Bifurcations of Periodic Solutions to Hamiltonian System with Symmetries

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Abstract. For an autonomous, real Hamiltonian system with two degrees of freedom in a generic setting, normal form (NF) theory and power-geometry techniques are employed to analyze bifurcations of periodic solutions associated with various groups of discrete linear automorphisms of the phase space. A Hamiltonian normal form is constructed in a neighborhood of a given periodic solution and, by means of an additional canonical transformation, is further reduced to an autonomous Hamiltonian system with a single degree of freedom and a single small parameter (the reduced normal form). The equilibrium points of this reduced normal form correspond to families of periodic orbits of the original system, and the resulting bifurcation scenarios depend crucially on the symmetry properties of the initial Hamiltonian system.

1. Introduction

The Poincaré approach for studying a non-integrable system of ODEs is based on finding a certain "**skeleton**" in the phase space around which the hierarchical structure of invariant manifolds of various dimensions is formed. Many problems in Celestial Mechanics can be reduced to Hamiltonian systems whose canonical equations admit a discrete group of symplectomorphisms. Each additional symmetry gives rise to new properties of the phase flow, which are closely related to bifurcations of periodic solutions of the equations of motion. This work is a continuation of [1], where considered only bifurcation scenarios of periodic solution to a generic time-independent Hamiltonian system with 2 degrees of freedom (DOF).

Consider a real Hamiltonian system with 2DOF and an analytic Hamiltonian $\gamma(\xi_1, \xi_2, \eta_1, \eta_2)$ defined in a domain $\mathcal{C}$. Suppose that the system has a periodic solution $\mathcal{M} \subset \mathcal{C}$ with period $T = T(\mathcal{M})$, and consider a sufficiently small neighborhood $\mathcal{U} \subset \mathcal{C}$ of $\mathcal{M}$. Then there exist analytic functions ξ, η, r, and ψ in $\mathcal{U}$ such that

1) the trajectory $\mathcal{M}$ is defined by the equalities $\xi = 0$, $\eta = 0$, $r = 0$;

2) the function ψ is cyclic (angular) modulo 2π;

3) the neighborhood $\mathcal{U}$ is a skew product of a 3-dimensional ball and the cycle $\psi \in [0, 2\pi]$;

4) the coordinates ξ, η and ψ, r are canonically conjugate.

The Hamiltonian γ is 2π-periodic in ψ in the neighborhood $\mathcal{U}$ and can be expanded into the convergent Taylor series

$$\gamma = \sum \gamma_{pq\ell}(\psi)\, \xi^p \eta^q r^\ell,$$

where $p, q, \ell \in \mathbb{N}_0$. In these coordinates, γ defines the Hamiltonian system

$$\dot\xi = \frac{\partial\gamma}{\partial\eta}, \quad \dot\eta = -\frac{\partial\gamma}{\partial\xi}, \quad \dot r = \frac{\partial\gamma}{\partial\psi}, \quad \dot\psi = -\frac{\partial\gamma}{\partial r}. \tag{1}$$

By applying standard complex and nonlinear transformations [2], the Hamiltonian γ is reduced to the complex normal form

$$h(\tilde u, \tilde v, \tilde r, \tilde\varphi) = \tilde h_{20} + \tilde h_{01} + \sum \tilde h_{pq\ell m} \tilde u^p \tilde v^q \tilde r^\ell \mathrm{e}^{im\tilde\varphi}, \tag{2}$$

where the main diagonal of the matrix JG contains its eigenvalues ν and $-\nu$, with $\nu = \exp(2\pi\lambda)$. The Hamiltonian (2) is called a *complex normal form* (NF) if:

1) its quadratic part $\tilde h_{20} + \tilde h_{01}$ is in NF;
2) the expansion (2) contains only resonant terms satisfying so called *resonant equation*

$$(p - q)\,\lambda + i\,m = 0. \tag{3}$$

Theorem 1 ([3]). *The canonical transformation*

$$\tilde u = u \exp(-i\ \mathrm{Im}\,\lambda\varphi), \quad \tilde v = v \exp(i\ \mathrm{Im}\,\lambda\varphi), \quad \tilde r = \rho - i\,\mathrm{Im}\,\lambda uv, \quad \tilde\varphi = \varphi, \tag{4}$$

reduces the NF of the Hamiltonian (2) *to a constant power series*

$$h(u, v, \rho) = \sum h_{pq\ell m} u^p v^q \rho^\ell,\ (p - q)\,\mathrm{Re}\,\lambda = 0,\ (p - q)\,\mathrm{Im}\,\lambda = -m, \tag{5}$$

where p, q, ℓ, m are integers, $p, q, \ell \geqslant 0$. Here $h_{pq\ell m} = \tilde h_{pq\ell m}$ if $p + q \geqslant 3$, the quadratic terms have the form $h_{20} + h_{01} = \dfrac{2\pi}{T}\left(\dfrac{1}{2}\,\langle \mathbf{w}, G\mathbf{w}\rangle - \rho\right)$, where the matrix $JG = JG - i\ \mathrm{Im}\,\Lambda$ with the diagonal matrix $\Lambda = \mathrm{diag}(\boldsymbol{\nu}, -\boldsymbol{\nu})$ and there are no linear terms in $\mathbf{w} = (u, v)$.

