

On Refining the Applicability Conditions of Power Series in the Restricted Circular Three-Body Problem

A. V. Dobroslavskiy

Moscow Aviation Institute
(National Research University)

Analytical Methods of Celestial Mechanics

St. Petersburg, August 17–21, 2026

- 1 Introduction
- 2 Averaging of the disturbing function
- 3 Numerical quadrature
- 4 Phase portraits
- 5 Linked orbits
- 6 Estimation of the radius of convergence
- 7 Conclusion

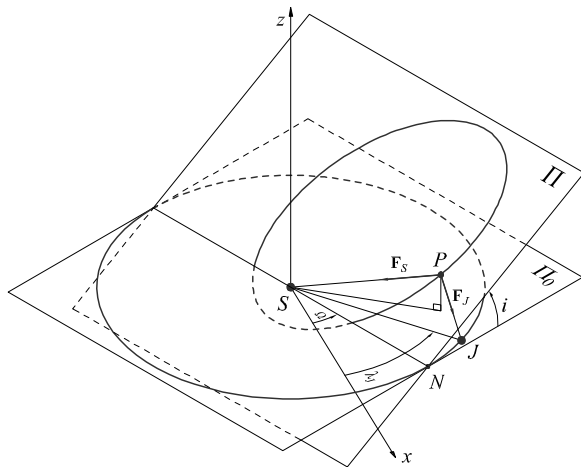
Selected works on the doubly averaged circular three-body problem

-  Gauß J.C.F. *Theoria Motus Corporum Coelestium in sectionibus conicis solem ambientium*, 1809.
-  von Zeipel H. *Sur l'application des series de M. Lindstedt a l'etude du mouvement des cometes periodiques.* // *Astronomische Nachrichten* v. 183, 1910, p. 345–418.
-  Vashkovyak M.A. *Evolution of orbits in the restricted circular twice-averaged-three body problem. I - Qualitative investigations.* // *Cosmic Research*, v. 19, 1981, p. 1–10.
-  Ito T. *High-Order Analytic Expansion of Disturbing Function for Doubly Averaged Circular Restricted Three-Body Problem.* // *Advances in Astronomy*, 2017
-  Krasilnikov P.S., Dobroslavskiy A.V. *On the Asymptotic Behavior of the Secular Perturbation Function in the Circular Restricted Three-Body Problem.* // *Cosmic Res.* 62, 2024, p. 266–275.

Statement of the problem

Consider the spatial circular three-body problem:

- S – the central body (the Sun) of mass m_S
- J – the perturbing body (Jupiter) of mass m_J , orbiting about S in a circular orbit of radius r_J in the ecliptic plane Π_0
- P – the asteroid, moving about S along an unperturbed elliptic orbit with semi-major axis a and eccentricity e in the plane Π
- F_S, F_J – the gravitational forces exerted by the Sun and by Jupiter
- Ω – the longitude of the ascending node N
- i – the inclination
- fast variables: λ_J – the longitude of Jupiter in the plane Π_0 , and λ – the mean longitude of the asteroid on the unperturbed orbit



Averaging of the disturbing function

the first averaging

The perturbing factor is the gravitational attraction exerted by Jupiter:

$$R = f m_J \left(\frac{1}{\Delta} - \frac{(\vec{r}_J, \vec{r})}{r_J^3} \right) = \frac{f m_J}{r_J} \sum_{n=2}^{\infty} \left(\frac{r}{r_J} \right)^n P_n(\cos \gamma),$$

where $\vec{r}_J$ – the radius vector of Jupiter, $\vec{r}$ – the radius vector of the asteroid, Δ – the distance from Jupiter to the asteroid, f – the gravitational constant, m_J – the mass of Jupiter, P_n – the Legendre polynomial, γ – the angle between r_J and r . The series converges if and only if $r/r_J < 1$, since the singular points of Δ^{-1} , that is, the points

$$\cos \gamma \pm j \sqrt{1 - \cos^2 \gamma}, (j^2 = -1),$$

lie on the unit circle.

