



Санкт - Петербургский Государственный Университет

МАТЕМАТИКО - МЕХАНИЧЕСКИЙ
ФАКУЛЬТЕТ



Cosmology on the extended Kaluza–Klein spacetime

Analytical Methods of Celestial Mechanics 2026

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Introduction

The Universe is homogeneous and isotropic on large scales (cosmological principle). Observations of CMB show temperature fluctuations of order $\frac{\Delta T}{T} \approx 10^{-5}$. Density perturbations $\frac{\Delta \rho}{\rho} \lesssim 10^{-4}$. This suggests that our 3-dim space is one of the three spaces of constant curvature:

- ▶ Positive curvature — 3-sphere S^3 .
- ▶ Zero curvature — Euclidean space $\mathbb{R}^3$.
- ▶ Negative curvature — hyperbolic 3-space.

Metric

The Friedmann–Lemaître–Robertson–Walker metric is

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right)$$

or in the conformal time η

$$ds^2 = a(\eta)^2 \left(d\eta^2 - \frac{dr^2}{1 - \kappa r^2} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right)$$

with the signature $(+, -, -, -)$.

Christoffel symbols

Nonzero Christoffel symbols are

$$\Gamma_{0\nu}^{\mu} = \Gamma_{\nu 0}^{\mu} = \frac{a'}{a} \delta_{\nu}^{\mu}, \quad \Gamma_{\mu\nu}^0 = -\frac{a'}{c^2 a} g_{\mu\nu}$$

and spatial components

$$\Gamma_{11}^1 = \frac{\kappa r}{1 - \kappa r^2}, \quad \Gamma_{22}^1 = -r(1 - \kappa r^2), \quad \Gamma_{33}^1 = -r(1 - \kappa r^2) \sin^2 \theta,$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \quad \Gamma_{33}^2 = -\frac{1}{2} \sin 2\theta,$$

$$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{r}, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \cot \theta.$$

Sectional curvature

The sectional curvature $K_{u \wedge v}$ is defined as

$$K_{u \wedge v} = \frac{\langle R(u, v)v, u \rangle}{\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2},$$

where R is the Riemann curvature tensor. For FLRW, the sectional curvature in the planes 0–1, 0–2, 0–3 equals $\frac{a''}{c^2 a}$.

In spatial planes 1–2, 1–3, 2–3 it equals $\frac{\kappa c^2 + a'^2}{c^2 a^2}$.

Ricci tensor and scalar curvature

The Ricci tensor is

$$R_{00} = -\frac{3a''}{a}, \quad R_{\mu\nu} = \left(-\frac{2\kappa}{a^2} - \frac{2a'^2}{c^2 a^2} - \frac{a''}{c^2 a} \right) g_{\mu\nu}.$$

The scalar curvature is

$$R = -\frac{6\kappa}{a^2} - \frac{6a'^2}{c^2 a^2} - \frac{6a''}{c^2 a}.$$

Einstein equations

The Einstein equations are obtained by varying the action

$$-\frac{1}{2\kappa} \int R \sqrt{|\det(g_{\mu\nu})|} d^4x + S_m,$$

where S_m is the matter action. They are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}.$$

The left-hand side is the Einstein tensor $G_{\mu\nu}$.

With the cosmological constant Λ , these equations become

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} + \Lambda g_{\mu\nu},$$

where $\kappa = \frac{8\pi G_N}{c^2} = 1.8659218 \cdot 10^{-26}$ m/kg is Einstein's gravitational constant. The stress-energy tensor is

$$(T_{\nu}^{\mu})_{\mu,\nu=0,\dots,3} = \text{Diag}(\rho, -P, -P, -P),$$

where ρ is the energy density and P is the pressure. The conservation law $\nabla_{\mu} T_{\nu}^{\mu} = 0$ gives

$$\rho' + 3\frac{a'}{a}(\rho + P) = 0.$$

Friedmann equations

Assuming an equation of state of a perfect fluid $P = (\gamma - 1)\rho$ with $1 \leq \gamma \leq 4/3$, we get

