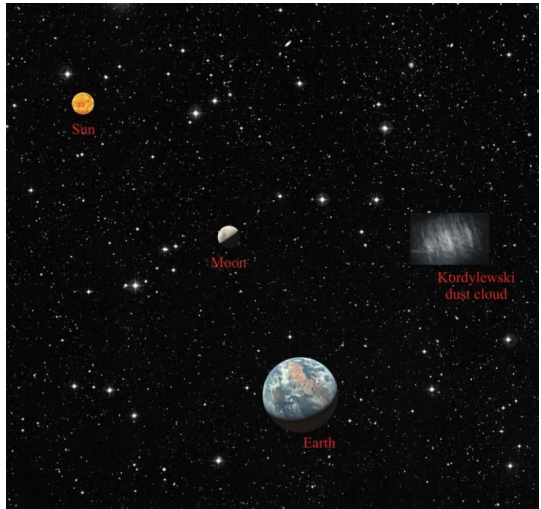
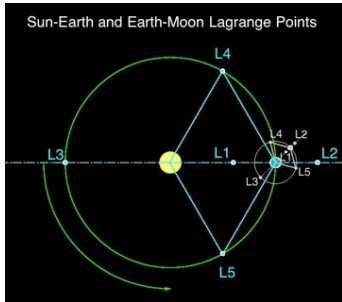
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Dynamically asymmetric space stations



Libration points of the Earth-Moon system



Gyroscopic forces with constant intensity vector

- We consider the problem of the motion of a rigid body with a fixed point under the action of gyroscopic forces with a constant intensity vector.

A similar problem arises in the analysis of the motion of a rigid body relative to its center of mass in the extended circular three-body problem.

$\bar{F} = m\bar{c} \times \bar{v}$, где $\bar{c}$ constant vector in absolute space

$\bar{r} = (x_1, x_2, x_3)$, $\bar{\omega} = (\omega_1, \omega_2, \omega_3)$ - in the principal axes of inertia of a rigid body

$$\bar{M}_O = \int_V \bar{r} \times d\bar{F} = \int_V \bar{r} \times (\bar{c} \times (\bar{\omega} \times \bar{r})) dm,$$

$$J\dot{\bar{\omega}} + \bar{\omega} \times J\bar{\omega} = \bar{M}_O, \quad \dot{\bar{c}} + \bar{\omega} \times \bar{c} = 0.$$

$$\bar{M}_O = -\bar{\omega} \times J^* \bar{c}, \quad J^* = \text{diag}\{I_1^*, I_2^*, I_3^*\} = \frac{(I_1 + I_2 + I_3)}{2} E - J$$

$$J\dot{\bar{\omega}} + \bar{\omega} \times (J\bar{\omega} + J^* \bar{c}) = 0, \quad \dot{\bar{c}} + \bar{\omega} \times \bar{c} = 0.$$

Stationary rotations

$$\bar{\omega}(t) = \text{const}$$

$$\bar{\omega} \times (J\bar{\omega} + J^*\bar{c}) = 0, \quad \dot{\bar{c}} + \bar{\omega} \times \bar{c} = 0$$

$\bar{\omega}$ is collinear with the vector $\bar{c}$, i.e. $\bar{c} = \mu\bar{\omega}$

$$\bar{\omega} \times \tilde{J}\bar{\omega} = 0, \quad \tilde{J} = J + \mu J^*$$

Since J and $\tilde{J}$ are diagonal matrices and $\det J \neq 0$, then the eigenvectors of matrix J are also eigenvectors of matrix $\tilde{J}$. Consequently, rotations around an arbitrary principal axis of inertia with an arbitrary angular velocity are stationary rotations of a rigid body.

Stationary rotations

If μ is such that the matrix $\tilde{J}$ is non-singular (the non-critical case), then the set of its eigenvectors coincides with the set of eigenvectors of the matrix J . Then there are no other stationary motions of the rigid body.