Thus, we obtain an autonomous 1DOF Hamiltonian system $\dot u = \ \ h_v$, $\dot v = -h_u$ with an additional equation $\dot\varphi = -h_\rho$ and an additional small parameter $\rho = \mathrm{const}$. We call (5) the *reduced normal form* (RNF). It allows us to study the bifurcations of families of periodic solutions of the system (1) in a neighborhood of the resonant periodic solution, because each periodic solution $\mathcal{M}$ to system (1) corresponds to a stationary point (SP) of the RNF Hamilton system.

For that to each germ of family of periodic solutions of the truncated system we make transformations:

- reverse to (4) and
- reverse to the standard transformation for given λ [2, S. 1.4].

Hypothesis 1. *If result of these transformations is a real germ, then corresponding family of periodic solutions is real.*

2. Classes of Symmetries

A periodic solution $\mathcal{M}$ has a monodromy matrix $\mathcal{N}$ with characteristic polynomial $\chi(\nu) = (\nu-1)^2\left(\nu^2 - 2S\nu + 1\right)$, where S is the stability index. The solution is linearly stable if $|S| \leqslant 1$, with $S = \operatorname{Re}\exp(2\pi\lambda)$, $\nu_{3,4} = \exp(\pm 2\pi\lambda)$. If the argument of the non-trivial multipliers $\nu_{3,4}$ is commensurable with π, then, according to Poincaré [4, Ch. XXX], additional periodic solutions may bifurcate near the generating solution. Their period T' is a multiple of the original period T.

In this case, the resonant equation (3) becomes

$$(p-q)\mathfrak{r} + \mathfrak{s}m = 0, \qquad \lambda = \frac{\mathfrak{r}}{\mathfrak{s}}, \quad |\mathfrak{r}| < \mathfrak{s}, \quad \mathfrak{r} \in \mathbb{Z}, \quad \mathfrak{s} \in \mathbb{N}. \tag{6}$$

Let $F_k^{(\mathfrak{s})}(u,v)$ denote a homogeneous resonant form of degree k in the canonical variables (u,v):

$$F_k^{(\mathfrak{s})}(u,v) \stackrel{\text{def}}{=} \sum_{p+q=k} a_{pq} u^p v^q, \qquad a_{pq} = \text{const}, \tag{7}$$

and p, q satisfy (6). Accordingly, the RNF can be represented as

$$h(u,v,\rho) = \sum_{\ell=0}^{\infty} \rho^\ell \sum_{k=1}^{\infty} F_k^{(\mathfrak{s})}(u,v), \tag{8}$$

where triple indices $a_{pq\ell}$ are used to distinguish the coefficients corresponding to different powers of ρ.

We consider finite groups of linear canonical automorphisms of the two-dimensional phase space that preserve the homogeneity of the forms (7) and the integral character of their monomial exponents. Such transformations change the signs of u and v, interchange the variables, or combine these operations. The full group of these transformations is the dihedral group D_4 of order 8,

$$D_4 = \langle r, s \mid r^4 = s^2 = e,\ srs = r^{-1} \rangle,$$

with generators $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $r = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

A linear map A is called a Hamiltonian symmetry if $H \circ A = H$ and A are symplectic. A linear map $\varkappa$ is called a reversing symmetry, or reversor, if $H \circ \varkappa = H$ and $\varkappa$ is anti-symplectic. In the latter case, $\varkappa \circ \Phi_t = \Phi_{-t} \circ \varkappa$. For a homogeneous form, invariance under a subgroup $G \subset D_4$ always means literal invariance $F_k^{(\mathfrak{s})} \circ A = F_k^{(\mathfrak{s})}$ for all $A \in G$ including anti-symplectic elements.

The subgroups of D_4 relevant to classification are the trivial subgroup $\{e\}$; five cyclic subgroups of order 2: $\langle r^2 \rangle$, $\langle s \rangle$, $\langle rs \rangle$, $\langle r^2 s \rangle$, $\langle r^3 s \rangle$; the cyclic subgroup of order 4: $C_4 = \langle r \rangle$; the two Klein four subgroups $V_4^{\text{ax}} = \langle r^2, s \rangle$, $V_4^{\text{diag}} = \langle r^2, rs \rangle$; and the full group D_4. The elements of C_4 are symplectic, whereas the four reflections are anti-symplectic.

