

Bifurcations of Periodic Solutions to Hamiltonian Systems with Symmetries

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“...ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable.”

Les méthodes nouvelles de la mécanique céleste.
Vol I, Ch. III, § 36
HENRI POINCARÉ

«... особая ценность этих периодических решений заключается в том, что они являются единственной брешью, через которую мы могли бы попытаться проникнуть в область, считавшуюся недоступной.»

Новые методы небесной механики.
Том I, Глава III, п.37
АНРИ ПУАНКАРЕ

1. Theory: reduced normal form and symmetries
2. Bifurcation scenarios for $\varepsilon = 1$
3. Bifurcation scenarios for $\varepsilon = 2$
4. Conclusions

1. Theory: reduced normal form and symmetries

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Continuation of the previous talk

Previous talk

The previous talk treated the *generic case*: no additional discrete symmetry was imposed on the reduced normal form.

This talk

We keep the same *Reduced Normal Form* (RNF) and power-geometry framework, but add finite groups of linear canonical automorphisms of the reduced phase plane. The goal is to understand how symmetry restricts the RNF support and changes the bifurcation scenarios.

Main focus

Resonances with denominators $k = 1, 2$ and three symmetry types: *axial symmetry* (AS), *central symmetry* (CS), and *double symmetry* (DS).

Local coordinates near a periodic solution

Let an autonomous Hamiltonian system with two degrees of freedom have a periodic solution $\mathcal{M}$ of period T . In a neighbourhood of $\mathcal{M}$ one introduces canonical coordinates

$$(\xi, \eta, r, \psi), \quad \psi \bmod 2\pi,$$

so that $\mathcal{M}$ is defined by $\xi = 0$, $\eta = 0$, $r = 0$.

Hamiltonian expansion

$$\gamma = \sum \gamma_{pq\ell}(\psi) \xi^p \eta^q r^\ell, \quad \gamma_{pq\ell}(\psi + 2\pi) = \gamma_{pq\ell}(\psi).$$

Hamiltonian equations

$$\dot{\xi} = \gamma_\eta, \quad \dot{\eta} = -\gamma_\xi, \quad \dot{r} = \gamma_\psi, \quad \dot{\psi} = -\gamma_r.$$

Nonlinear normalization

Complex normal form

After linear and nonlinear canonical transformations the Hamiltonian is reduced to

$$\tilde{h}(\tilde{u}, \tilde{v}, \tilde{r}, \tilde{\varphi}) = \tilde{h}_{20} + \tilde{h}_{01} + \sum \tilde{h}_{pq\ell m} \tilde{u}^p \tilde{v}^q \tilde{r}^\ell e^{im\tilde{\varphi}}.$$

Resonant monomials

If $\nu = \exp(2\pi\lambda)$ is the non-trivial multiplier, only terms satisfying *resonant equation*

$$(p - q)\lambda + im = 0$$

remain in the complex NF.

Interpretation

Normalization reduces the study of *periodic-solution families* to the study of *stationary points* (SP) of a 1DOF Hamiltonian system.

Reduced Normal Form

Autonomous RNF

The transformation

$$\tilde{u} = ue^{-i \operatorname{Im} \lambda \varphi}, \quad \tilde{v} = ve^{i \operatorname{Im} \lambda \varphi}, \quad \tilde{r} = \rho - i \operatorname{Im} \lambda uv, \quad \tilde{\varphi} = \varphi$$

reduces the NF to

$$h(u, v, \rho) = \sum h_{pq\ell m} u^p v^q \rho^\ell,$$

where $(p - q) \operatorname{Re} \lambda = 0$, $(p - q) \operatorname{Im} \lambda = -m$.

Reduced Hamiltonian system

$$\dot{u} = h_v, \quad \dot{v} = -h_u, \quad \dot{\varphi} = -h_\rho, \quad \rho = \text{const.}$$

Stationary points and reality selection

Stationary point condition

Families of periodic solutions correspond to germs of solutions of

$$h_u(u, v, \rho) = 0, \quad h_v(u, v, \rho) = 0.$$

Computational steps

- 1 Solve truncated stationary systems over $\mathbb{C}$.
- 2 Select germs compatible with the normal cone.
- 3 Apply the inverse RNF and standard transformations.
- 4 Retain the germs that correspond to real periodic solutions.
- 5 Count orbits under the inherited symmetry group.