The double averaging following the Gauss scheme has the form:

$$R^{**} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R d\lambda_J d\lambda$$

After the first averaging over the longitude of Jupiter λ_J we obtain for the disturbing function:

$$R^* = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{f m_J}{r_J} \sum_{n=2}^{\infty} \left(\frac{r}{r_J} \right)^n P_n(\cos \gamma) \right) d\lambda_J = \frac{f m_J}{r_J} \sum_{n=1}^{\infty} \left(\frac{r}{r_J} \right)^{2n} P_{2n}(0) P_{2n}(\sin i \sin(\nu + \omega)) \quad (1)$$

here ν , ω – the true anomaly and the argument of pericentre of the asteroid P .

Averaging of the disturbing function

the second averaging

As a result of the repeated averaging over λ and the use of Parseval's formula, we obtain a power series in even powers of $\mu = a/r_J$:

$$R^{**} = \frac{fm_J}{r_J \sqrt{1-e^2}} \sum_{n=1}^{\infty} D_n \mu^{2n},$$

here

$$D_n = (1+e)^{2n+2} P_{2n}(0) \left(P_{2n}(0) P_{2n}(\cos i) {}_2F_1 \left(\frac{1}{2}, 2n+2; 1; \frac{2e}{e-1} \right) + \sum_{m=1}^n (-1)^m A_{2n}^{(2m)} \cos 2m\omega \right),$$
$$A_{2n}^{(2m)} = {}_3\tilde{F}_2 \left(\frac{1}{2}, 1, 2n+2; 1-2m, 1+2m; \frac{2e}{e-1} \right) \frac{(2n-2m)!}{(2n+2m)!} P_{2n}^{(2m)}(0) P_{2n}^{(2m)}(\cos i) \quad (m \leq n),$$

e , a – the eccentricity and the semi-major axis of the asteroid's orbit, $P_{2n}^{(2m)}$ – the associated Legendre functions, ${}_2F_1$ – the Gauss hypergeometric function, and ${}_3\tilde{F}_2$ – the regularized Clausen function.

Averaging of the disturbing function

the second averaging, a new result

The transition from the generalized Clausen hypergeometric functions to the Gauss hypergeometric functions has not changed the structure of the series

$$R^{**} = \frac{fm_J}{r_J} \sum_{n=1}^{\infty} D_n \mu^{2n}, \quad (2)$$

but the coefficients D_n have changed

$$D_n = (P_{2n}(0))^2 P_{2n}(\cos i) F_{n,0} + 2P_{2n}(0) \sum_{m=1}^n (-1)^m \binom{2n+2m+1}{2m} F_{n,m} \left(\frac{e}{2}\right)^{2m} P_{2n}^{(-2m)}(0) P_{2n}^{(2m)}(\cos i) \cos 2m\omega, \quad (3)$$
$$F_{n,m} = {}_2F_1\left(m-n, m-n-\frac{1}{2}; 2m+1; e^2\right).$$

In the case of the doubly averaged disturbing function, the system of Lagrange equations in orbital elements possesses a complete set of first integrals:

$$a = c_0, \quad (1 - e^2) \cos^2 i = c_1, \quad R^{**} = c_2$$

and admits a reduction to a system with one degree of freedom in (e, ω) .

Numerical quadrature

determination of the potential of a ring

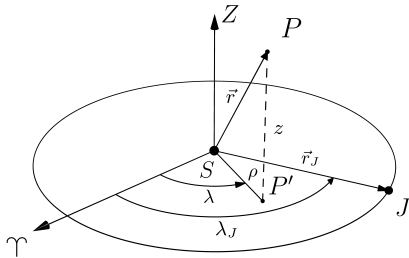
Another representation of the averaged disturbing function is possible:

$$R^{**} = \frac{fm_J}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d\lambda_J d\lambda}{|\vec{r} - \vec{r}_J|},$$

where $\vec{r}_J, \vec{r}$ – the radius vectors of Jupiter and of the asteroid.

