



A Covariant Poisson-Bracket Form of the Levi-Civita Separability Condition

Vladimir Martinusi

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Euler International Mathematical Institute, St. Petersburg

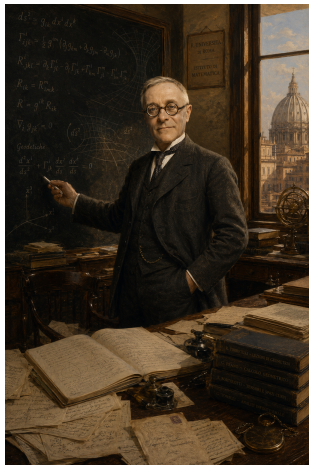
Dr. Vladimir Martinusi

Stephen B. Klein Faculty of Aerospace Engineering, Technion – Israel Institute of Technology

Email: vmartinusi@technion.ac.il

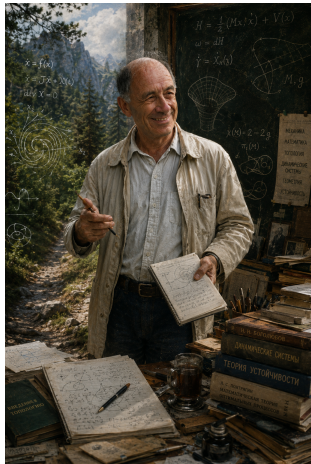
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- **1904 Landmark Result:** In a celebrated letter to Paul Stäckel, Levi-Civita established the necessary and sufficient conditions for orthogonal Hamilton–Jacobi separability.
- **The Levi-Civita Equations:** Formulated as a system of $\frac{1}{2}n(n-1)$ non-linear second-order PDEs governing the Hamiltonian $H(p, x) = h$.
- **Historical Bottleneck:** Traditional evaluation relies strictly on partial derivative strings in a pre-selected coordinate chart, making calculations labor-intensive.
- **The Diagnostic Ambiguity:** Hard to distinguish whether a non-zero evaluation signals a true physical obstruction or merely an ill-chosen coordinate system.



In the Spirit of Vladimir Arnold (1937–2010)

- **The Arnoldian Philosophy:** "Every mathematician knows that it is impossible to understand any concept if it is restricted to its coordinate representation."
- **Intrinsic Geometric Mechanics:** Re-formulating analytical dynamics directly on symplectic manifolds $(\mathcal{M}, \omega)$ via coordinate-free operators.
- **Objective of This Work:** Translate Levi-Civita's classical partial derivative condition into an intrinsic, covariant operator hierarchy of nested Poisson brackets.
- **Symplectic Symmetry:** Eliminate preferred charts and allow diagnostic tests to track geometric foliations across canonical transformations.



- **Classical Criterion:** For two distinct degrees of freedom (i, j) , orthogonal separability requires:

$$\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x_j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_i \partial p_j} - \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x_j} + \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial p_i \partial p_j} = 0$$

- **Limitations:**

- 1 Strongly bound to local Darboux coordinates (x_i, p_i, x_j, p_j) .
- 2 Highly non-linear derivative strings obscure underlying geometric invariants.
- 3 Difficult to automate cleanly without manual differential-algebraic expansions.

- **Definition 1 (Four-Bracket Minor):** For $a, b, c, d \in C^\infty(\mathcal{M})$, define the 2×2 Poisson minor:

$$\{a, b, c, d\}_4 = \det \begin{pmatrix} \{a, c\} & \{a, d\} \\ \{b, c\} & \{b, d\} \end{pmatrix} = \{a, c\} \{b, d\} - \{a, d\} \{b, c\}$$

- **Definition 2 (Recursive Five-Bracket):** For five scalar functions $a, b, c, d, H \in C^\infty(\mathcal{M})$:

$$\{a, b, c, d\}_5(H) = \det \begin{pmatrix} \{H, c\} & \{H, d\} \\ \{a, b, \{H, c\}, H\}_4 & \{a, b, \{H, d\}, H\}_4 \end{pmatrix}$$

- **Expanded Operational Form:**

$$\{a, b, c, d\}_5(H) = \{H, c\} \{a, b, \{H, d\}, H\}_4 - \{H, d\} \{a, b, \{H, c\}, H\}_4$$

- **Alternating Canonical Quartet:** Four functions (a, b, c, d) satisfying $\{a, b\} = 1$, $\{c, d\} = 1$, with all other pairwise brackets vanishing (local Darboux pairs).

- **Covariant Separability Theorem:** H satisfies the Levi-Civita condition for the (i, j) sector if and only if:

$$\{a, b, c, d\}_5(H) = 0$$

- **Symplectic Curvature Representation:** The bracket factorizes into a symmetric quadratic form:

$$\{a, b, c, d\}_5(H) = -\frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v}$$

where $\mathbf{v} = (\{H, p_i\}, \{H, p_j\}, \{H, x_i\}, \{H, x_j\})^T$ and $\mathbf{M}$ is a connection matrix of iterated Poisson brackets.

- **Foliation Invariance:** $\{a, b, c, d\}_5(F(H)) = (F'(H))^3 \{a, b, c, d\}_5(H)$, isolating energy leaf geometry.