Stability of the stationary rotations

Without loss of generality, we assume that the absolute coordinate system is chosen such that in it $\bar{c} = (c_0, 0, 0)$, $c_0 > 0$. Consider the stationary solution

$$\bar{\omega}(t) = (\omega_0, 0, 0), \quad \bar{c}(t) = (c_0, 0, 0), \quad c_0 > 0 \quad (1)$$

projected onto the principal axes of inertia of the body.

Stability of the stationary rotations

First integrals:

$$H = (J\bar{\omega}, \bar{\omega})/2 = h = \text{const}$$

.

$$F = (\bar{c}, \bar{c}) = c_0^2 = \text{const}$$

.

$$K = (J(\bar{\omega} - \bar{c}/2), \bar{c}) = k$$

Then

$$h = I_1 \omega_0^2 / 2, \quad k = I_1 c_0 \omega_0 - I_1 c_0^2 / 2.$$

Stability of the stationary rotations

Assuming that the quantities $\omega_2, \omega_3, c_2, c_3$ are infinitesimal, we can express the quantities ω_1 and c_1 from the integrals H and F to third order. Substituting them into the integral K , we obtain to third order:

$$K = f_1(\omega_2, c_2) + f_2(\omega_3, c_3) + O(\omega_2^4 + \omega_3^4 + c_2^4 + c_3^4),$$

where

$$f_1(\omega_2, c_2) = - \left(\frac{l_2 c_0}{2\omega_0} \right) \omega_2^2 + l_2 c_2 \omega_2 + \left(-\frac{l_1 \omega_0}{2c_0} + \frac{l_1 - l_2}{2} \right) c_2^2$$

$$f_2(\omega_3, c_3) = - \left(\frac{l_3 c_0}{2\omega_0} \right) \omega_3^2 + l_3 c_3 \omega_3 + \left(-\frac{l_1 \omega_0}{2c_0} + \frac{l_1 - l_3}{2} \right) c_3^2.$$

Stability of the stationary rotations

According to Routh's theory, for the stationary motion (1) to be Lyapunov stable, it is sufficient that at the point $(\omega_2, \omega_3, c_2, c_3) = (0, 0, 0, 0)$ the quadratic part of K reaches a strict minimum or strict maximum. But the sign of the coefficient of ω_2^2 in the form f_1 is the same as the sign of the coefficient of ω_3^2 in the form f_2 . This means that the type of extremum of these forms (maximum or minimum) is the same. Therefore, a sufficient condition for stability is that both forms have definite signs. Using the fact that $I_2 > 0$ and $I_3 > 0$, we find that this condition consists of the simultaneous satisfaction of two inequalities:

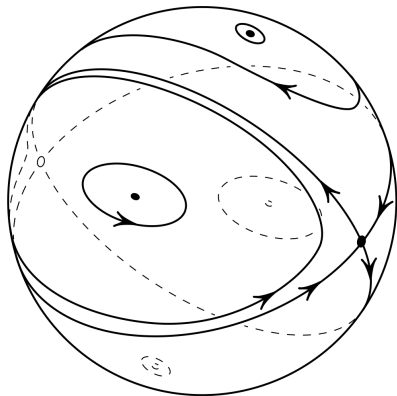
$$(I_1 - I_2) \left(\frac{c_0}{\omega_0} - 1 \right) < 0, \quad (I_1 - I_3) \left(\frac{c_0}{\omega_0} - 1 \right) < 0 \quad (2)$$

Stability of the stationary rotations — sufficient conditions

Assuming that $I_2 \leq I_3$ is certain, we obtain three cases:

1. Stationary rotation around the axis with the smallest moment of inertia ($I_1 < I_2 \leq I_3$) is stable for $0 < \omega_0 < c_0$.
2. Stationary rotation about an axis with the intermediate moment of inertia ($I_2 < I_1 < I_3$) is **possibly unstable** since the conditions (2) cannot be performed simultaneously.
3. Stationary rotation about an axis with the largest moment of inertia ($I_2 \leq I_3 < I_1$) is stable in two cases. If $\omega_0 > 0$, then it is sufficient that $c_0 < \omega_0 < +\infty$. If $\omega_0 < 0$, then it is sufficient that $-\infty < \omega_0 < 0$.