Let us introduce:

- ρ – the distance from the origin of coordinates to the projection P' of the asteroid's position onto the plane of Jupiter's orbit
- z – the height of the point P above this plane



Then R^{**} may be rewritten in the form:

$$R^{**} = \frac{fm_J}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d\lambda_J d\lambda}{\sqrt{r_J^2 + \rho^2 + z^2 - 2r_J\rho \cos \Delta\lambda}}, \quad \Delta\lambda = \lambda - \lambda_J.$$

Numerical quadrature

representation in terms of an elliptic integral

The inner integral

$$I = \int_0^{2\pi} \frac{d\lambda_J}{\sqrt{r_J^2 + \rho^2 + z^2 - 2r_J\rho \cos \lambda_J}}$$

admits a representation in terms of the complete elliptic integral of the first kind K with the argument

$$\varkappa^2 = \frac{4r_J\rho}{(r_J + \rho)^2 + z^2},$$

which allows it to be written in the form

$$I = \frac{2\varkappa}{\sqrt{r_J\rho}} \int_0^{\pi/2} \frac{ds}{\sqrt{1 - \varkappa^2 \sin^2 s}} = \frac{2\varkappa}{\sqrt{r_J\rho}} K(\varkappa).$$

Returning to the averaged force function and taking into account the differential substitution of the true anomaly for the mean anomaly

$$R^{**} = \frac{fm_J}{4\pi^2 a^2 \sqrt{1 - e^2}} \int_0^{2\pi} \frac{2r^2 \varkappa K(\varkappa)}{\sqrt{r_J\rho}} d\nu$$

and passing to the Keplerian elements, we obtain

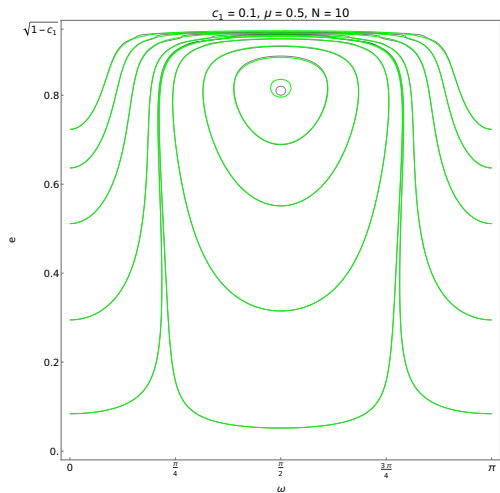
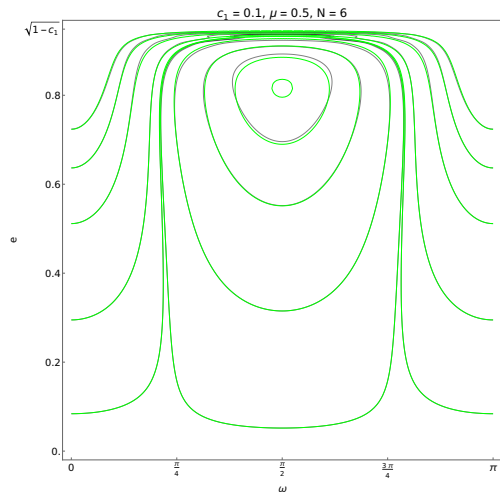
$$R^{**} = \frac{fm_J (1 - e^2)}{4\pi^2 a^{1/2} r_J^{1/2}} \int_0^{2\pi} \frac{2\varkappa K(\varkappa) d\nu}{(1 - \sin^2(\nu + \omega) \sin^2 i)^{1/4} (1 + e \cos \nu)^{3/2}},$$

where

$$\varkappa^2 = \frac{4r_J r \sqrt{1 - \sin^2(\nu + \omega) \sin^2 i}}{r_J^2 + r^2 + 2r_J r \sqrt{1 - \sin^2(\nu + \omega) \sin^2 i}}.$$