Definition 1. *Let* $\mathrm{Sym}(z) \subset D_4$ *be the isotropy subgroup of a solution* z*. The solution is called* ***asymmetric*** *if* $\mathrm{Sym}(z) = \{e\}$*,* ***singly symmetric*** *if* $\mathrm{Sym}(z)$ *is cyclic, and* ***doubly symmetric*** *if* $\mathrm{Sym}(z)$ *requires two generators.*

Thus, singly symmetric solutions correspond to the cyclic subgroups of orders 2 or 4, while doubly symmetric solutions correspond to V_4^{ax}, V_4^{diag}, or D_4. The symmetry subgroup imposes algebraic restrictions on the coefficients $a_{pq\ell}$; consequently, some coefficients may vanish and the admissible RNF terms, and hence the possible bifurcation scenarios of periodic solutions, become symmetry-dependent. For $\rho = 0$, the degrees of $F_k^{(\mathfrak{s})}$ begin with $k = 3$ for the prime elementary divisors of the matrix M and with $k = 2$ otherwise.

3. Results

Here we give some results of the computations for singly and doubly symmetric cases for $\mathfrak{s} = 1, 2$, whereas a generic case with a trivial group of symplectomorphisms was described in [1]. The scheme of computation presented in [1] is slightly modified here and is mainly based on the power geometry technique applied rather to Hamiltonain then to its canonical system. Briefly, the main steps of the computations are the following: getting the support $\mathbf{S}$ of the RNF H, calculating its convex hull Γ, selecting the facets $\Gamma_j^{(2)}$ with negative normal vectors N_j and obtaining the corresponding truncated Hamiltonians $\hat{H}_j$. Solutions to the truncated system $\hat{\mathfrak{S}}_j \stackrel{\mathrm{def}}{=} \left\{\partial\hat{H}_j/\partial u = \partial\hat{H}_j/\partial v = 0\right\}$ provide the germs of periodic solution. After that only those germs should be selected that give real periodic solutions.

In the following, we consider only the Klein four subgroup V^{ax} with axial symmetries s, r^2s and central symmetry r^2. The case of axial symmetry with generator r^2s is the same as the case with generator s. For cases $\mathfrak{s} = 1, 2$ we should distinguish sub cases for the presence and absence of the Jordan block (JB).

For $\mathfrak{s} = 1$ with JB we consider single axial symmetry (AS), central symmetry (CS) and double symmetry (DS).

Symmetry AS. There is the only normal vector $N_1 = -(2, 3, 4)$, corresponding truncated Hamiltonian is $\hat{H}_1 = \frac{1}{2}\sigma v^2 + a_{300}u^3 + a_{101}\rho u$, and truncated system $\hat{\mathfrak{S}}_1 = \left\{\sigma v = 0, -3u^2a_{300} - \rho a_{101} = 0\right\}$. The family of SP solutions takes the form $\left\{u = \pm\sqrt{\beta\rho}, v = 0\right\}$, where β depends on the NF coefficients. Here we have a *saddle-node* bifurcation.

Symmetries CS and DS give us the only normal vector $N_1 = -(1, 2, 2)$, truncated Hamiltonian $\hat{H}_1 = \frac{1}{2}\sigma v^2 + a_{201}u^2\rho + a_{400}u^4$ and truncated system $\hat{\mathfrak{S}}_1 = \left\{\sigma v = 0, -4a_{400}u^3 - 2a_{201}\rho u = 0\right\}$. Here we have the same asymptotic form of SP solutions and the same saddle-node bifurcation.

For $\mathfrak{s} = 1$ without JB we get the following.

Symmetry AS gives two normal vectors $N_1 = -(1,1,2)$ and $N_2 = -(2,1,2)$. The corresponding Hamiltonians are the following: $\hat{H}_1 = a_{101}u\rho + a_{120}uv^2 + a_{300}u^3$ and $\hat{H}_2 = a_{101}u\rho + a_{021}v^2\rho + a_{120}uv^2 + a_{040}v^4$. The first system $\hat{\mathfrak{S}}_1 = \{2uva_{120} = 0, -3u^2a_{300} - v^2a_{120} - \rho a_{101} = 0\}$ has only symmetrical solution $\{u = \pm\sqrt{\beta_1\rho}, v = 0\}$, whereas the second system $\hat{\mathfrak{S}}_2 = \{4v^3a_{040} + 2\rho va_{021} + 2uva_{120} = 0, -v^2a_{120} - \rho a_{101} = 0\}$ has the solutions of the form $\left\{u = \beta_1\rho, v = \pm\sqrt{\beta_2\rho}\right\}$, which are asymmetrical. So here we have a *pitch-fork* or *symmetric break* bifurcation. Along the first solution the Hamiltonian value changes monotonically and along the second one the Hamiltonian has an extrema.

