

A Covariant Poisson-Bracket Form of the Levi-Civita Separability Condition

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Abstract. We introduce a coordinate-free formulation of the classical Levi-Civita separability condition for Hamiltonian systems. By organizing candidate separation variables into an alternating canonical quartet (x_i, p_i, x_j, p_j) , a recursive sequence of nested Poisson minors fully reproduces the classical separability criteria without invoking a preferred coordinate chart. Furthermore, we demonstrate that this five-bracket factorizes under local canonical scalings and maps directly onto a quadratic form governed by iterated Poisson brackets, offering an intrinsic diagnostic tool for tracking both separable and non-separable dynamics.

Introduction

The integration of the Hamilton–Jacobi equation by separation of variables is fundamental to analyzing conservative dynamical systems. In 1904, Levi-Civita [1] established the classical necessary and sufficient structural criteria for orthogonal separability. For an arbitrary Hamiltonian $H(p, x) = h$, the phase space geometry must satisfy:

$$\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x_j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_i \partial p_j} - \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x_j} + \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial p_i \partial p_j} = 0. \quad (1)$$

Because classical representations rely strictly on standard partial derivatives, they cannot easily distinguish between intrinsic non-integrability and a suboptimal choice of configuration coordinates. To resolve this limitation, this note introduces a coordinate-free, recursive operator formulation of the classical Levi-Civita condition using nested Poisson minors.

1. The Functional Brackets

Let $(\mathcal{M}, \omega)$ be a symplectic phase space with the standard Poisson bracket $\{f, g\} = \omega^{-1}(df, dg)$.

Definition 1 (The Four-Bracket Minor). For four scalar functions $a, b, c, d \in C^\infty(\mathcal{M})$, the four-bracket is defined as the 2×2 Poisson minor:

$$\{a, b, c, d\}_4 = \det \begin{pmatrix} \{a, c\} & \{a, d\} \\ \{b, c\} & \{b, d\} \end{pmatrix} = \{a, c\} \{b, d\} - \{a, d\} \{b, c\}.$$

Definition 2 (The Recursive Five-Bracket). For five functions $a, b, c, d, H \in C^\infty(\mathcal{M})$, the five-bracket is built recursively via:

$$\{a, b, c, d\}_5(H) = \det \begin{pmatrix} \{H, c\} & \{H, d\} \\ \{a, b, \{H, c\}, H\}_4 & \{a, b, \{H, d\}, H\}_4 \end{pmatrix}.$$

Definition 3 (Alternating Canonical Quartet). Four functions $a, b, c, d \in C^\infty(\mathcal{M})$ form an alternating canonical quartet if $\{a, b\} = 1$, $\{c, d\} = 1$, and all other pairwise brackets vanish. Locally, these functions act as Darboux coordinate pairs (x_i, p_i, x_j, p_j) .

2. Main Results

Theorem 1 (Covariant Separability). Let $H \in C^\infty(\mathcal{M})$ be a Hamiltonian on a symplectic phase space. Suppose $a, b, c, d \in C^\infty(\mathcal{M})$ form an alternating canonical quartet. Then H satisfies the Levi-Civita separability condition for the (i, j) sector if and only if

$$\{a, b, c, d\}_5(H) = 0. \quad (2)$$

Theorem 2 (Symplectic Curvature Representation). Let (x_i, p_i, x_j, p_j) be an alternating canonical quartet. The recursive five-bracket can be expressed intrinsically as a quadratic form:

$$\{x_i, p_i, x_j, p_j\}_5(H) = -\frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v},$$

where $\mathbf{v} = (H_{x_i}, H_{x_j}, H_{p_i}, H_{p_j})^T$ is the structural gradient vector and $\mathbf{M}$ is a symmetric connection matrix constructed from iterated Poisson brackets [2]:

$$\mathbf{M} = \begin{pmatrix} 0 & \{\{H, x_i\}, x_j\} & 0 & -\{\{H, x_i\}, p_j\} \\ \{\{H, x_i\}, x_j\} & 0 & -\{\{H, x_j\}, p_i\} & 0 \\ 0 & -\{\{H, x_j\}, p_i\} & 0 & \{\{H, p_i\}, p_j\} \\ -\{\{H, x_i\}, p_j\} & 0 & \{\{H, p_i\}, p_j\} & 0 \end{pmatrix}.$$

3. Applications

The recursive five-bracket serves both as a diagnostic tool for mapping cross-coupling boundaries and as an algorithmic investigator for discovering adapted canonical frames.

3.1. Cartesian Systems

For a natural Cartesian Hamiltonian $H = \frac{1}{2}(X^2 + Y^2) + V(x, y)$, evaluating the five-bracket isolates mixed potential curvature:

$$\{x, X, y, Y\}_5(H) = -YX \left(\frac{\partial^2 V(x, y)}{\partial x \partial y} \right).$$

The bracket vanishes globally if and only if $V(x, y) = f_1(x) + f_2(y)$. On non-separable profiles—such as Hénon-Heiles or Toda lattice potentials—the bracket maps exact submanifolds where separability breaks [4].

3.2. Polar-Nodal (Whittaker) Systems

In polar-nodal coordinates $(r, \theta, \nu, R, \Theta, N)$ with $H = \frac{1}{2}(R^2 + \Theta^2/r^2) - \mu/r + V(\theta, \Theta, N)/r^q$ [5], the radial-angular coupling yields:

$$\{r, R, \theta, \Theta\}_5(H) = R\Theta r^{-3-q}(-2 + q) \left(\frac{\partial V}{\partial \theta} \right).$$

For $q = 2$, the bracket vanishes identically, isolating angularly separable extensions such as MICZ-Kepler systems [3]. For the uniform Stark effect ($V = \epsilon r \cos \theta$), the non-vanishing bracket formally proves the necessity of transitioning to parabolic coordinates.

Conclusion

We have reformatted the Levi-Civita orthogonal separability condition into a coordinate-free framework using nested Poisson minors. By replacing classical partial derivative strings with recursive bracket operators, integrability checks become independent of pre-selected charts and are optimized for direct implementation in computer algebra systems.

References

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