

On Refining the Applicability Conditions of the Power Series in the Circular Restricted Three-Body Problem

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Abstract. The secular evolution of an asteroid orbit linked with the circular orbit of Jupiter is studied in the inner restricted three-body problem. It is shown that the power series expansion of the doubly averaged perturbing function has the radius of convergence $\rho(e) = 1/(1 + e)$ and is therefore inapplicable in the region $a(1 + e) > r_J$, where spurious equilibria appear on the phase portraits. This region is wider than the one indicated by von Zeipel.

Keywords. restricted three-body problem, secular perturbations, averaging, power series, radius of convergence.

We consider the classical problem of analyzing the secular perturbations of the Keplerian orbital elements of an asteroid in the inner restricted three-body problem, when the asteroid's orbit, with its focus at the Sun, is linked with the circular orbit of the perturbing body (Jupiter). As is well known, the linking condition reads

$$\frac{a(1 - e^2)}{1 + e \cos \omega} < r_J < \frac{a(1 - e^2)}{1 - e \cos \omega}, \quad (1)$$

where a , e , ω are the semi-major axis, the eccentricity, and the argument of pericenter of the asteroid's orbit, and r_J is the radius of Jupiter's orbit. Based on condition (1), von Zeipel [1] introduces the parameter region A , where both nodes of the asteroid's orbit lie inside Jupiter's orbit; the orbit-linking regions B and B' , where one of the nodes of the asteroid's orbit lies outside Jupiter's orbit; and the region C , where both nodes of the asteroid's orbit lie outside Jupiter's orbit.

Averaging over the fast variables (λ_J , the longitude of Jupiter, and l , the mean longitude of the asteroid) yields the following expression for the perturbing function in region A :

$$R^{**} = \frac{fm_J}{r_J} \sum_{n=1}^{\infty} D_n \mu^{2n}, \quad (2)$$

where

$$\mu = \frac{a}{r_J}, \quad D_n = P_{2n}(0) \left(P_{2n}(0) P_{2n}(\cos i) \Phi(0, n) + \right. \\ \left. 2 \sum_{m=1}^n (-1)^m \binom{2(n+m)+1}{2m} \Phi(m, n) \left(\frac{e}{2} \right)^{2m} P_{2n}^{(-2m)}(0) P_{2n}^{(2m)}(\cos i) \cos 2m\omega \right), \\ \Phi(m, n) = {}_2F_1\left(m - n, m - n - \frac{1}{2}; 2m + 1; e^2\right).$$

Here i is the inclination of the asteroid's orbit, m_J is the mass of Jupiter, f is the gravitational constant, P_{2n} and $P_{2n}^{(2m)}$ are the Legendre polynomials, and ${}_2F_1$ is the Gauss hypergeometric function.

In the case of the doubly averaged perturbing function, the system of Lagrange equations in the orbital elements possesses a complete set of first integrals:

$$a = c_0, \quad (1 - e^2) \cos^2 i = c_1, \quad R^{**} = c_2,$$

and admits reduction to a system with one degree of freedom in (e, ω) . Analysis of the phase portraits in (e, ω) has shown that the phase curves in the region $A' : a(1 + e) > r_J$ are unreliable: spurious equilibria appear, and their number grows as the number of retained terms of the series increases.

This fact has been confirmed analytically: it was established that the power series (2) converges absolutely in the interval

$$0 < \mu < \frac{1}{1 + e},$$

so that the radius of convergence is

$$\rho(e) = \frac{1}{1 + e}.$$

This result was also verified numerically: relying on the asymptoticity estimate from [2], the following quantity was computed for each point of the parameter space (e, ω, μ, c_1) :

$$n_{\max} = \sup \left\{ n \in \mathbb{N} \mid |R^{(i)} - R_n^{**}| < O(\mu^{2n+2}) \right\},$$

where $R^{(i)}$ is the numerically computed value of the doubly averaged perturbing function and R_n^{**} is the truncation of the series R^{**} to n terms. Maps of $n_{\max}$ in the (e, ω) plane were constructed for various values of c_1 and μ , confirming the non-asymptotic behavior of the series R_n^{**} in the region A' . It should be noted that the region A' is wider than the region indicated in [1], for which von Zeipel proposes using the power series, since it includes both the regions B and B' and a part of the region A .

References

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