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0} \right)^{-3\gamma}.$$

The Friedmann equations are

$$\begin{cases} G_0^0 = \frac{3\kappa}{a^2} + \frac{3a'^2}{c^2 a^2} = \varkappa \rho + \Lambda \\ G_1^1 = G_2^2 = G_3^3 = \frac{\kappa}{a^2} + \frac{a'^2}{c^2 a^2} + \frac{2a''}{c^2 a} = -\varkappa P + \Lambda \end{cases}$$

which are equivalent to

$$\begin{cases} \frac{a'^2}{c^2 a^2} = \frac{\varkappa}{3} \rho - \frac{\kappa}{a^2} + \frac{\Lambda}{3} \\ \frac{a''}{c^2 a} = -\frac{\varkappa}{6} (\rho + 3P) + \frac{\Lambda}{3} \end{cases}$$

Density parameters

The critical density is $\rho_c = \frac{3H_0^2}{8\pi G_N} = \frac{3H_0^2}{8\pi G_N}$. We define

$$\Omega_m = \frac{\rho}{\rho_c}, \quad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \quad \text{and} \quad \Omega_\kappa = -\frac{\kappa c^2}{H_0^2 a^2}.$$

Then the Friedmann equation becomes

$$\left(\frac{a'}{a}\right)^2 = H^2 = H_0^2 (\Omega_m + \Omega_\Lambda + \Omega_\kappa),$$

and hence $\Omega_{0m} + \Omega_\Lambda + \Omega_{0\kappa} = 1$.

Solution

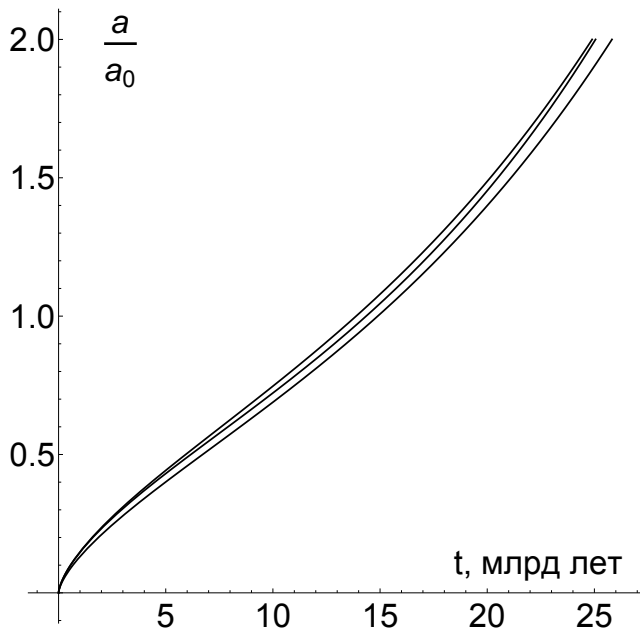
The cosmic time as a function of the scale factor

$$t = \frac{1}{H_0} \int_0^a \frac{da}{a \sqrt{\Omega_{0m} (a/a_0)^{-3\gamma} + \Omega_\Lambda + \Omega_{0\kappa} (a_0/a)^2}}.$$

For a flat universe ($\kappa = 0$)

$$t = \frac{2}{3\gamma H_0 \sqrt{\Omega_\Lambda}} \operatorname{arsinh} \sqrt{\frac{a^{3\gamma} \Omega_\Lambda}{a_0^{3\gamma} (1 - \Omega_\Lambda)}}.$$

$$\text{At present, } t_0 = \frac{2}{3\gamma H_0 \sqrt{\Omega_\Lambda}} \operatorname{arsinh} \sqrt{\frac{\Omega_\Lambda}{1 - \Omega_\Lambda}} = \frac{2}{3\gamma H_0 \sqrt{\Omega_\Lambda}} \operatorname{artanh} \sqrt{\Omega_\Lambda}.$$



