Comparison of D–criteria and metrics in the space of Keplerian orbits

Local Extrema and Asymptotic Behaviour

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Introduction

Problem

Assessment of Keplerian orbit proximity for identifying common-origin bodies (asteroid families, meteor streams).

Tools:

- Classical D-criteria (Southworth–Hawkins, Drummond, Jopek)
- Kholshevnikov metrics (ρ_1–ρ_5)
- New metric g based on integrals of motion

Goal of this work:

- Analysis of local extrema with respect to all elements
- Asymptotic behaviour in limiting cases
- Comparison of mathematical complexity and stability

Classical D-criteria

$$D_1^2 = \frac{(q_1 - q_2)^2}{L^2} + 4\sin^2\frac{I}{2} + (e_1 - e_2)^2 + (e_1 + e_2)^2 \sin^2\frac{\Pi}{2},$$

$$D_2^2 = \frac{(q_1 - q_2)^2}{L^2} + 4\sin^2\frac{I}{2} + (e_1 - e_2)^2 + (e_1 + e_2)^2 \sin^2\frac{\Pi'}{2},$$

$$D_3^2 = \left(\frac{q_1 - q_2}{q_1 + q_2}\right)^2 + \left(\frac{e_1 - e_2}{e_1 + e_2}\right)^2 + I^2 + \left(\frac{e_1 + e_2}{2}\right)^2 P^2,$$

$$D_J^2 = \left(\frac{q_1 - q_2}{q_1 + q_2}\right)^2 + (e_1 - e_2)^2 + 4\sin^2\frac{I}{2} + (e_1 + e_2)^2 \sin^2\frac{\Pi}{2}.$$

Here $\cos I = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_1 - \Omega_2)$, Π and P are angles between pericentres and Laplace–Runge–Lenz vectors.

Kholshevnikov metrics ρ_1–ρ_5

$$\rho_1^2 = \frac{1}{L}(p_1 + p_2 - 2\sqrt{p_1 p_2}\cos I) + (e_1^2 + e_2^2 - 2e_1 e_2 \cos P),$$
$$\rho_2^2 = (1 + e_1^2)p_1 + (1 + e_2^2)p_2 - 2\sqrt{p_1 p_2}(\cos I + e_1 e_2 \cos P),$$
$$\rho_3^2 = (1 + e_1^2)p_1 + (1 + e_2^2)p_2 - 2\sqrt{p_1 p_2}\,\zeta_3,$$
$$\rho_4^2 = (1 + e_1^2)p_1 + (1 + e_2^2)p_2 - 2\sqrt{p_1 p_2}(e_1 e_2 + \cos I),$$
$$\rho_5^2 = (1 + e_1^2)p_1 + (1 + e_2^2)p_2 - 2\sqrt{p_1 p_2}(e_1 e_2 + \cos(i_1 - i_2)).$$

ζ_3 is a complex function of angles providing factorization over Ω.

Metric g based on integrals of motion

Squared distance

$$g^2 = \left(\frac{E_1 - E_2}{E_{\text{scale}}}\right)^2 + \left(\frac{L_1 - L_2}{L_{\text{scale}}}\right)^2 + d_q^2(q_1, q_2),$$

where $E = -\mu/(2a)$, $L = \sqrt{\mu p}$, and d_q is the quaternion orientation distance:

$$d_q(q_1, q_2) = \min\left(\|q_1 - q_2\|, \|q_1 + q_2\|\right).$$

- Satisfies all metric axioms.
- Well-defined for circular and coplanar orbits.
- Simple extremum condition for e_1: $e_1 = 0$ or $L_1 = L_2$.

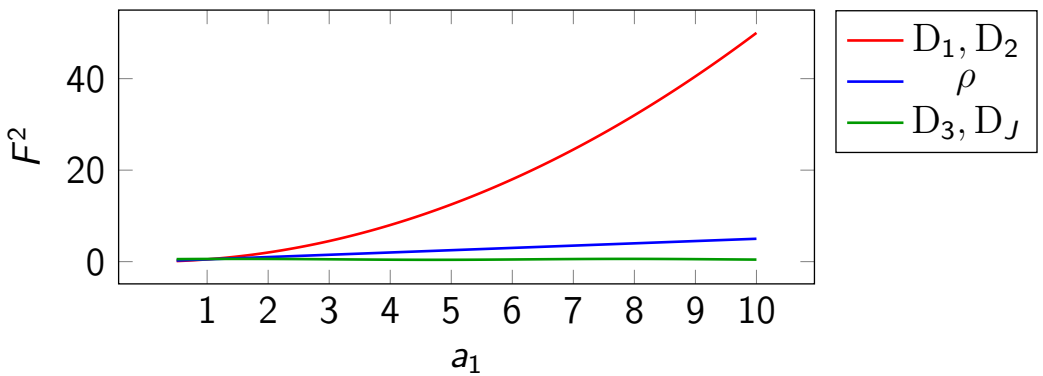
Variations of semimajor axis a_1

Extremum conditions:

- $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_J$: $q_1 = q_2$
- ρ: $1 + e_1^2 = \sqrt{p_2/p_1}\,\mathcal{F}$
- g: polynomial of degree 6

Asymptotics as $a_1 \to \infty$:

- $\mathrm{D}_1, \mathrm{D}_2 \sim a_1^2$ (over-dominance)
- $\mathrm{D}_3, \mathrm{D}_J \to$ const (loss of sensitivity)
- ρ, $g \sim a_1$ (balanced sensitivity)



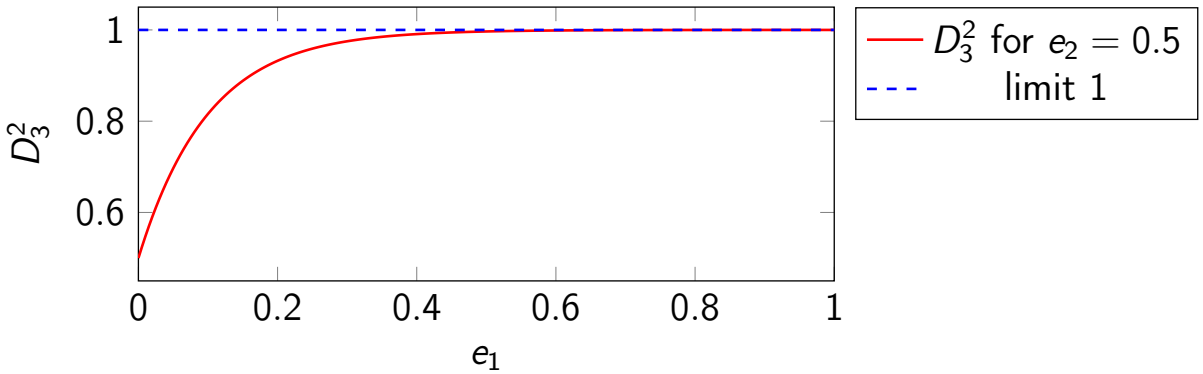
Variations of eccentricity e_1

Equation complexity:

- $\mathrm{D}_1, \mathrm{D}_2$: linear (analytic solution)
- D_J: polynomial degree 4
- D_3: polynomial degree 7
- ρ: polynomial degree 8
- g: $e_1 = 0$ or $L_1 = L_2$ (simplest!)

Singularities:

- D_3: as $e_1 \to 0^+$ with $e_2 > 0$, $\left(\frac{e_1 - e_2}{e_1 + e_2}\right)^2 \to 1$ (unphysical)
- ρ and g: derivative $\sim 1/\sqrt{1 - e_1}$ as $e_1 \to 1^-$ (high sensitivity near parabolic orbits)



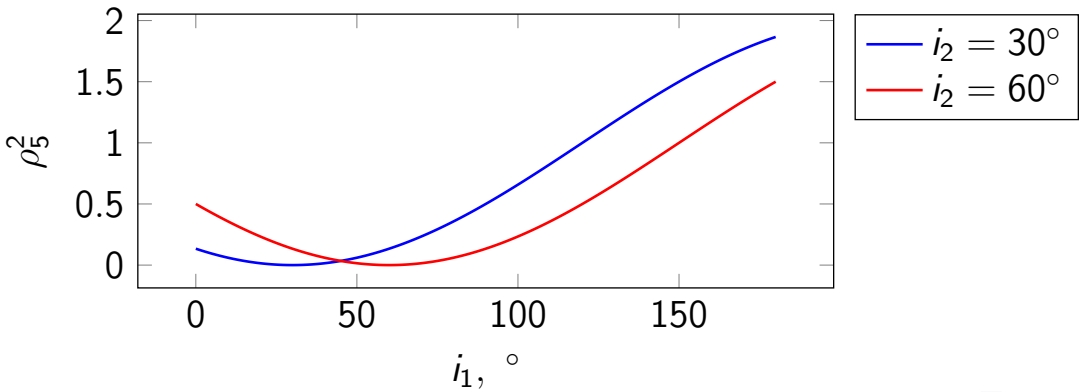
Variations of inclination i_1

Extremum conditions:

- D_2: $i_1 = \arctan(\tan i_2 \cos \Delta\Omega) + \pi k$
- ρ_4: $\tan i_1 = \tan i_2 \cos \Delta\Omega$
- ρ_5: $i_1 = i_2$
- $\mathrm{D}_1, \mathrm{D}_3, \mathrm{D}_J, \rho_1, \rho_2, \rho_3, g$: transcendental equations

Asymptotics:

- All metrics have finite limits as $i_1 \to 0$ and $i_1 \to \pi$
- Derivatives proportional to $\sin i_2$, vanish for $i_2 = 0$
- g: in coplanar limit, derivative depends only on quaternions (no Euler angle ambiguities)



Variations of longitude of ascending node Ω_1

Extremum conditions:

- $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_J$: transcendental equations
- ρ_4: $\sin i_1 \sin i_2 \sin(\Omega_1 - \Omega_2) = 0$
- ρ_3, ρ_5: independent of Ω_1
- g: transcendental (but simpler than for D)

Periodicity:

$$F^2(\Omega_1) = F^2(\Omega_1 + 2\pi),$$

$$\left.\frac{\partial F^2}{\partial \Omega_1}\right|_0 = -\left.\frac{\partial F^2}{\partial \Omega_1}\right|_{2\pi}$$

Derivatives contain $\sin i_1 \sin i_2$, so for $i = 0$ or π metrics are independent of Ω_1.