Power geometry: support and truncations

Support

Each monomial of the RNF is associated with

$$Q = (p, q, \ell) \in \mathbb{Z}^3, \quad u_Q = h_{pq\ell} u^p v^q \rho^\ell.$$

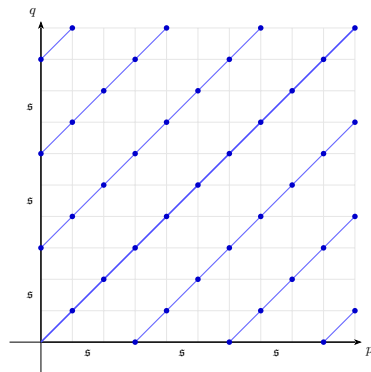
The support is $\mathbf{S}(h) = \{Q : h_Q \neq 0\}$.

Truncated Hamiltonian

A face of the Newton polyhedron gives

$$\widehat{h}_j = \sum_{Q \in \mathbf{S}_j} h_Q u^p v^q \rho^\ell.$$

Then solve $\partial_u \widehat{h}_j = \partial_v \widehat{h}_j = 0$.



Resonance denominator and RNF support

Resonant equation

For a rational resonance $\lambda = i\frac{\mathfrak{r}}{\mathfrak{s}}$, $|\mathfrak{r}| < \mathfrak{s}$, $\mathfrak{r} \in \mathbb{Z}$, $\mathfrak{s} \in \mathbb{N}$, the resonant equation becomes

$$(p - q)\mathfrak{r} + \mathfrak{s}m = 0.$$

Homogeneous resonant form

$$F_k^{(\mathfrak{s})}(u, v) = \sum_{p+q=k} a_{pq} u^p v^q, \quad (p - q)\mathfrak{r} + \mathfrak{s}m = 0.$$

RNF representation

$$h(u, v, \rho) = \sum_{\ell=0}^{\infty} \rho^{\ell} \sum_{k=1}^{\infty} F_k^{(\mathfrak{s})}(u, v).$$

Symmetry classes in the reduced phase plane

The group of automorphisms

We use the *dihedral* group

$$D_4 = \langle r, s \mid r^4 = s^2 = e, srs = r^{-1} \rangle,$$

with

$$s : (u, v) \mapsto (u, -v), \quad r : (u, v) \mapsto (-v, u), \quad r^2 : (u, v) \mapsto (-u, -v).$$

Relevant subgroups

$$V_4^{\text{ax}} = \langle r^2, s \rangle = \{e, r^2, s, r^2 s\}, \quad V_4^{\text{diag}} = \langle r^2, rs \rangle = \{e, r^2, rs, r^3 s\}.$$

Three symmetry types

Isotropy of a solution

For a SP (germ) $z = (u, v)$ define

$$\text{Sym}(z) = \{A \in D_4 : Az = z\}.$$

Type	Isotropy	Typical meaning
Asymmetric	$\{e\}$	no fixed symmetry
Singly symmetric	cyclic subgroup	one axial symmetry
Doubly symmetric	$V_4^{\text{ax}}, V_4^{\text{diag}}, \text{ or } D_4$	two generators

Inherited versus accidental RNF symmetry

A reduced Hamiltonian may have extra symmetries not present in the original system. In bifurcation interpretation one must mark the inherited symmetry subgroup.

1. Theory: reduced normal form and symmetries
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$\mathfrak{s} = 1$: double multiplier $+1$

Two linear cases

For the non-trivial monodromy block with multiplier $+1$ one have to distinguish two cases:

a) Jordan block: $(\nu - 1)^2$, b) semisimple: $(\nu - 1), (\nu - 1)$.

RNF consequence

- With JB the quadratic part contributes a term $\frac{1}{2}\sigma v^2$.
- Without JB, the quadratic part can be removed and the leading terms start at higher degrees.

Case $\varepsilon = 1$ with JB: axial symmetry (AS)

Truncated Hamiltonian

$$\hat{H} = \frac{1}{2}\sigma v^2 + a_{300}u^3 + a_{101}\rho u.$$

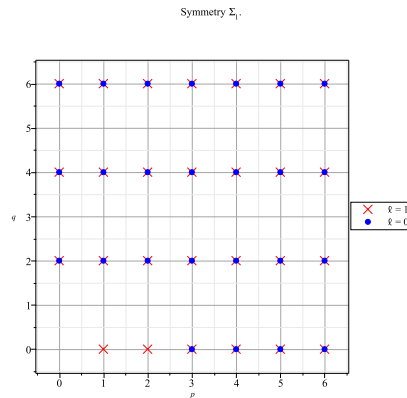
Stationary equations

$$\sigma v = 0, \quad 3a_{300}u^2 + a_{101}\rho = 0.$$

Branch

$$v = 0, \quad u = \pm\sqrt{\beta\rho}.$$

This is a *fold/saddle-node* type bifurcation in the fixed subspace of the axial symmetry.



