Families of asymmetric periodic solutions to the Hill 3-body problem

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Abstract. We study families of asymmetric periodic solutions in the planar Hill 3-body problem (Hill3BP), a parameter-free limiting form of the circular R2BP with a Klein four-group of discrete symmetries. This structure allows periodic-orbit families to be organized by symmetry type and limiting generating solutions. We combine the symbolic theory of second-species generating solutions with a bifurcation-based continuation method applied to critical singly symmetric periodic orbits. We identify ten previously unknown families of asymmetric planar periodic solutions, belonging to two main classes: bridge families linking distinct symmetric families, and families that, in the second-species limit, continue to asymmetric generating sequences. We also examine the spatial stability of these planar families. Our results extend the known network of periodic-orbit families in the Hill3BP and provide a systematic path from highly symmetric backbone families to much less symmetric dynamics.

1. Motivation and description of the Hill3BP

Periodic solutions form a natural organizing framework for the phase space of a non-integrable Hamiltonian system. In the classical Poincaré viewpoint, they constitute a skeleton around which invariant manifolds of different dimensions are arranged. When the equations of motion possess additional discrete symmetries, the periodic solutions are divided into classes with distinct global properties. Families having the highest degree of symmetry frequently act as a backbone of a more complicated network: less symmetric families emerge from them through bifurcations, connect them to one another, or terminate at singular limiting solutions. The purpose of this work is to study how asymmetric families arise in this network for the planar Hill3BP and to develop a procedure by which such families can be found systematically.

The circular Hill3BP was originally introduced by G. W. Hill in lunar theory [1]. It is obtained as a limiting case of the circular restricted three-body problem by translating the origin to the smaller primary and blowing up the local coordinates. In this limit, the second primary is displaced infinitely far away, while its action remains as a velocity-independent tidal perturbation of the rotating Kepler problem. The model therefore gives an adequate local description of motion inside and near the Hill sphere of the smaller primary. Its principal advantages are the absence of a mass parameter, the appearance of an additional discrete symmetry, and the possibility of continuing its solutions to the circular restricted problem and, further, to the general three-body problem.

In canonical variables $\mathbf{z} = (x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, the planar problem is governed by the Hamiltonian

$$H(\mathbf{z}) = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 - x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{r}, \qquad r = \sqrt{x_1^2 + x_2^2}. \tag{1}$$

The Jacobi integral can be written as

$$\mathcal{J} \equiv 3x_1^2 + \frac{2}{r} - \dot{x}_1^2 - \dot{x}_2^2 = \Gamma = -2H. \tag{2}$$

The equations are invariant under four transformations

$$\rho_{(\alpha,\beta)} : (t, x_1, x_2, y_1, y_2) \longmapsto (\alpha t, \beta x_1, \alpha\beta x_2, \alpha\beta y_1, \beta y_2), \qquad \alpha, \beta = \pm 1, \tag{3}$$

which form the Klein group $V_4 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Depending on its isotropy subgroup, a periodic orbit may be doubly symmetric, singly symmetric, or asymmetric. The symmetry type is a global invariant along a regular one-parameter family, although it may change when families meet at a bifurcating critical orbit.

The limiting variants of the Hill3BP can be used to obtain integrable approximations. One truncation is the Kepler Problem in the rotating frame Hamiltonian

$$\hat{H}_1 = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 - \frac{1}{r}, \tag{4}$$

whereas another is the integrable Hénon problem

$$\hat{H}_2 = \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 - x_1^2 + \frac{1}{2}x_2^2. \tag{5}$$

Power Geometry provides a systematic derivation of the limiting variant by associating exponent vectors with the monomials of the Hamiltonian and selecting truncations from the faces of its Newton polygon [3]. The resulting integrable limits provide the asymptotic building blocks used in the theory of generating solutions.

2. Theory of generating solutions [4]

As $\Gamma \to -\infty$, a family of periodic orbits that approaches a second-species solution can be encoded by a finite symbolic sequence called a generating solution (GS). The sequence is composed of two kinds of elementary arcs. Type-I arcs are denoted by $\{\pm j\}$, $j = 1, 2, 3, \ldots$, and type-II arcs are denoted by i and e. Two identical arcs

$\{ii\}$ or $\{ee\}$ are not allowed consecutively. According to this restriction, each family that converges to a second-species limit is represented by a generating sequence.

A GS contains considerably more information than a label. It determines the limiting symmetry type of the family, its global multiplicity, the asymptotic form of the initial conditions, and the leading stability index. A sequence is defined only up to cyclic permutation; for example, $\{+1,+2\} \equiv \{+2,+1\}$. The generated solutions can be divided into asymmetric, singly symmetric, and doubly symmetric classes. The latter two are associated with the symmetry sets usually denoted by Σ_1, Σ_2, and their composition. Sequences containing only type-II arcs have a particularly restricted structure, while mixed sequences allow a much richer set of symmetry types.