Mode A: Diagnostic Sensor

- Evaluated over standard configuration charts (e.g., Cartesian).
- Non-vanishing five-bracket isolates exact algebraic boundaries where coordinate coupling destroys separability.
- Identifies singular zero-momentum sub-manifolds.

Mode B: Separability Investigator

- Operates purely on primitive symplectic brackets.
- Evaluates candidates across canonical transformations.
- Explicit vanishing under transformed frames reveals hidden symmetries and valid uncoupled integration charts.

Application 1: Cartesian and Perturbed Systems

- **Natural Cartesian Potentials:** For $H = \frac{1}{2}(X^2 + Y^2) + V(x, y)$:

$$\{x, X, y, Y\}_5(H) = -YX \left(\frac{\partial^2 V}{\partial x \partial y} \right)$$

Vanishes globally if and only if $V(x, y) = f_1(x) + f_2(y)$.

- **Generalized Hénon-Heiles System:** $V(x, y) = \frac{1}{2}(Ax^2 + By^2) + \alpha x^2 y + \beta y^3$:

$$\{x, X, y, Y\}_5(H) = -2YX\alpha x$$

Confirms cubic coupling $\alpha \neq 0$ breaks separability globally, vanishing only along $x = 0$ or zero-momentum lines.

- **Two-Particle Toda Lattice:** $H = \frac{1}{2}(X^2 + Y^2) + \exp(x - y) \implies \{x, X, y, Y\}_5(H) = YX \exp(x - y) \neq 0$.
Signals necessity of center-of-mass / relative canonical transformation.

- **Polar-Nodal Formulation:** In Whittaker variables $(r, \theta, \nu, R, \Theta, N)$:

$$H = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} + \frac{V(\theta, \Theta, N)}{r^q}$$

- **Five-Bracket Diagnostic Evaluation:**

$$\{r, R, \theta, \Theta\}_5(H) = R \Theta r^{-3-q} (-2 + q) \left(\frac{\partial V}{\partial \theta} \right)$$

- **Astrodynamical Takeaways:**

- 1 Isolates $q = 2$ as the unique physical scaling exponent permitting global orthogonal separability (e.g., MICZ-Kepler and Vinti systems).
- 2 Velocity-dependent perturbations $V = f(\Theta, N)$ yield $\partial_\theta f = 0 \implies \{r, R, \theta, \Theta\}_5(H) = 0$ for any q .

Application 3: The Stark-Kepler Problem

- **Uniform Electric Field Perturbation:** $H = \frac{1}{2} \left(R^2 + \frac{\Theta^2}{r^2} \right) - \frac{\mu}{r} + \epsilon r \cos \theta.$

- **Obstruction Diagnostic:**

$$\{r, R, \theta, \Theta\}_5(H) = \frac{3\Theta R \epsilon \sin \theta}{r^2} \neq 0$$

Proves non-separability in polar-nodal frames and dictates transition to parabolic coordinates $(\xi, \eta).$

- **General Potential Scanning:** For arbitrary $F(r, \theta)$, requiring $\{r, R, \theta, \Theta\}_5(H) = 0$ yields the PDE:

$$r \frac{\partial^2 F}{\partial r \partial \theta} + 2 \frac{\partial F}{\partial \theta} = 0 \implies F(r, \theta) = f_1(r) + \frac{f_2(\theta)}{r^2}$$

- **Cartesian Obstruction:** Evaluating 3D Kepler problem in standard Cartesian frame (x, X, y, Y) :

$$\{x, X, y, Y\}_5(H) = \frac{3YX\mu yx}{(x^2 + y^2 + z^2)^{5/2}} \neq 0$$

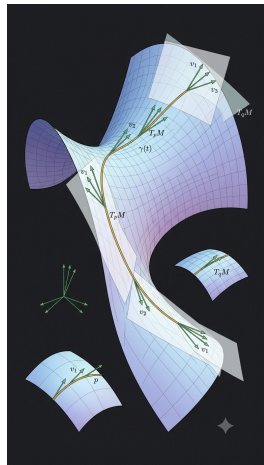
- **Shift to Angular Momentum Quartet:** Define canonical quartet matched to total angular momentum magnitude $L = \|\mathbf{r} \times \mathbf{p}\|$:

$$a = r, \quad b = p_r, \quad c = \theta, \quad d = L$$

- **Algorithmic Verification:**

$$\{a, b, c, d\}_5(H) = 0$$

- **Conclusion:** Algorithmic confirmation of integrability under canonical shifts.



- **Symbolic Efficiency:** The bracket framework relies strictly on primitive Poisson derivations, avoiding high-order tensor derivatives and Killing tensor derivations.
- **Computer Algebra Systems (CAS):** Perfectly adapted for automated Maple / Mathematica / SymPy worksheets to scan high-degree gravitational or orbital perturbations.
- **Outlook and Frontiers:**
 - Extension to Eisenhart-metric and Benenti manifold configurations in astrodynamics.
 - Testing separability in non-bi-Hamiltonian invariant systems.
 - Quantum operator generalizations replacing Poisson brackets with commutators $[\cdot, \cdot]$.



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