Accelerating expansion

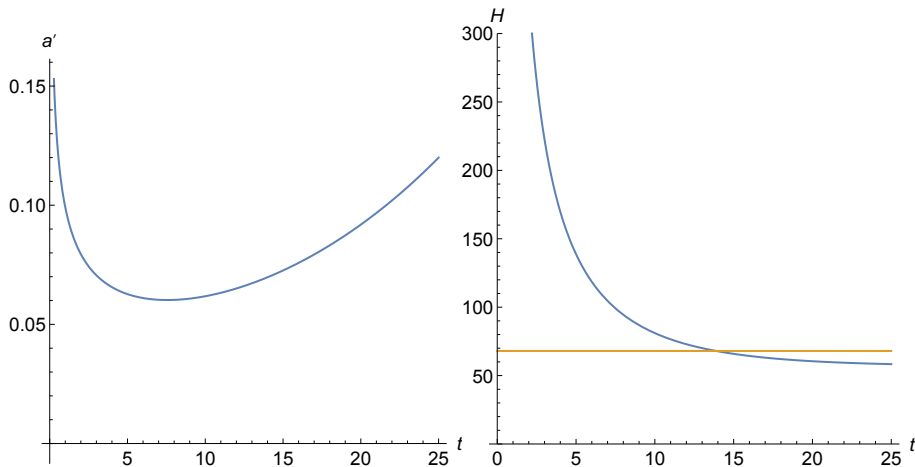


Figure 1: Cosmic time t in Gyr

Luminosity distance

For a source with luminosity $\mathcal{L}$, the observed flux is $\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2}$, where d_L is the luminosity distance. In an expanding Universe,

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi(1+z)^2 a_0^2 r^2}.$$

Thus $d_L = (1+z)a_0 r$. Using the comoving coordinate χ defined by $d\chi^2 = \frac{dr^2}{1-\kappa r^2}$, we have $r = \sin_\kappa \chi$, where

$$\sin_\kappa \chi = \begin{cases} \sin \chi, & \kappa = 1, \\ \chi, & \kappa = 0, \\ \sinh \chi, & \kappa = -1. \end{cases}$$

The comoving distance is

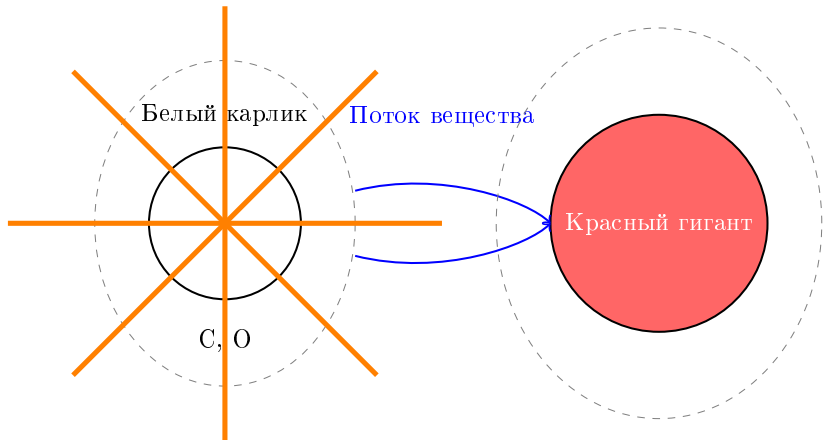
$$\chi = \frac{c}{a_0} \int_0^z \frac{dz}{H(z)}.$$

Hence

$$d_L = (1 + z) \frac{c}{H_0 \sqrt{|\Omega_{0\kappa}|}} \sin_{\kappa} \left(\sqrt{|\Omega_{0\kappa}|} \int_0^z \frac{dz_1}{E(z_1)} \right),$$

where $E(z) = \frac{H(z)}{H_0}$. For $\kappa = 0$ the limit is taken. The distance modulus is $\mu = 5 \log_{10}(d_L/10 \text{ pc})$.