$\mathfrak{s} = 1$ with JB: central (CS) and double symmetry (DS)

Truncated Hamiltonian

For CS and DS the leading Hamiltonian is $\hat{H} = \frac{1}{2}\sigma v^2 + a_{201}u^2\rho + a_{400}u^4$.

Stationary equations: $\sigma v = 0$, $u(2a_{201}\rho + 4a_{400}u^2) = 0$

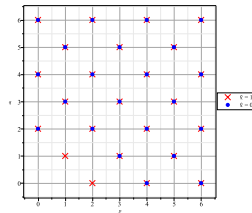
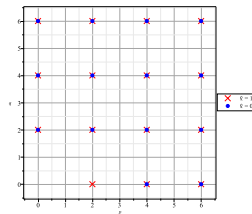
Interpretation

There is a persistent branch $u = v = 0$ and a nontrivial pair

$$v = 0, \quad u = \pm\sqrt{\beta\rho}.$$

Thus the scaling is the same as in the AS case, but the local scenario is better interpreted as **pitchfork-type symmetry breaking**.

Symmetry CS.

Symmetry Σ_{12} .

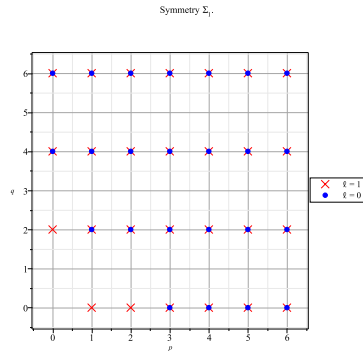
$\mathfrak{s} = 1$ without JB: axial symmetry

Two relevant normals: $N_1 = -(1, 1, 2)$

$$\hat{H}_1 = a_{101}u\rho + a_{120}uv^2 + a_{300}u^3,$$

$N_2 = -(2, 1, 2)$

$$\hat{H}_2 = a_{101}u\rho + a_{021}v^2\rho + a_{120}uv^2 + a_{040}v^4.$$



$\mathfrak{s} = 1$ without JB: axial branch refinement I

First normal truncation

$$2a_{120}uv = 0, \quad 3a_{300}u^2 + a_{120}v^2 + a_{101}\rho = 0.$$

It gives a symmetric branch

$$v = 0, \quad u = \pm\sqrt{\beta_1\rho}.$$

Boundary solution of the first normal truncation

The first normal truncation also detects the direction

$$u = 0, \quad v^2 = -\frac{a_{101}}{a_{120}}\rho.$$

This is not rejected; it is an incomplete leading germ because $u = 0$ is not a fixed subspace of the axial symmetry.

$\mathfrak{s} = 1$ without JB: axial branch refinement II

Second normal truncation resolves the asymmetric branch

For $N_2 = -(2, 1, 2)$ one obtains

$$v = \pm \sqrt{\beta_2 \rho}, \quad u = \beta_1 \rho.$$

This branch breaks the axial symmetry $s : (u, v) \mapsto (u, -v)$.

$\mathfrak{s} = 1$ without JB: central symmetry I

Generic CS has quartic truncation

CS requires even total degree, not separate evenness in u and v :

$$\hat{H} = \rho Q_2(u, v) + Q_4(u, v),$$

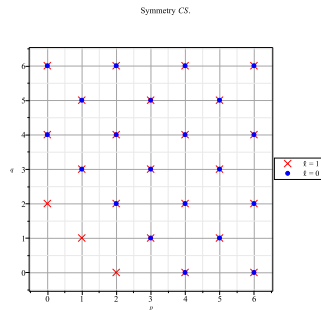
where

$$Q_2 = a_{201}u^2 + a_{111}uv + a_{021}v^2$$

and Q_4 is a general binary quartic.

