

Cosmology on the extended spacetime

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Abstract. We consider a multidimensional cosmological model with spacetime $M^4 \times SU(2)$, where the extra dimensions form the Lie group $SU(2)$ equipped with a bi-invariant metric. The Riemannian curvature of $SU(2)$ contributes to the scalar curvature of the total manifold, naturally generating the cosmological constant $\Lambda = 3/b_0^2$, where b_0 is the stabilized scale factor of the internal space. The generalized Friedmann equations are reduced to a form suitable for numerical integration and solved.

We also propose the concept of a *quasiperiodic time manifold*. It is generally accepted that if the time is a coordinate on a compact manifold then it is cyclic and there is the problem of closed timelike paths. But the time coordinate can also be defined as a coordinate on the *universal covering space of a compact manifold* G . This eliminates closed timelike curves without changing the metric. Hence the spacetime of the general relativity theory becomes $G \times \mathbb{R}^3$ with the covering space $\mathbb{R}_+^4$.

Keywords: Kaluza–Klein cosmology, Supernova Ia, Dark energy, Cosmological constant.

1. Introduction

The accelerated expansion of the Universe is a central problem in modern cosmology. Within the standard Λ CDM model, this acceleration is attributed to a cosmological constant Λ , which is treated as a free parameter. Multidimensional Kaluza–Klein theories offer a geometric explanation for Λ arising from the curvature of extra dimensions [1]. In this work, we consider the spacetime $M^4 \times SU(2)$, where M^4 is the Friedmann–Lemaître–Robertson–Walker (FLRW) space and the extra dimensions form the compact Lie group $SU(2)$ (the 3-sphere) with a bi-invariant metric. It was previously shown [2] that the curvature of $SU(2)$ yields a nonzero contribution to the scalar curvature, exactly corresponding to Λ .

A known issue in higher-dimensional cosmologies is the collapse of extra dimensions as in the work of D. Sahdev [4], where the internal scale factor $b(t)$ inevitably tends to zero. In our model, the intrinsic geometry of $SU(2)$ provides a

natural stabilization mechanism, allowing $b(t)$ to asymptotically approach a finite value b_0 , thereby generating the effective cosmological constant.

The objectives of this work are: (i) to reduce the generalized Friedmann equations to a form suitable for numerical integration; (ii) to study the dynamical stabilization of the internal dimensions; (iii) to estimate cosmological parameters from Type Ia supernova data using a nonlinear regression method.

2. Generalized Metric and Friedmann Equations

The FLRW metric is

$$ds_{\text{FRW}}^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right), \quad (1)$$

and the bi-invariant metric on $SU(2)$ is

$$ds_G^2 = b(t)^2 (d\xi^2 + \sin^2 \xi d\psi^2 + \sin^2 \xi \sin^2 \psi d\zeta^2), \quad (2)$$

where $a(t)$ and $b(t)$ are the scale factors of the four-dimensional space and the internal space, respectively. The total metric is $ds^2 = ds_{\text{FRW}}^2 + ds_G^2$.

The stress-energy tensor is assumed to be

$$T_{\nu}^{\mu} = \text{Diag}(\rho, -P, -P, -P, \sigma, \sigma, \sigma), \quad (3)$$

where ρ is the mass density, P is the pressure, and σ is the density associated with the extra dimensions. From the Einstein equations, we obtain the generalized Friedmann equations [2]:

$$\frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{c^2 a^2} + \frac{9\dot{a}\dot{b}}{c^2 ab} + \frac{3\dot{b}^2}{c^2 b^2} - \frac{3}{b^2} = \kappa\rho, \quad (4)$$

$$\frac{\kappa}{a^2} + \frac{\dot{a}^2}{c^2 a^2} + \frac{2\ddot{a}}{c^2 a} + \frac{6\dot{a}\dot{b}}{c^2 ab} + \frac{3\ddot{b}}{c^2 b} + \frac{3\dot{b}^2}{c^2 b^2} - \frac{3}{b^2} = -\kappa P, \quad (5)$$

$$\frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{c^2 a^2} + \frac{3\ddot{a}}{c^2 a} + \frac{6\dot{a}\dot{b}}{c^2 ab} + \frac{2\ddot{b}}{c^2 b} + \frac{\dot{b}^2}{c^2 b^2} - \frac{1}{b^2} = \kappa\sigma. \quad (6)$$

The consistency condition from the Bianchi identity yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) + 3\frac{\dot{b}}{b}(\rho - \sigma) = 0. \quad (7)$$

Assuming $P = (\gamma - 1)\rho$ and $\sigma = (1 - \gamma_1)\rho$, we obtain

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0} \right)^{-3\gamma} \left(\frac{b}{b_0} \right)^{-3\gamma_1}. \quad (8)$$

In what follows, we take $\gamma = 1$ (dust-like matter).