Stability of the stationary rotations



*Unlike Euler's problem for a triaxial rigid body, in which stable rotations around the largest or smallest axis of the ellipsoid of inertia occur at any value of angular velocity, in this formulation, the ranges of admissible angular velocity values for stationary states are not arbitrary! We note that **sufficient conditions** for the stability of stationary rotation have been obtained.*

The goal of the study is to find stable rotations of the station relative to the center of mass, located at the libration point, that require minimal energy expenditure for stabilization.

In the satellite approximation model, when calculating the moments of two gravitational forces, the centrifugal force and the Coriolis force relative to the libration point, it was found that in the linear approximation, the Coriolis moment has a dominant effect.

Intensity vector of gyroscopic forces $\bar{c} = (0, 0, -2\Omega)$.

$\bar{\Omega}$ is the orbital angular velocity vector.

Moments of centrifugal and gravitational forces of the order $\approx \Omega^2$, and the moment of gyroscopic forces is of the order of $\approx \Omega$, both in the vicinity of triangular and collinear libration points.

Analytically obtained conditions for stable rotation

$\bar{c} = (0, 0, -2\Omega)$, $\bar{\omega} = (0, 0, \omega_0)$ — vector of angular velocity of the satellite's proper rotation in the orbital system.

1. Let the direction $O\bar{\omega}$ be the largest axis of the ellipsoid of inertia, corresponding to the smallest moment of inertia.

Stability is achieved for any $\omega_0 < 0$ such that $0 < |\omega_0| < 2\Omega$.

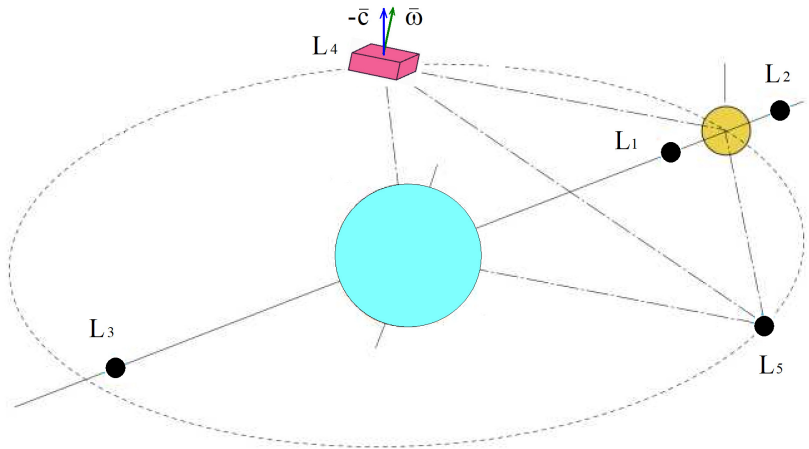
2. Let the direction $O\bar{\omega}$ be the smallest axis of the ellipsoid of inertia, corresponding to the largest moment of inertia.

Stability is achieved for any $\omega_0 < 0$ such that $|\omega_0| > 2\Omega$.

3. Let the direction $O\bar{\omega}$ be the smallest axis of the ellipsoid of inertia, corresponding to the greatest moment of inertia.

Stability is achieved for any $\omega_0 > 0$.