Phase portraits

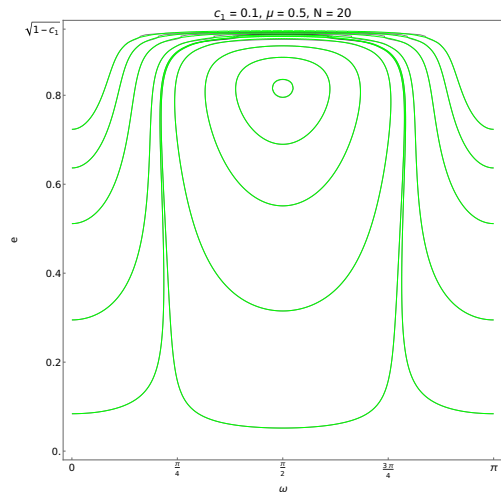
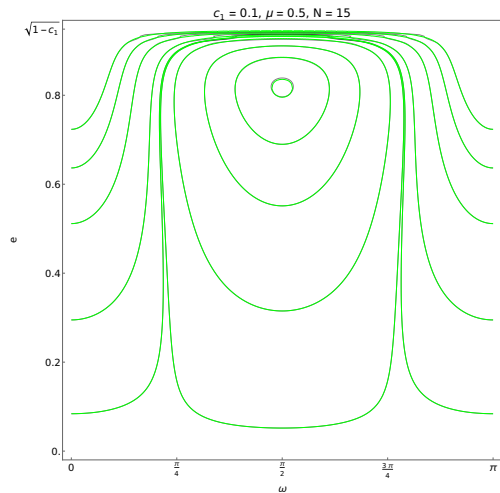
$c_1 = 0.1$, $\mu = 0.5$, $n = 6$ and $n = 10$



Green phase curves – numerical quadrature, grey ones – truncation of the series

Phase portraits

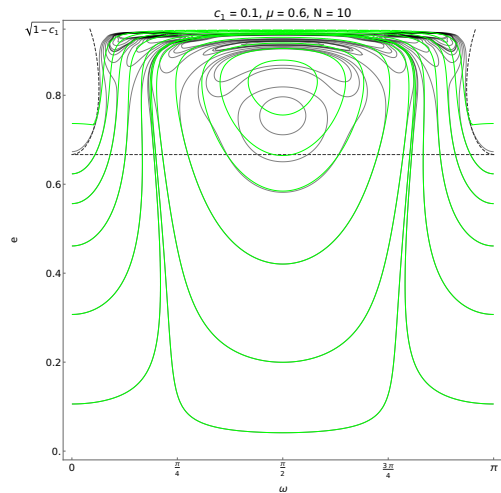
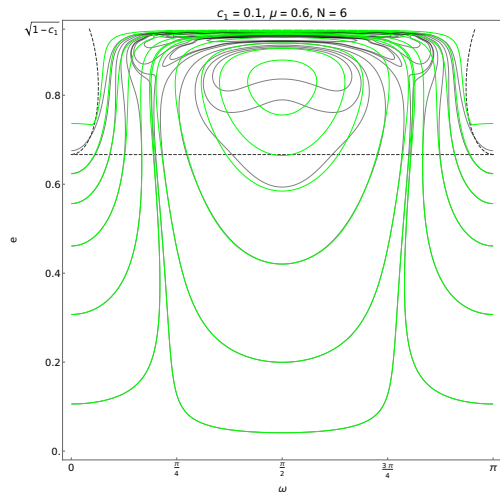
$c_1 = 0.1$, $\mu = 0.5$, $n = 15$ and $n = 20$