Why direct solving becomes complicated

Eliminating ρ leads to a homogeneous quartic in (u, v) with coefficients composed from the RNF coefficients. It is the natural projective direction equation for the central case.



$\mathfrak{s} = 1$ without JB: central symmetry II

Projective direction equation

For nonzero branches set $[u : v] \in \mathbb{CP}^1$. Directions satisfy

$$D(u, v) = Q_{2,u}Q_{4,v} - Q_{2,v}Q_{4,u} = 0,$$

a homogeneous polynomial of degree 4.

Realness and counting

Real projective roots give candidates of real branches; nonreal roots are discarded. Nonzero branches are generically asymmetric, and (u, v) and $(-u, -v)$ are symmetry-related representatives.

$\varepsilon = 1$ without JB: double symmetry I

DS quartic truncation

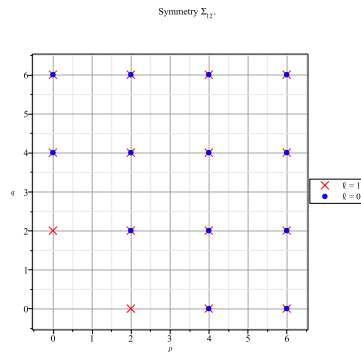
$$\hat{H} = \rho(Au^2 + Bv^2) + Eu^4 + Du^2v^2 + Cv^4.$$

Stationary equations

$$u(A\rho + 2Eu^2 + Dv^2) = 0, \quad v(B\rho + Du^2 + 2Cv^2) = 0.$$

Branches

- $u = v = 0$: parent doubly symmetric branch
- $v = 0, u^2 = -A\rho/(2E)$: singly symmetric branch
- $u = 0, v^2 = -B\rho/(2C)$: singly symmetric branch
- $u \neq 0, v \neq 0$: possible asymmetric branch



$\mathfrak{s} = 1$ without JB: double symmetry II

Solving for the mixed branch

Put $U = u^2$, $V = v^2$. For $U, V \neq 0$:

$$2EU + DV = -A\rho, \quad DU + 2CV = -B\rho.$$

Let $\Delta = 4EC - D^2$. Then $U = \rho \frac{DB - 2CA}{\Delta}$, $V = \rho \frac{DA - 2EB}{\Delta}$.

Reality and symmetry

The mixed branch is real only if $U > 0$ and $V > 0$. If real, all four sign choices form one V_4^{ax} -orbit and represent one asymmetric family type.

$\mathfrak{s} = 1$: working summary

Linear case	Symmetry	Leading branches	Interpretation
JB	AS	$v = 0, u \sim \sqrt{\rho}$	saddle-node/fold in fixed subspace
JB	CS/DS	$u = v = 0$ plus $v = 0, u \sim \sqrt{\rho}$	pitchfork-type symmetry reduction
No JB	AS	$v = 0, u \sim \sqrt{\rho}; u \sim \rho, v \sim \sqrt{\rho}$	symmetric and symmetry-breaking branches
No JB	CS	projective quartic directions	generically asymmetric branches
No JB	DS	parent, two axial branches, possible mixed branch	single-symmetry and asymmetric branches

Hamiltonian Value at the Bifurcation

Bifurcation type	Local behavior of the family	Hamiltonian value
Saddle-node	$\rho - \rho_* \sim t^2$	$H - H_* \sim ct^2$ – Extremum of H
Pitchfork / symmetry breaking	$\rho - \rho_* \sim t$	$H - H_* \sim ct$ – Monotonic change of H

Important

The truncated bifurcation Hamiltonian $\hat{H}(u, v, \rho)$ does not contain the full contribution depending only on ρ . Therefore the behavior of $\hat{H}$ on a stationary branch should not be identified directly with the behavior of the physical Hamiltonian along the continued periodic family.

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$\varsigma = 2$: double multiplier -1

Two linear cases

a) Jordan block: $(\nu + 1)^2$, b) semisimple: $(\nu + 1), (\nu + 1)$

Period doubling

At $\varsigma = 2$ the generated solutions have period

$$T' = 2T.$$

Important distinction

The semisimple case may have accidental extra symmetries in the RNF. Therefore the inherited symmetry subgroup must be tracked separately from the full symmetry of the polynomial RNF.