A periodic orbit of an autonomous 2DOF Hamiltonian system belongs to a one-parameter family. Its period, integral value, geometry, and stability vary smoothly along a regular branch, whereas the symmetry type is preserved. Hence, symmetry-breaking critical solutions of symmetric families provide natural entry points to asymmetric branches.

3. Systematic search for asymmetric solutions and results

Two complementary search strategies were implemented. The first starts from asymmetric generating solutions and attempts to continue them from their second-species limits. The second starts from critical members of known singly symmetric families and detects branches created by symmetry-breaking bifurcations. The latter proved to be more robust and produced the main new results reported here.

Let $\mathcal{M}$ be the monodromy matrix of a periodic orbit. It is obtained from the fundamental matrix $Z(t)$ of the variational system

$$\dot{Z} = J \operatorname{Hess} H(\mathbf{z}(t)) Z, \qquad Z(0) = E^4, \qquad Z(t+T) = Z(t)\mathcal{M}. \tag{6}$$

Because $\mathcal{M}$ is symplectic, its multipliers occur in reciprocal and complex-conjugate pairs. In an autonomous 2DOF Hamiltonian system, M has the eigenvector $\mathbf{v}_1 = J\nabla H(\mathbf{z}_0)$ associated with the double multiplier 1. Its characteristic polynomial has the form

$$\chi(\lambda) = (\lambda - 1)^2(\lambda^2 - 2S\lambda + 1), \tag{7}$$

where S is the stability index. A periodic orbit is called critical when $S = \pm 1$. At $S = +1$, two qualitatively different cases occur. If S has an extremum as the family parameter passes through the critical orbit, the event is normally a saddle-node bifurcation. If S crosses the value $+1$ monotonically, a pitchfork-type loss of symmetry is possible. Hénon used such critical symmetric orbits to discover four asymmetric families [2].

For a periodic solution of period T, small corrections $\delta\mathbf{z}$ and δT satisfy the linearized continuation equation

$$(\mathcal{M} - E)\delta\mathbf{z}(T) + J\nabla H(\mathbf{z}_0)\,\delta T = 0. \tag{8}$$

Direct work with this singular equation is inconvenient near a critical solution. Therefore, we employ an orthogonal-symplectic transformation A of the monodromy matrix. A generic four-dimensional form is

$$\mathcal{A} = \begin{pmatrix} a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & -a_1 \\ -a_1 & -a_2 & a_3 & -a_4 \\ -a_2 & a_1 & a_4 & a_3 \end{pmatrix}, \qquad \mathcal{A}^{\mathsf{T}}\mathcal{A} = E, \qquad \mathcal{A}^{\mathsf{T}} J \mathcal{A} = J, \tag{9}$$

with the third column chosen parallel to the normalized gradient $\nabla H(\mathbf{z}_0)$. This construction is a special case of the parametric orthogonal-symplectic representation developed in [5]. In the transformed coordinates, the monodromy matrix can be reduced to a sparse form, and the solution of (8) becomes a two-parameter family. At a singly symmetric critical solution undergoing a pitchfork bifurcation, the reduction separates two directions: one continues the original symmetric family, whereas the second generates a new family with the same period but lower symmetry. Starting the differential correction along this second direction gives a stable numerical mechanism for entering the asymmetric branch.

More than 30 families of singly symmetric periodic solutions were processed using this procedure. Ten new families of planar asymmetric solutions were found. They can be divided into two groups.

- The first group consists of bridge families joining distinct symmetric families. A typical connection starts from a multiply covered doubly symmetric family, from which two pairs of singly symmetric branches emerge. Each singly symmetric branch contains a critical orbit with a pitchfork bifurcation. The resulting asymmetric family successively connects the singly symmetric branches and changes the symmetry set associated with its limiting endpoints. Thus, asymmetric solutions are not isolated curiosities; they provide missing links in the global network of symmetric families.
- The second group consists of families that can be continued toward asymmetric second-species generating solutions. These branches connect the local bifurcation picture with symbolic asymptotic theory. They demonstrate that the two search approaches – continuation from a critical symmetric orbit and continuation from an asymmetric GS – describe different parts of the same global family. In practical computations, the critical-orbit approach provides accurate initial conditions in a moderately unstable region, while the generating sequence identifies the limiting structure and global multiplicity of the branch.

All newly found planar asymmetric families were also tested for spatial stability, since their critical members may generate three-dimensional branches. For the families of periodic symmetric solutions, a kind of spatial bridge was studied in [6]. The results show that symmetry-breaking bifurcations create asymmetric bridges between backbone families, whereas second-species limits organize their remote endpoints. Symbolic asymptotics and adapted symplectic continuation therefore provide complementary parts of a systematic search method.

References

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