Green phase curves – numerical quadrature, grey ones – truncation of the series

Phase portraits

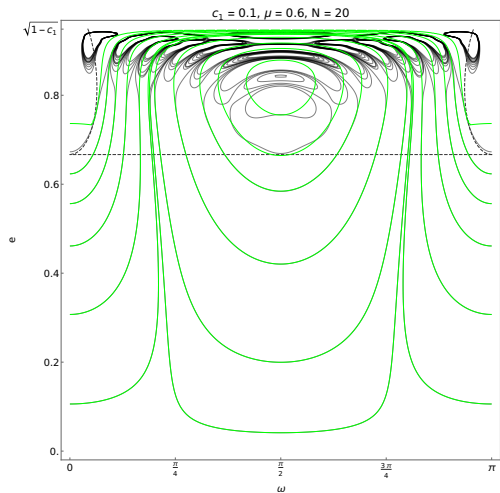
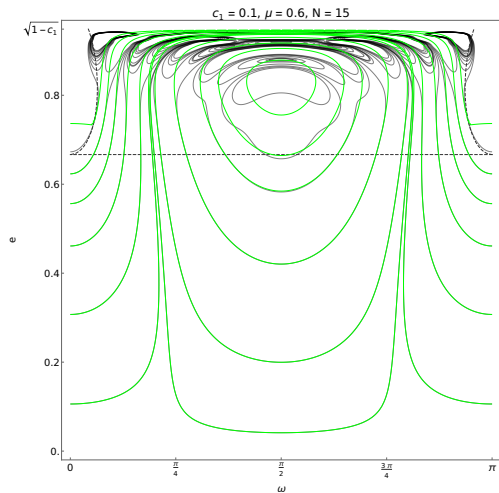
$c_1 = 0.1$, $\mu = 0.6$, $n = 6$ and $n = 10$



Green phase curves – numerical quadrature, grey ones – truncation of the series

Phase portraits

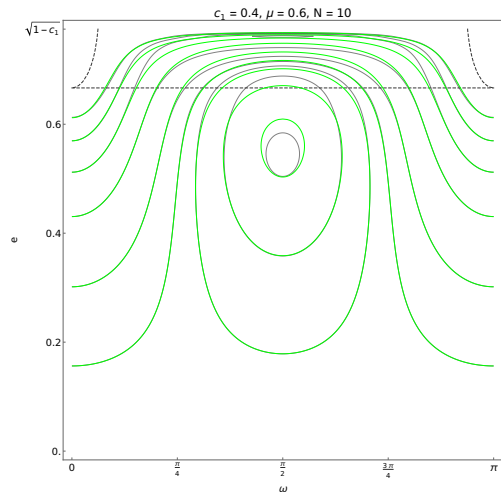
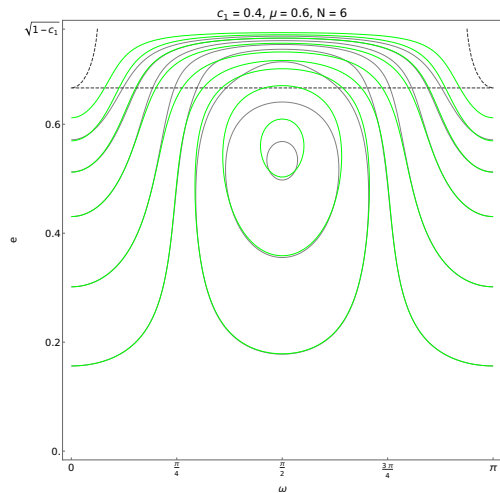
$c_1 = 0.1$, $\mu = 0.6$, $n = 15$ and $n = 20$