$\mathfrak{s} = 2$ with JB: axial symmetry

Truncated Hamiltonian

$$\widehat{H} = \frac{1}{2}\sigma v^2 + a_{201}u^2\rho + a_{400}u^4.$$

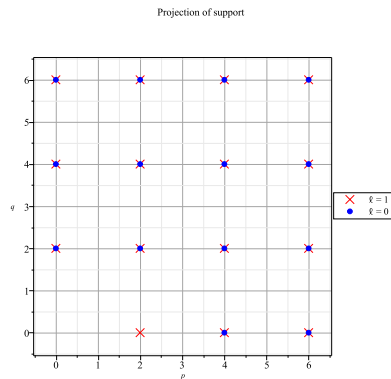
Stationary points

$$u = v = 0$$

for the parent family, and

$$v = 0, \quad u = \pm\sqrt{\beta\rho}$$

for the generated family.



$s = 2$ without JB: axial symmetry

Parity mechanism

For $s = 2$, $\tau = 1$, the resonance equation implies

$$p - q \equiv 0 \pmod{2}, \quad p + q \equiv 0 \pmod{2}.$$

If axial symmetry further imposes q even, then automatically p is even.

Consequence

The semisimple axial RNF has the same support as the V_4^{ax} -invariant RNF: $\mathcal{R}_{\text{res}}^{\langle s \rangle} = \mathcal{R}_{\text{res}}^{V_4^{\text{ax}}}$

Interpretation

This is a degeneracy of the RNF classification. Distinguish the inherited symmetry $\langle s \rangle$ from accidental RNF symmetry V_4^{ax} .

$\mathfrak{s} = 2$ without JB: central symmetry

Expected structure

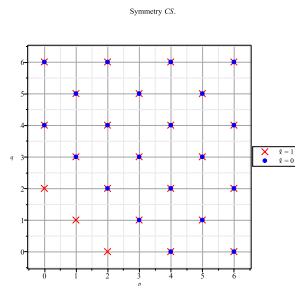
CS truncation is

$$\hat{H} = \rho Q_2(u, v) + Q_4(u, v),$$

with a generic quadratic Q_2 and a binary quartic Q_4 .

Interpretation

As in the $\mathfrak{s} = 1$ central case, nonzero stationary points are generically not fixed by r^2 as points. They are expected to represent asymmetric generated families, after the real-germ selection.



$\mathfrak{s} = 2$ without JB: double symmetry

Truncated Hamiltonian

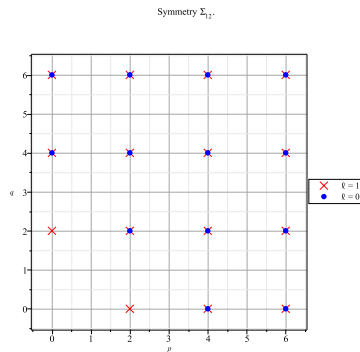
$$\hat{H} = \rho(Au^2 + Bv^2) + Cu^4 + Du^2v^2 + Ev^4.$$

Stationary equations

$$u(A\rho + 2Cu^2 + Dv^2) = 0, \quad v(B\rho + Du^2 + 2Ev^2) = 0.$$

Generated singly symmetric families

$$\mathcal{F}_u : v = 0, u^2 = -\frac{A}{2C}\rho, \quad \mathcal{F}_v : u = 0, v^2 = -\frac{B}{2E}\rho.$$



$s = 2$ without JB: two variants in DS case

Variant 1: simultaneous birth/death

If the two axial branches exist on the same side of $\rho = 0$, two singly symmetric double-period families are born or die simultaneously. Their stability depends on the types of extrema of $\hat{H}$ along the branches.

Variant 2: branch replacement

If the two axial branches exist on opposite sides of $\rho = 0$, one observes the death of one singly symmetric double-period family and the birth of another family with the other axial symmetry.

Counting

The signs $\pm\sqrt{\cdot}$ are representatives in the same symmetry orbit; they should not automatically be counted as separate physical families.

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Main conclusions

- 1 Symmetry restrictions change not only coefficients, but also the bifurcation topology.
- 2 The same Newton support can correspond to different physical scenarios if the inherited symmetry differs.
- 3 For $s = 1$, JB and no-JB cases require separate treatment for AS, CS and DS.
- 4 For $s = 2$, the generic AS period-doubling mechanism is the JB case, while semisimple cases produce richer symmetry-breaking scenarios.
- 5 Real branch selection and quotienting by symmetry are essential before interpreting algebraic roots as families of periodic solutions.

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Thanks you for your attention