Сверхновая Ia



Двойная система: белый карлик + красный гигант
формирование сверхновой Ia

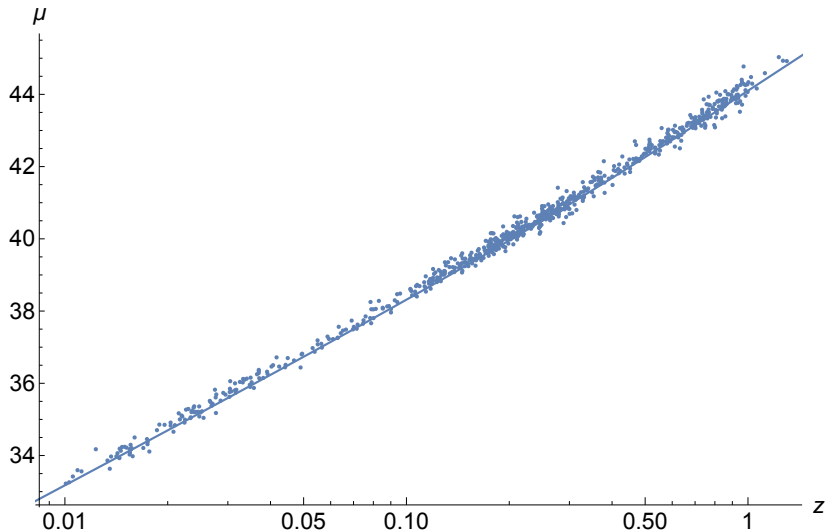


Figure 2: The Hubble diagram.

Regression analysis

This is still an introductory part.

To check my algorithm I repeated M. Betoule calculations using Wolfram Mathematica. The χ^2 function used to estimate cosmological parameters:

$$\chi^2 = \sum_i \frac{(\mu_i - \mu(d_L^{\text{theor}}(z_i, H_0, \Omega_{0m}, \Omega_\Lambda, \gamma)))^2}{\sigma_i^2},$$

where σ_i are the uncertainties. More generally, with the covariance matrix $\mathcal{C}$ published in [6, 7]:

$$\chi^2 = \sum_{i,j} \Delta\mu_i (\mathcal{C}^{-1})_{ij} \Delta\mu_j.$$

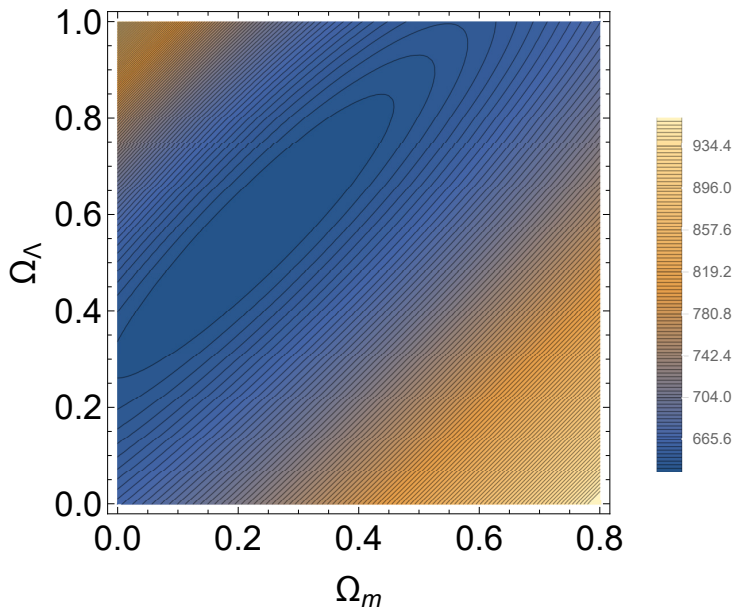


Figure 3: The classical result.