Green phase curves – numerical quadrature, grey ones – truncation of the series

Phase portraits

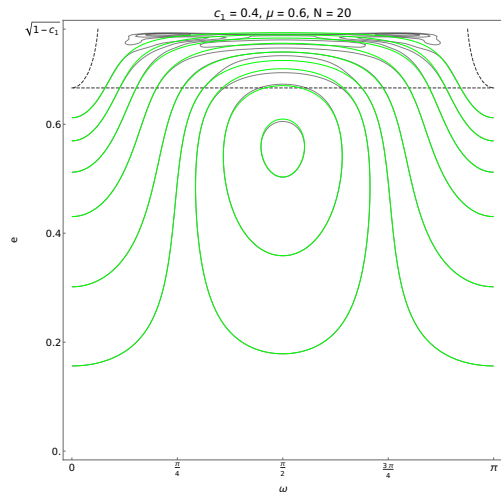
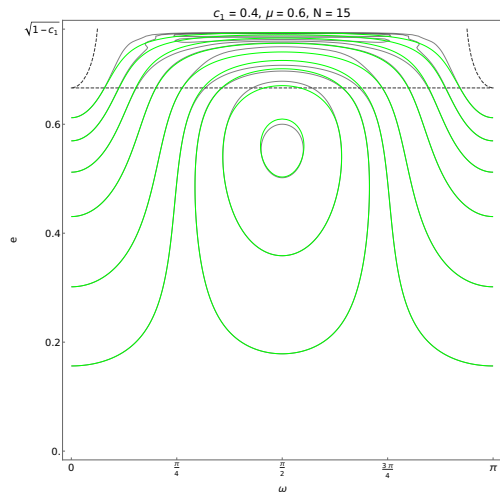
$c_1 = 0.4$, $\mu = 0.6$, $n = 6$ and $n = 10$



Green phase curves – numerical quadrature, grey ones – truncation of the series

Phase portraits

$c_1 = 0.4$, $\mu = 0.6$, $n = 15$ and $n = 20$

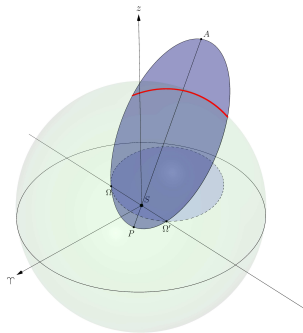


Green phase curves – numerical quadrature, grey ones – truncation of the series

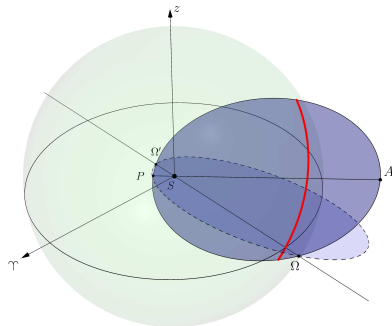
Linked orbits

As is well known, the condition for the orbits to be linked has the form

$$\frac{a(1 - e^2)}{1 + e \cos \omega} < r_J < \frac{a(1 - e^2)}{1 - e \cos \omega},$$



(a) The orbit reaching the sphere of the radius of Jupiter's orbit



(b) The orbits of Jupiter and of the asteroid are linked

The von Zeipel regions

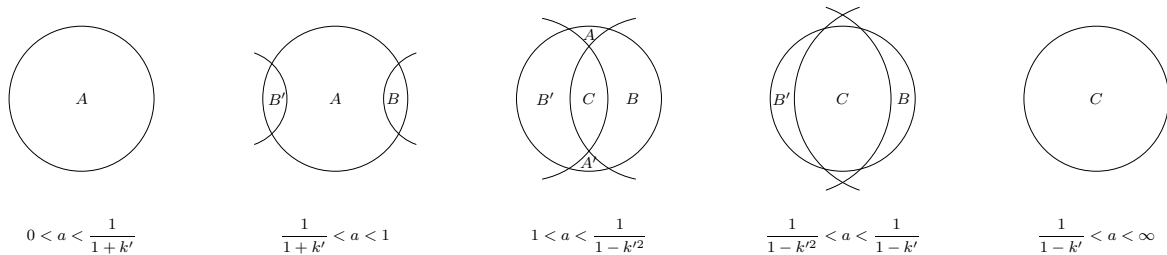


Figure 2: The regions A , A' , B , B' , C in von Zeipel's work

Here a – the ratio of the semi-major axis of the asteroid's orbit to the radius of Jupiter's orbit, $(k')^2 = 1 - c_1$.

the parameter region

- A , A' – both nodes of the asteroid's orbit lie inside Jupiter's orbit, the argument of pericentre being $0 < \omega < \pi$ and $\pi < \omega < 2\pi$ respectively
- B , B' – interlocking of the orbits, one of the nodes of the asteroid's orbit lies outside Jupiter's orbit
- C – both nodes of the asteroid's orbit lie outside Jupiter's orbit.