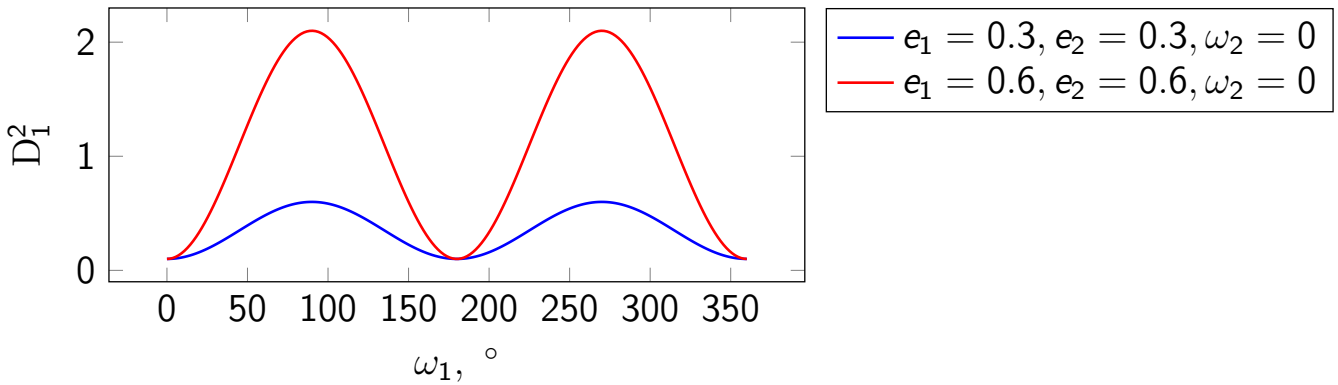
Variations of argument of pericentre ω_1

Extremum conditions:

- $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_J$: $\Pi = \pi n$ or $\Pi' = \pi n$ (simplest!)
- D_3: $P \cdot \partial P / \partial \omega_1 = 0$ (transcendental)
- ρ_4, ρ_5: independent of ω_1
- $\rho_1, \rho_2, \rho_3, g$: transcendental

Special features:

- 2π-periodicity
- Derivatives of ρ proportional to $e_1 e_2 \Rightarrow$ vanish for $e = 0$
- D-criteria are inapplicable for $e = 0$ (singularity of ω)



Comparative table of properties

	$\mathrm{D}_1, \mathrm{D}_2$	D_3	D_J	ρ	g
Metric	no	no	no	yes	yes
Valid for $e = 0$	no	no	no	yes	yes
Valid for $I = \pi$	no	yes	no	yes	yes
Simplicity a_1	simple	simple	simple	medium	complex
Simplicity e_1	linear	poly.7	poly.4	poly.8	$e_1 = 0$ or $L_1 = L_2$
Simplicity i_1	transc.	transc.	transc.	ρ_4, ρ_5 anal.	transc.
Simplicity Ω_1	transc.	transc.	transc.	ρ_4 anal.	transc.
Simplicity ω_1	$\Pi = \pi n$	transc.	$\Pi = \pi n$	ρ_4, ρ_5 indep.	transc.

Main conclusions

1. **Classical D-criteria**:
   - Simple for a and ω, but have singularities and are not metrics.
   - D_3 gives unphysical behaviour as $e \to 0$.
2. **Kholshevnikov metrics**:
   - Universal, valid for all orbits.
   - ρ_4, ρ_5 provide simple analytic conditions for angular elements.
   - Derivatives diverge as $e \to 1$ (high sensitivity near parabolic orbits).
3. **Metric g**:
   - Simplest extremum condition for e_1.
   - Stable in coplanar limit.
   - Requires numerical solution for a and angular elements.

Recommendations for choice

- **For quick analysis of close circular orbits**: avoid D-criteria, use ρ_2 or g.
- **For highly eccentric orbits**: ρ gives high sensitivity, g is more stable.
- **If simplicity for angular elements matters**: ρ_4 (factor by ω) or ρ_5 (factor by Ω, ω).
- **For rigorous mathematical justification**: only metrics (ρ or g).
- **When dealing with large semimajor axes**: avoid $\mathrm{D}_3, \mathrm{D}_J$ (saturation), prefer ρ and g (linear growth).

References

Thanks for your attention!

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