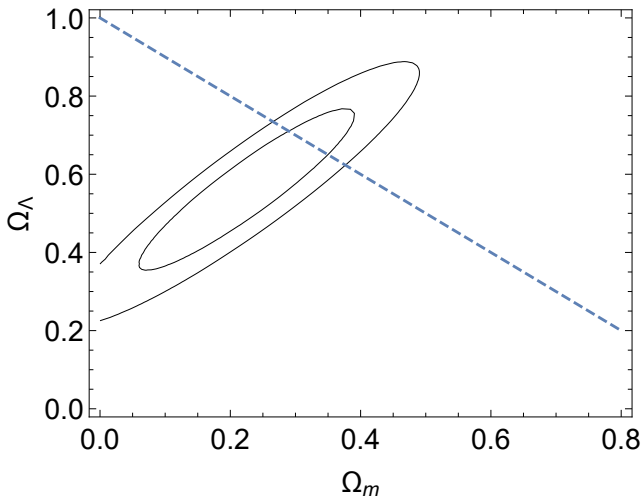


Figure 4: Confidence contours at 68% and 95% in the $(\Omega_m, \Omega_\Lambda)$ plane.

Problems of Modern Cosmology

- ▶ Matter is assumed to be created from “pure energy”. However, particles are born in particle–antiparticle pairs. Why does matter dominate while antimatter is practically nonexistent? This constitutes a violation of the baryon number conservation law.
- ▶ Too much energy is required to produce particles in pairs first and then preserve only ΔM .
- ▶ Unknown factors:
 - ▶ The nature of dark matter and dark energy.
 - ▶ The cause of discrepancies in measuring the Hubble constant, $H_0 = 67.4 \dots 73 \frac{\text{km/s}}{\text{Mpc}}$.
 - ▶ The cause of anomalies in the CMB.
 - ▶ The cause of anomalies in the large-scale structure.
 - ▶ The cause of the emergence of “early galaxies”.

Generalized Metric

Let us consider the space $M^4 \times SU(2)$ where the FRW metric on M^4 is as usual

$$ds_{\text{FRW}}^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right),$$

and the bi-invariant metric on $SU(2)$ is

$$ds_G^2 = b(t)^2 (d\xi^2 + \sin^2 \xi d\psi^2 + \sin^2 \xi \sin^2 \psi d\zeta^2),$$

where $a(t)$ and $b(t)$ are the scale factors of the four-dimensional space and the internal space, respectively. The total metric is

$$ds^2 = ds_{\text{FRW}}^2 + ds_G^2.$$

The stress-energy tensor

$$T_{\hat{\nu}}^{\hat{\mu}} = \text{Diag}(\rho, -P, -P, -P, \sigma, \sigma, \sigma),$$

where ρ is the mass density, P is the pressure, and σ is the density associated with the extra dimensions. From the Einstein equations, we obtain the generalized Friedmann equations [2]:

$$\frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{c^2 a^2} + \frac{9\dot{a}\dot{b}}{c^2 ab} + \frac{3\dot{b}^2}{c^2 b^2} - \frac{3}{b^2} = \kappa\rho, \quad (1)$$

$$\frac{\kappa}{a^2} + \frac{\dot{a}^2}{c^2 a^2} + \frac{2\ddot{a}}{c^2 a} + \frac{6\dot{a}\dot{b}}{c^2 ab} + \frac{3\ddot{b}}{c^2 b} + \frac{3\dot{b}^2}{c^2 b^2} - \frac{3}{b^2} = -\kappa P, \quad (2)$$

$$\frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{c^2 a^2} + \frac{3\ddot{a}}{c^2 a} + \frac{6\dot{a}\dot{b}}{c^2 ab} + \frac{2\ddot{b}}{c^2 b} + \frac{\dot{b}^2}{c^2 b^2} - \frac{1}{b^2} = \kappa\sigma. \quad (3)$$

Matter density

The consistency condition from the Bianchi identity yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) + 3\frac{\dot{b}}{b}(\rho - \sigma) = 0.$$

Assuming perfect-fluid equations of state $P = (\gamma - 1)\rho$ and $\sigma = (1 - \gamma_1)\rho$, we obtain

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3\gamma} \left(\frac{b}{b_0}\right)^{-3\gamma_1}.$$

In what follows, we take $\gamma = 1$ (dust-like matter).