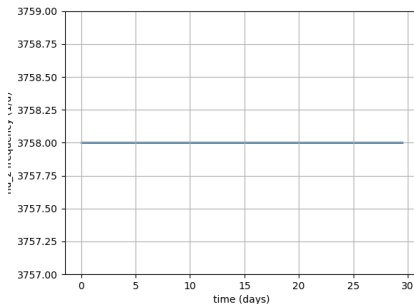
Numerical modeling of the Earth-Moon system



Intensity vector of the gyroscopic forces $\bar{c} = (0, 0, -2\Omega)$, $\bar{\Omega}$ is the orbital angular velocity vector.

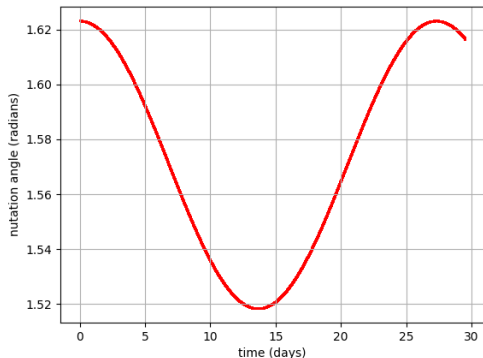
Numerical modeling (L₄)

Stable rotation around the largest and smallest axes of inertia. (Preserved even when taking solar disturbances into account!)



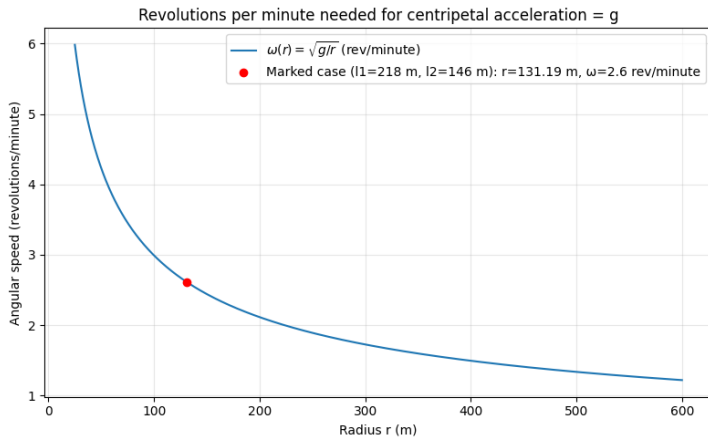
Stable rotation — initial value of the angular velocity vector remains virtually unchanged over time. At the initial moment of time, the angular velocity vector is directed along the smallest axis of inertia of the rigid body and is deflected from the z-axis by an angle $\theta_0 = \pi/60$.

Numerical modeling (L_4)

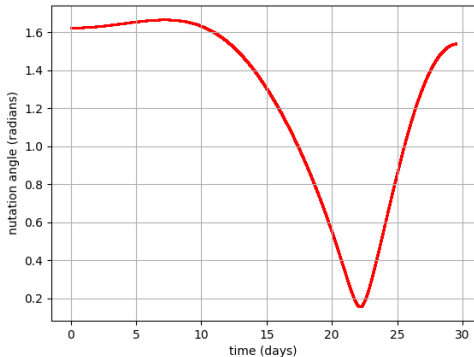


Stable rotation — nutation angle fluctuates in the vicinity of the angle $\pi/2$ (z-axis) with the amplitude of the initial deviation $\theta_0 = \pi/60$.

Numerical modeling — achieving the value g

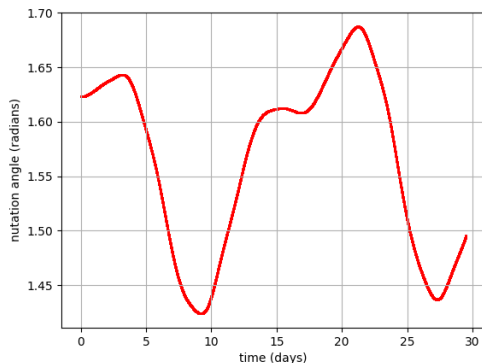


Numerical simulation - intermediate axis of inertia (L_4)