Conclusion. In the case of JB the symmetry of periodic solution does not change while crossing the bifurcation point. The birth of asymmetrical periodic solution appears in the case of JB absence and monotonical changes of the Hamiltonian value.

For $\mathfrak{s} = 2$ with JB we consider only the case with single axial symmetry, because the monodromy matrix $\mathcal{N}$ in the cases of central and double symmetries is the full square matrix, hence it cannot have a JB with negative eigenvalue. So the truncated Hamiltonian corresponding to the normal vector $N_1 = -(1,1,2)$ is $\hat{H}_1 = \frac{1}{2}\sigma v^2 + a_{201}u^2\rho + a_{400}u^4$, and we get 2 solutions: a trivial one $\{u = v = 0\}$ corresponding to the original periodic solution and a symmetrical one: $\left\{u = \pm\sqrt{\beta\rho}, v = 0\right\}$. This solution gives us a family of solutions with double period, i.e. here *period doubling* bifurcation takes place.

For $\mathfrak{s} = 2$ without JB we consider a doubly symmetrical situation. Convex hull gives us the only normal vector $N_1 = -(1,1,2)$ and to it the truncated Hamiltonian $\hat{H}_1 = (a_{021}v^2 + a_{201}u^2)\rho + a_{040}v^4 + a_{220}u^2v^2 + a_{400}u^4$ is corresponding. The truncated system

$$\hat{\mathfrak{S}}_1 = \left\{2a_{220}u^2v + 4a_{040}v^3 + 2\rho va_{021} = 0, -4a_{400}u^3 - 2a_{220}uv^2 - 2\rho ua_{201} = 0\right\}$$

has 4 series of solutions. The first solution $Sol_1 : \{u = v = 0\}$ corresponds to the original periodic solution. The second $Sol_2 : \left\{u = \pm\sqrt{\beta_2\rho}, v = 0\right\}$ and the third $Sol_3 : \left\{u = 0, v = \pm\sqrt{\beta_3\rho}\right\}$ correspond to new families of doubly periodic solutions with single symmetry each. Along each of the new families the Hamiltonian $\hat{H}_1$ reaches its extremum. There are possible 2 different scenarios depending on whether the extrema have the same types (both minima or maxima) or different types (one is minimum and another is maximum or vice versa). In scenario 1 one new family consists of stable solutions and the other consists of unstable ones. In scenario 2 both two new families consist of unstable solutions. Finally, the fourth solution of the form $Sol_4 : \left\{u = \pm\sqrt{\beta_4\rho}, v = \pm\sqrt{\beta_5\rho}\right\}$ does not correspond to real periodic solutions.

All the scenarios of periodic solution bifurcations described above were studied numerically in the Hill 3-body problem by M. Hénon [5, 6] and the second author [7, 8].

References

[1] A. D. Bruno and A. B. Batkhin. *International Conference Polynomial Computer Algebra '2026. St. Petersburg, May 25-30, 2026. Euler International Mathematical Institute*, chapter Studying Bifurcations of Periodic Solutions to Hamiltonian System, pages 20–24. VVM Publishing, St. Petersburg, Russia, 2026.

[2] A. D. Bruno. *The Restricted 3–body Problem: Plane Periodic Orbits*. Walter de Gruyter, Berlin, 1994. = Nauka, Moscow, 1990. 296 p. (in Russian).

[3] A. D. Bruno. Normalization of a periodic hamiltonian system. *Programming and Computer Software*, 46(2):76–83, March 2020. URL: `http://dx.doi.org/10.1134/S0361768820020048`, `doi:10.1134/s0361768820020048`.

[4] H. Poincaré. *Les Métods Nouvelles de la Mécanique Céleste*, volume 3. Gauthier-Villars, Paris, 1899.

[5] M. Hénon. Numerical exploration of the restricted problem V. Hill's case: periodic orbits and their stability. *Astron. & Astrophys.*, 1:223–238, 1969.

[6] M. Hénon. Families of asymmetric periodic orbits in Hill's problem of three bodies. *Celest. Mech. Dyn. Astr.*, 93:87–100, 2005. `doi:10.1007/s10569-005-3641-8`.

[7] A. B. Batkhin. Symmetric periodic solutions of the Hill's problem. II. *Cosmic Research*, 51(6):452–464, 2013. `doi:10.1134/S0010952513050018`.

[8] A. B. Batkhin. Bifurcations of periodic solutions of a Hamiltonian system with a discrete symmetry group. *Programming and Computer Software*, 46(2):84–97, 2020. `doi:10.1134/S0361768820020036`.

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