Dark energy

If $b \equiv b_0 = \text{const}$, equation (1) reduces to

$$\frac{\dot{a}^2}{c^2 a^2} = \frac{\kappa}{3} \rho + \frac{1}{b_0^2} - \frac{\kappa}{a^2}.$$

Comparing with the standard Λ CDM Friedmann equation,

$$\frac{\dot{a}^2}{c^2 a^2} = \frac{\kappa}{3} \rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2},$$

we obtain

$$\Lambda = \frac{3}{b_0^2}, \quad b_0 = \sqrt{\frac{3}{\Lambda}}.$$

Thus, the cosmological constant is naturally determined by the stabilized scale factor of the extra dimensions. The dark energy density parameter is

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = \frac{c^2}{H_0^2 b_0^2} = 1 - \Omega_m - \Omega_\kappa.$$

Evolution for $\Omega_\kappa = 0.1$

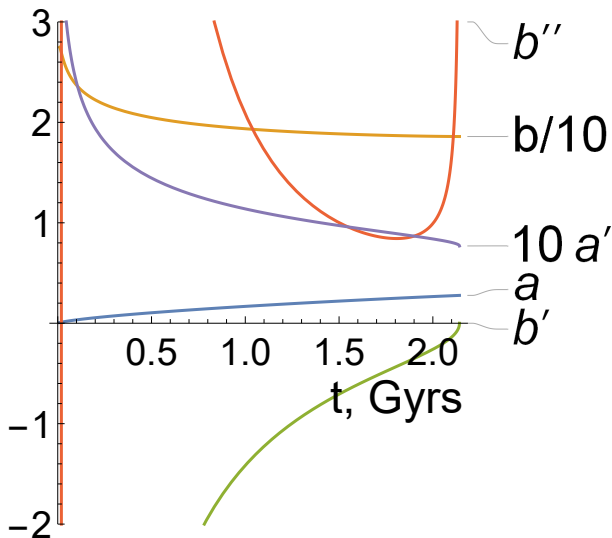


Figure 5: Scale factors $a(t)$, $b(t)$ and their derivatives for $\Omega_\kappa = 0.1$.

Evolution for $\Omega_\kappa = -0.1$

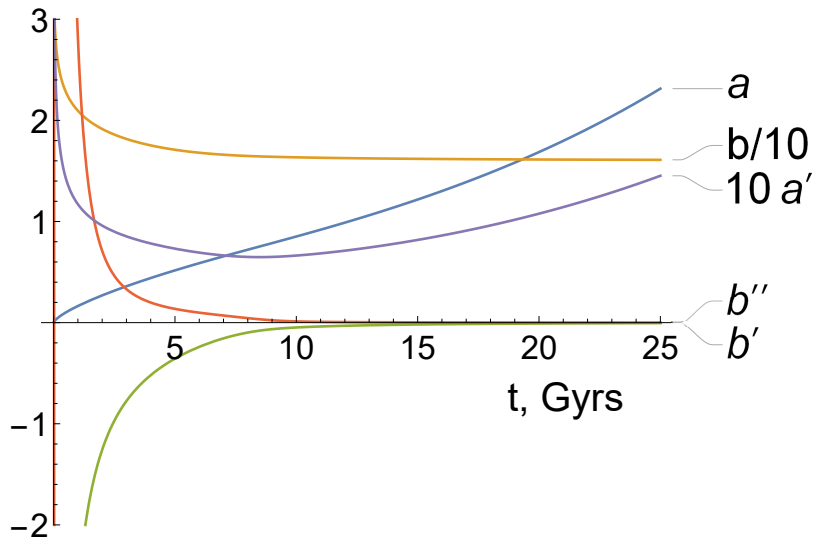


Figure 6: Scale factors $a(t)$, $b(t)$ and their derivatives for $\Omega_\kappa = -0.1$.