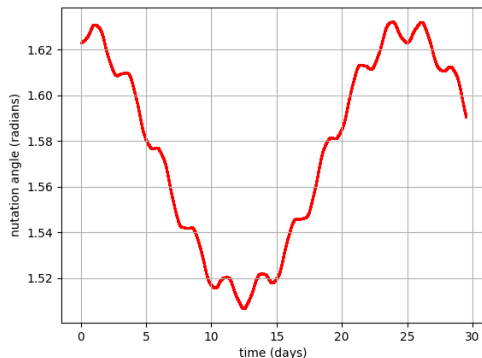
Unstable motion with zero initial angular velocity. Initial small deviation of the intermediate axis of inertia from the z-axis, collinear with the orbital angular velocity vector $\overline{\Omega}$, rapidly increases with time and changes in an unpredictable manner.

Numerical simulation (rotation around an intermediate axis of inertia)



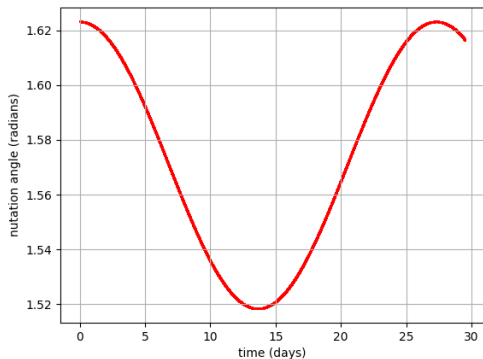
*Dependence on time
nutation angle for critical initial angular velocity 1 revolution per day.*

Numerical simulation (rotation around an intermediate axis of inertia)



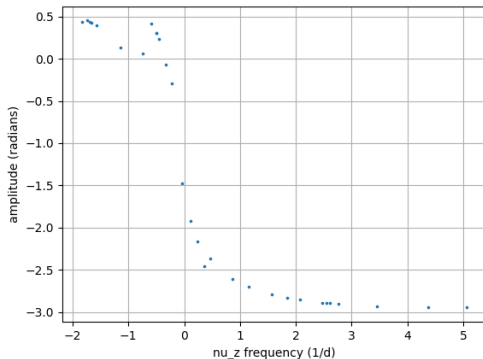
*Dependence on time
nutation angle for small initial angular velocity 8 revolutions per day.*

Numerical simulation (rotation around an intermediate axis of inertia)



Stable rotation with great enough initial angular velocity 3758 revolutions per day.

Numerical simulation (rotation around an intermediate axis of inertia)

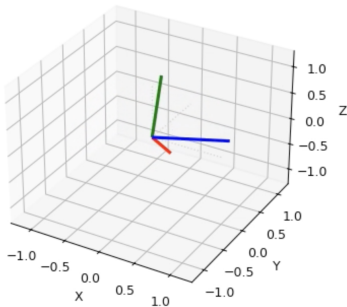


Dependence of the maximum deviation of the nutation angle from the initial value θ_0 on the initial angular velocity, logarithmic scale.

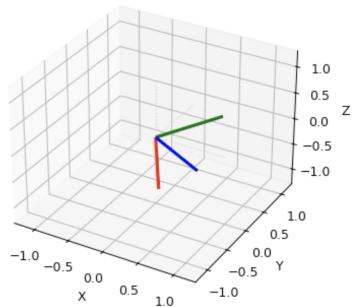
Stable and unstable dynamics

$t = 16.136$ days

Low initial angular velocity



High initial angular velocity



Numerical simulation — L_2

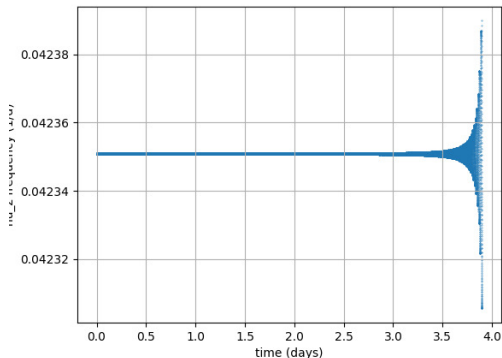
In the work of K. Howell, stable orbits were numerically found among a set of spatially periodic halo orbits covering different unstable libration points. (*Howell, K.C.: Three-dimensional, periodic, halo orbits. Celestial Mechanics. 32, (1984) 53–71.*)