3. Special Solution: $b = \text{const}$ and the Cosmological Constant

If $b \equiv b_0 = \text{const}$, equation (4) reduces to

$$\frac{\dot{a}^2}{c^2 a^2} = \frac{\varkappa}{3} \rho + \frac{1}{b_0^2} - \frac{\kappa}{a^2}. \quad (9)$$

Comparing with the standard Λ CDM Friedmann equation,

$$\frac{\dot{a}^2}{c^2 a^2} = \frac{\varkappa}{3} \rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2},$$

we obtain

$$\Lambda = \frac{3}{b_0^2}, \quad b_0 = \sqrt{\frac{3}{\Lambda}}. \quad (10)$$

Thus, the cosmological constant is naturally determined by the stabilized scale factor of the extra dimensions. The dark energy density parameter is

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = \frac{c^2}{H_0^2 b_0^2} = 1 - \Omega_m - \Omega_\kappa. \quad (11)$$

4. Numerical Solutions and Stabilization of Extra Dimensions

To solve the system numerically, we rewrite equations (4) and (5) in terms of the dimensionless Hubble parameter. From (4):

$$\frac{\dot{a}}{a} = \sqrt{\frac{\varkappa \rho c^2}{3} - \frac{\kappa c^2}{a^2} + \frac{c^2}{b^2} + \frac{5\dot{b}^2}{4b^2} - \frac{3}{2} \frac{\dot{b}}{b}}. \quad (12)$$

The second derivative of b is

$$\ddot{b} = \frac{1}{5} \left(-\frac{2H_0^2 \Omega_\kappa b^2 \dot{a}}{a^3 \dot{b}} + \frac{3b(6\dot{a}^2 - 7H_0^2 \Omega_\kappa)}{a^2} + \frac{3b(2\gamma_1 - 7)H_0^2 \Omega_m}{a^3} \left(\frac{b}{b_0} \right)^{-3\gamma_1} - \frac{8(c^2 - \dot{b}^2)}{b} + \frac{39a\dot{b}}{a} \right). \quad (13)$$

The initial conditions are set at the recombination epoch: $t_{\text{start}} = 0.00038$ billion years, $a_{\text{start}} = 1/(1 + 1089) = 0.000917$. The initial value b_{start} is chosen such that $b(t_0) = b_0$ at the present epoch.

We distinguish three cases based on the spatial curvature Ω_κ :

1. **Positive curvature** ($\Omega_\kappa > 0$): The function $b(t)$ decreases monotonically and asymptotically approaches b_0 . The derivative $\dot{b}$ reaches zero at finite time t_1 , where the solution must be glued to the special solution $b \equiv b_0$. This corresponds to the Λ CDM phase with accelerated expansion.
2. **Flat Universe** ($\Omega_\kappa = 0$): The function $b(t)$ first decreases, then increases. The derivative changes sign without singularities, but the expansion of $a(t)$ is not accelerated. This case is not cosmologically interesting.

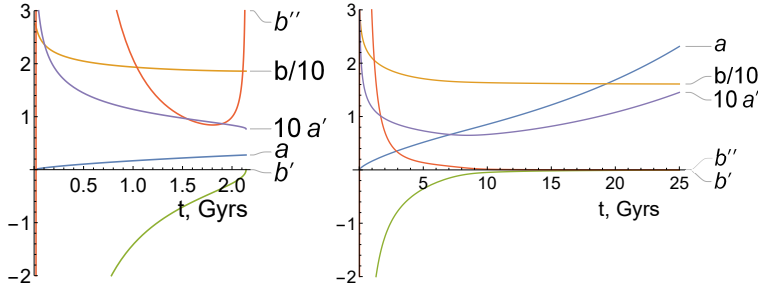


FIGURE 1. Scale factors $a(t)$, $b(t)$ and their derivatives for $\Omega_\kappa = 0.1$ (left) and $\Omega_\kappa = -0.1$ (right).

3. **Negative curvature** ($\Omega_\kappa < 0$): The function $b(t)$ decreases monotonically and asymptotically approaches b_0 as $t \rightarrow \infty$. The expansion is accelerated in the current epoch.

For $\Omega_\kappa < 0$, the asymptotic behavior is exponential:

$$b(t) - b_0 \sim e^{-ct/b_0}, \quad a(t) \sim e^{ct/b_0}.$$

Parameter Estimation for this 3-dimensional parameter space can be done based on the Type Ia Supernova Data [5, 6].

We can also propose the concept of a *quasiperiodic time manifold*. It is generally accepted that if the time is a coordinate on a compact manifold then it is cyclic and there is the problem of closed timelike paths. But the time coordinate can also be defined as a coordinate on the *universal covering space of a compact manifold* G . This eliminates closed timelike curves without changing the metric. Hence the spacetime of the general relativity theory becomes $G \times \mathbb{R}^3$ with the covering space $\mathbb{R}_1^4$ (with the classical Lorentz signature) [3].

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