Evolution asymptotic

1. **Positive curvature** ($\Omega_\kappa > 0$): The function $b(t)$ decreases monotonically and asymptotically approaches b_0 . The derivative $\dot{b}$ reaches zero at finite time t_1 , where the solution must be glued to the special solution $b \equiv b_0$. This corresponds to the Λ CDM phase with accelerated expansion.
2. **Flat Universe** ($\Omega_\kappa = 0$): The function $b(t)$ first decreases, then increases. The derivative changes sign without singularities, but the expansion of $a(t)$ is not accelerated. This case is not cosmologically interesting.
3. **Negative curvature** ($\Omega_\kappa < 0$): The function $b(t)$ decreases monotonically and asymptotically approaches b_0 as $t \rightarrow \infty$. The expansion is accelerated in the current epoch.

Confidence contours

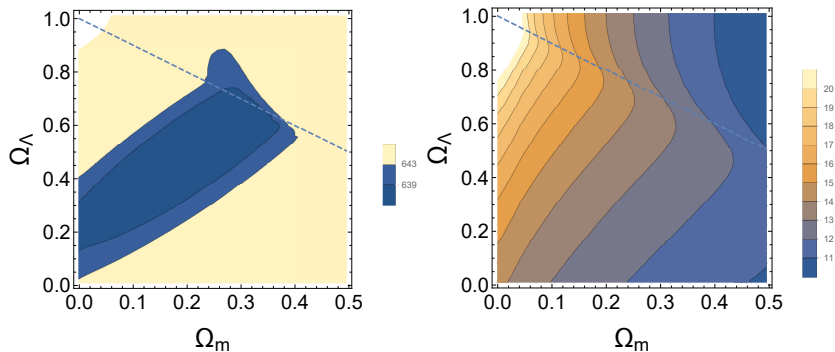
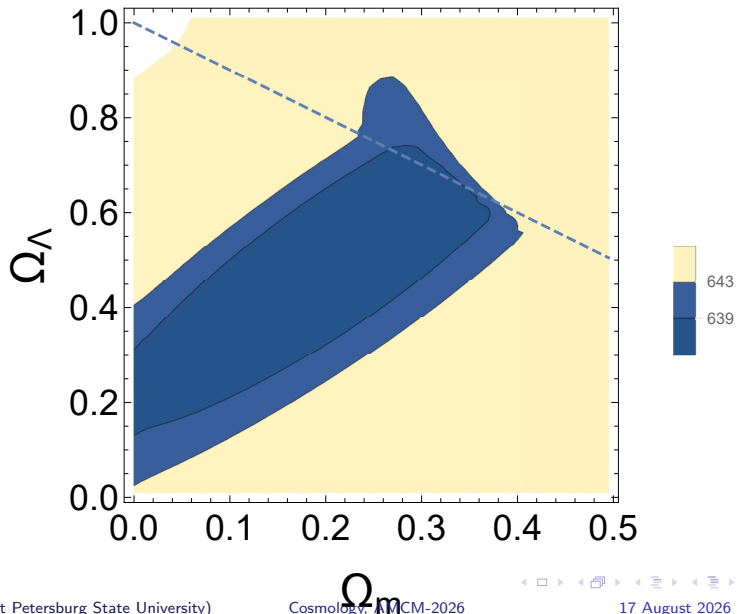
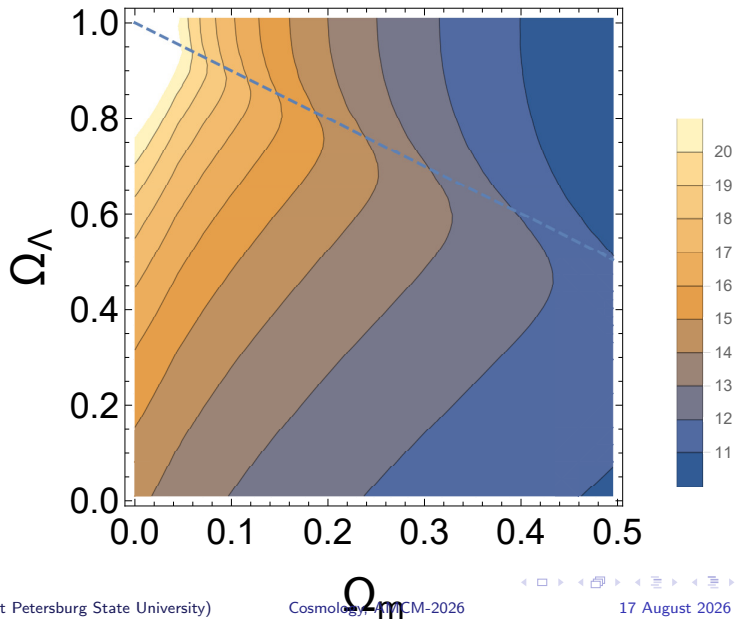