We used the relevant data and considered motion with initial conditions for the center of mass of a body in a halo orbit around point L_2 with the smallest z-coordinate of the center of mass and a deviation of the minor axis of inertia from the z-axis by the nutation angle, while moving near the halo orbit L_2 with zero initial angular velocity.

Numerical simulations show that the body maintains its orientation for some time—its angular velocity vector is constant, and the nutation angle does not change.

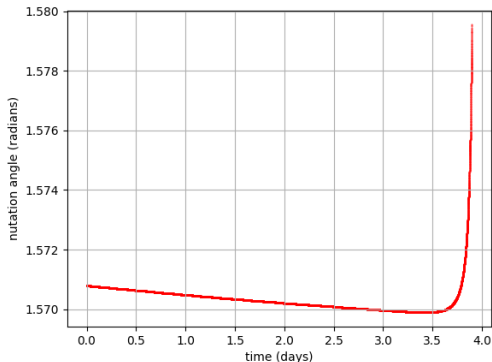
Further changes in orientation are associated with the deviation of the center of mass from the halo orbit.

Numerical simulation — L_2

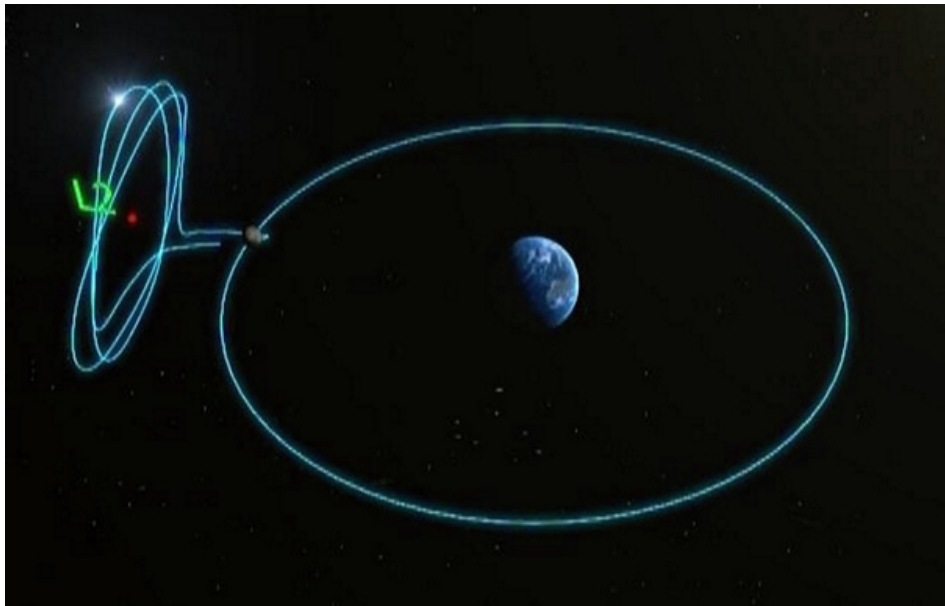


Dependence of the angular velocity on time. After approximately a time equal to the period of the center of mass motion along the halo orbit, stability is lost and angular velocity quickly moves away from the vicinity of the initial zero value.

Numerical simulation — L_2



Dependence of the nutation angle on time. After approximately a time equal to the period of the center of mass motion along the halo orbit, stability is lost and the nutation angle θ quickly moves away from the vicinity of the initial zero value.



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Thank you for your attention!