Theorem 1 (Estimate of the radius of convergence of the power series)

The power series (2) converges on the interval $0 < \mu < (1 + e)^{-1}$; its radius of convergence satisfies $\rho(e) \geq (1 + e)^{-1}$.

Proof.

1. Returning to the first averaging (1), note that $|P_{2n}(x)| \leq 1$ for $x \in [-1, 1]$, whence

$$\left| \left(\frac{r}{r_J} \right)^{2n} P_{2n}(0) P_{2n}(\sin i \sin(\nu + \omega)) \right| \leq \left| \left(\frac{r}{r_J} \right)^{2n} P_{2n}(0) \right|.$$

2. Factor out μ , pass to the eccentric anomaly ($r/a = 1 - e \cos E$, $dM = (1 - e \cos E) dE$), average and estimate:

$$\frac{1}{2\pi} \int_0^{2\pi} \mu^{2n} (1 - e \cos E)^{2n+1} P_{2n}(0) dE \leq P_{2n}(0) \mu^{2n} (1 + e)^{2n+1}.$$

(continued on the next slide) $\rightsquigarrow$

Estimate of the radius of convergence

Proof continued

Outcome of steps 1–2: $|D_n| \mu^{2n} \leq P_{2n}(0) \mu^{2n} (1+e)^{2n+1}$.

Proof (continued).

3. Apply Wallis' inequality $P_{2n}(0) = (2n)! 4^{-n} (n!)^{-2} \leq (\pi n)^{-1/2}$:

$$P_{2n}(0) \mu^{2n} (1+e)^{2n+1} \leq \mu^{2n} (1+e)^{2n+1} (\pi n)^{-1/2}.$$

4. This yields the bound on the coefficients D_n :

$$|D_n| \leq C(e, n) (1+e)^{2n}, \quad C(e, n) = (1+e) (\pi n)^{-1/2}.$$

5. Since $C(e, n)^{1/2n} \rightarrow 1$, we get $\limsup_n |D_n|^{1/2n} \leq 1+e$; by the Cauchy–Hadamard formula the radius of convergence satisfies $\rho(e) \geq (1+e)^{-1}$, and the series converges absolutely for $0 < \mu < (1+e)^{-1}$. □

Corollaries of the theorem on the radius of convergence

Corollary 1

The series converges uniformly in i and ω whenever $\mu(1+e) < 1$.

Corollary 2

The series converges uniformly in e as well whenever $\mu < 1/2$.

Remark 1

The uniform bound can be sharpened to $O(n^{-3/2})$. The polynomial factor $C(e, n)$ is not optimal: it can be improved from $n^{-1/2}$ to $n^{-3/2}$, and $n^{-3/2}$ is the best exponent attainable uniformly in i and ω . Specifically, we expect

$$|D_n| \leq \frac{(1+e)^{3/2}}{2\pi^{3/2}\sqrt{e}} \cdot \frac{(1+e)^{2n}}{n^{3/2}} (1+o(1)), \quad n \rightarrow \infty, \quad (4)$$

Remark 2 (at the cost of uniformity the gain is exponential, not polynomial)

For a *fixed* $\omega \neq \pi/2$ the estimate can be improved far beyond any power of n : what improves is the exponential rate itself, while the polynomial factor stays $n^{-3/2}$.

Results:

- The secular part of the disturbing function has been expressed in terms of the Gauss functions
- A numerical quadrature has been obtained for the averaged perturbation function
- Phase portraits have been constructed both for the numerical quadrature and for the truncated power series
- The radius of convergence of the power series has been determined