Figure 7: Left: Confidence contours at 68% and 95% in the parameter space $(\Omega_m, \Omega_\Lambda)$ for $\gamma_1 = 0.3$. Right: Isolines of the age of the Universe t_0 for $\gamma_1 = 0.25$.

χ^2 

t_0



Quasiperiodic time manifold

It is generally accepted that if the time is a coordinate on a compact manifold, then it is cyclic and there is the problem of closed timelike curves.

However, the time coordinate can also be defined as a coordinate on the **universal covering space** of a compact manifold G .

This eliminates closed timelike curves *without changing the metric*.

Hence the spacetime of general relativity can be taken as

$$G \times \mathbb{R}^3$$

with the covering space $\mathbb{R}_1^4$ (the Minkowski spacetime) [3].

Implications of the quasiperiodic time

- ▶ The compact direction in time is unobservable because it is of Planck scale.
- ▶ The universal cover removes periodic identification, so no closed timelike curves exist.
- ▶ The metric remains locally Minkowskian.
- ▶ This construction is analogous to the Kaluza–Klein compactification but applied to the time dimension.
- ▶ It provides a topological justification for the usual 4D spacetime while allowing compact extra dimensions without time-like pathologies.

Deshdeep Sahdev, 1984

Perfect-fluid higher-dimensional cosmologies.

A range of solutions to Einstein's equations in $1 + d + D$ dimensions is presented for a perfect-fluid Zeldovich stiff matter. The extended dimensions (D) are a *space of constant curvature*. Solutions for D dimensions are collapsing and for d dimensions are expanding on a Planck time scale. It is explained how the higher-dimensional universe passes into a $(1 + d)$ -dimensional Friedmann–Robertson–Walker phase. The modifications in the thermal history of the universe are traced. It is found that this altered scenario offers a resolution to the horizon problem. [5]

Zeldovich's Hypothesis

Ultrarelativistic Gas

In an ultrarelativistic gas, particles move chaotically in all directions. Averaging over these spatial directions yields a pressure coefficient of $1/3$:

$$p = \frac{1}{3}\varepsilon$$

Yakov Zeldovich questioned what would happen if the baryon density in the early Universe became so extreme that particles continuously interacted via massive vector fields.

- ▶ In such an extreme medium, all particles move in the same direction.
- ▶ As a result, pressure rises to its absolute theoretical limit, becoming equal to the energy density: $p = \varepsilon$

Summary

With Sahdev's notations I study the case with $d = 3$ and $D = 3$. This extended spacetime can collapse as described by Sahdev however there is a wide range of initial conditions where the evolution before stabilization takes several Gyears. The Riemannian curvature of extra dimensions contributes to the scalar curvature of the total manifold, naturally generating the cosmological constant $\Lambda = 3/b_0^2$, where b_0 is the stabilized scale factor of the internal space. The generalized Friedmann equations are reduced to a form suitable for numerical integration and solved.

Comparison with the Kaluza–Klein theory

The Kaluza–Klein theory assumes that our spacetime is $M^4 \times U(1) \approx M^4 \times S^1$ hence we have just one extra dimension. Since it is cyclic, the electric charge is quantized [4]. To use the Riemannian curvature we should use an internal space with dimension $D \geq 2$. Elementary particle physics gives $M^4 \times U(1) \times SU(2) \times \dots$ with some 4-dim nonholonomic distribution (a field of subspaces of the tangent bundle). The modern Kaluza–Klein cosmology deals with this extended spacetime however there are no explicit solutions published.

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