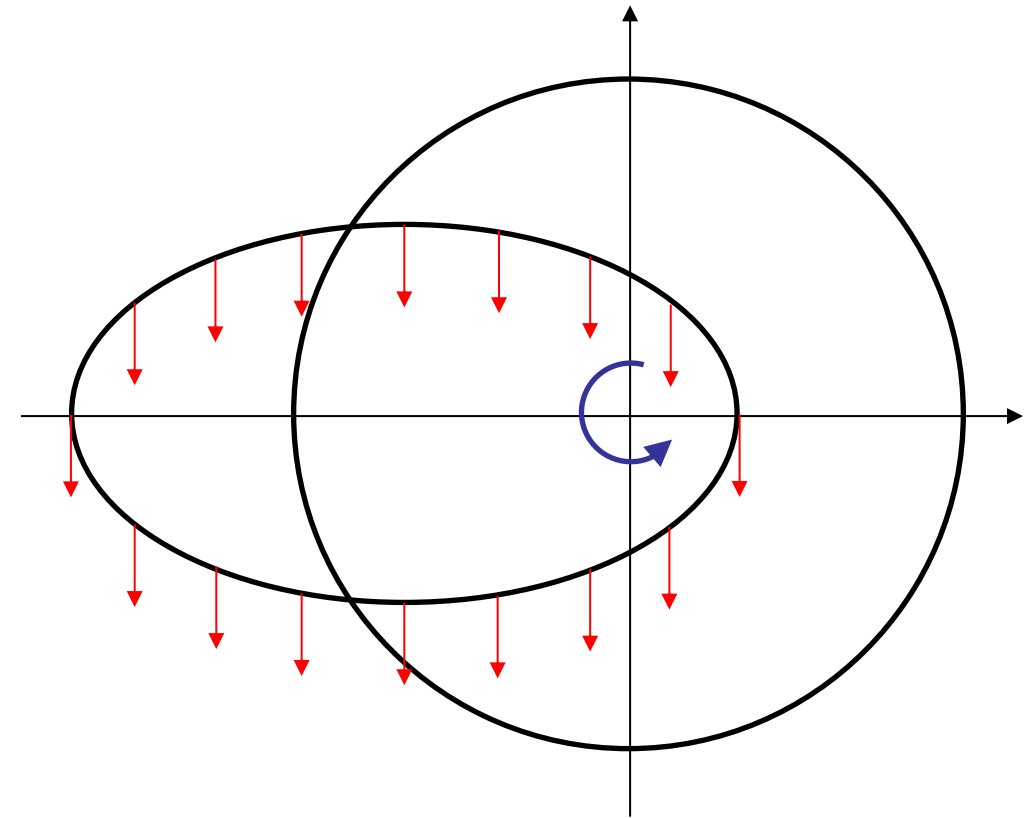




Low-Thrust Inclination Control on Elliptical Orbits with Constant Orbital Period

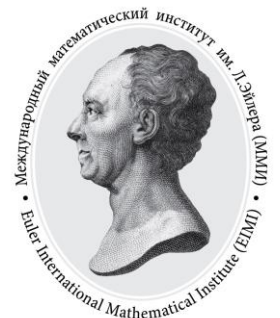
A.V. Ivanyukhin



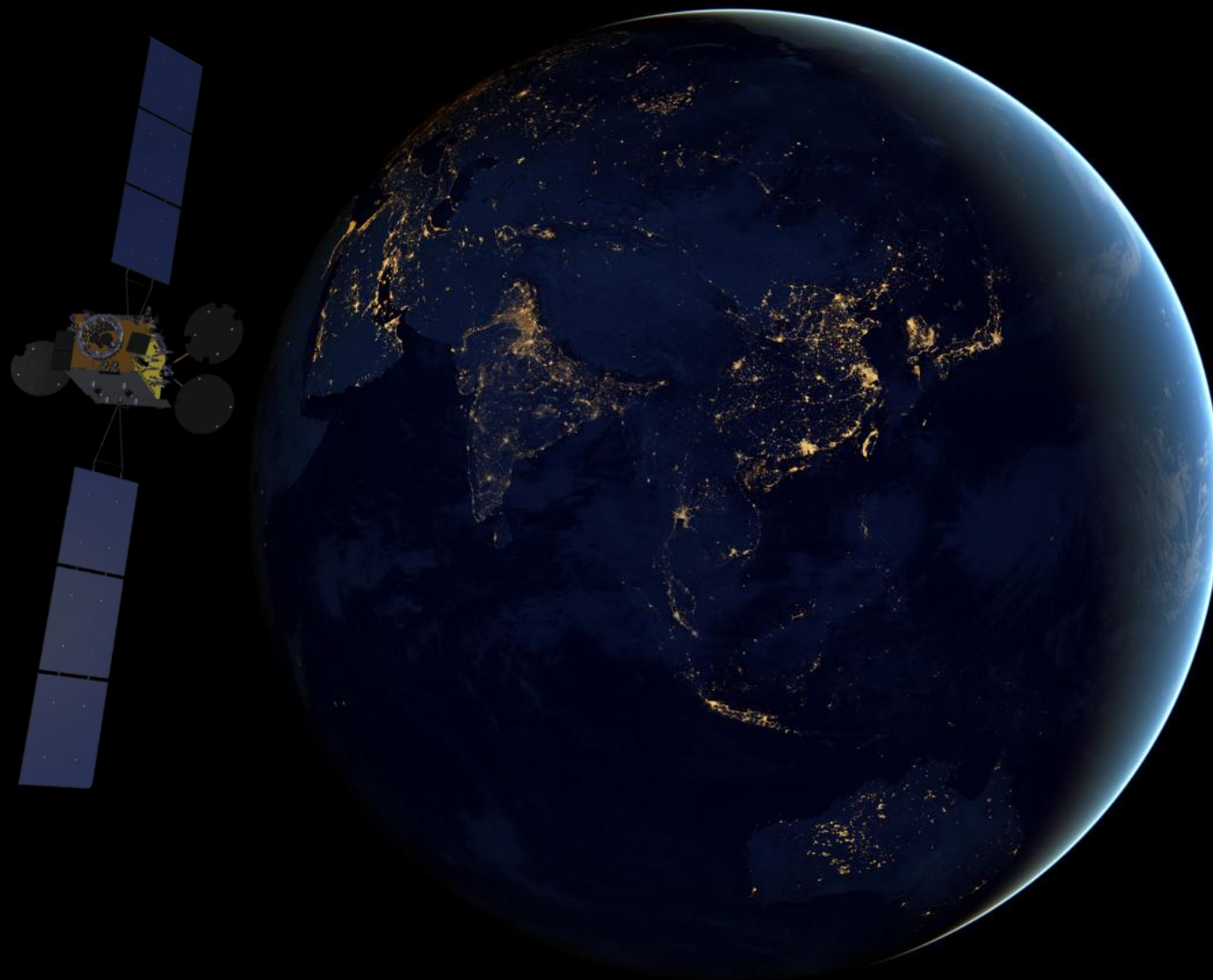
Analytical Methods of Celestial Mechanics 2026

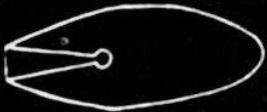
August 17-21, 2026

Euler International Mathematical Institute, St. Petersburg, RUSSIA



Motivation




$$V = V_i \ln \left(1 + \frac{M_2}{M_1} \right)$$
$$V = V_i \ln \left(1 + \frac{M_2}{M_1} \right) \left(\frac{p - q}{p} \right)$$
$$M_2 = M_1 \left[e^{\sqrt{\frac{\gamma}{\beta} \left(\frac{p}{p - q} \right)}} - 1 \right]$$

"Raketa," May 1897



Equations of Motion

The perturbed equations of motion (with low thrust) in osculating elements:

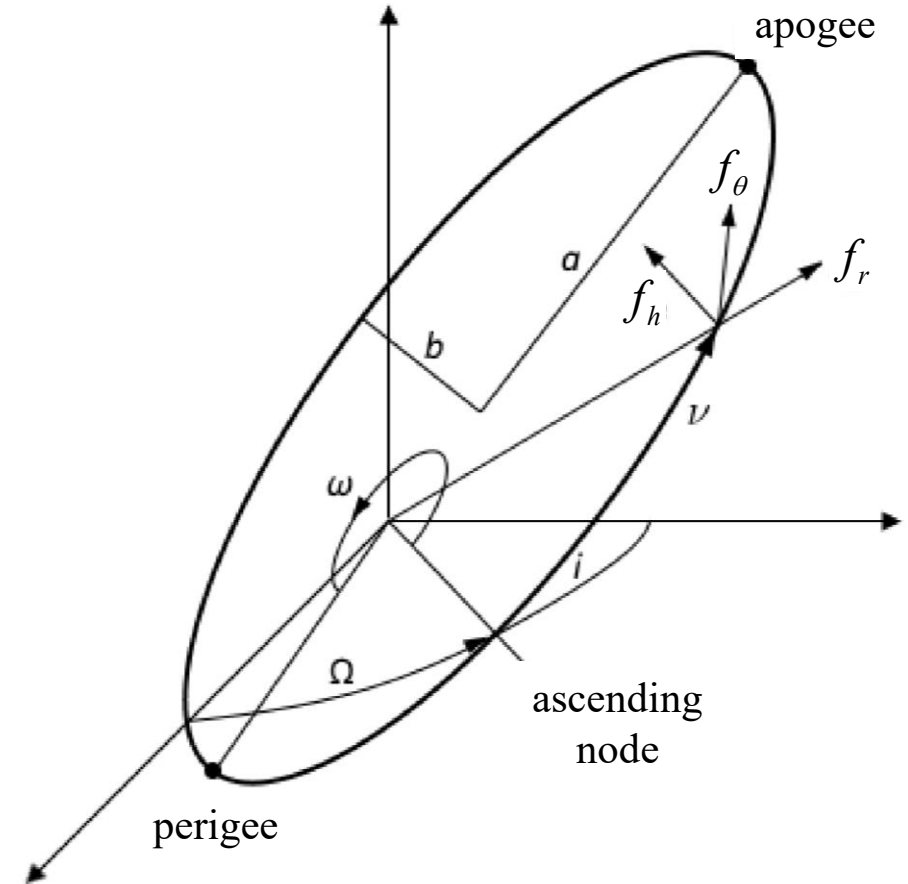
$$\frac{da}{dE} = \frac{2a^3}{\mu} \left(e \sin E f_r + \sqrt{1-e^2} f_\theta \right),$$

$$\frac{de}{dE} = \frac{a^2 \sqrt{1-e^2}}{\mu} \left[\sqrt{1-e^2} \sin E f_r + (2 \cos E - e - e \cos^2 E) f_\theta \right],$$

$$\frac{di}{dE} = \frac{a^2}{\mu} (1 - e \cos E) \left(\frac{\cos E - e}{\sqrt{1-e^2}} \cos \omega - \sin E \sin \omega \right) f_h,$$

$$\frac{d\Omega}{dE} = \frac{a^2}{\mu} (1 - e \cos E) \left(\frac{\cos E - e}{\sqrt{1-e^2}} \sin \omega + \sin E \cos \omega \right) \frac{f_h}{\sin i},$$

$$\frac{d\omega}{dE} = \frac{a^2}{\mu e} \left[\sqrt{1-e^2} (e - \cos E) f_r + (2 - e^2 - e \cos E) \sin E f_\theta \right] - \cos i \frac{d\Omega}{dE},$$



where a – semi-major axis, e – eccentricity, i – inclination, Ω – longitude of the ascending node, ω – argument of pericenter, E – eccentric anomaly.

Analytical Solutions

The perturbed equations of motion (with low-thrust) in osculating elements:

- Transversal acceleration (perpendicular to the radius vector),
- Tangential acceleration (along the velocity vector),
- Radial acceleration (along the radius vector),
- Acceleration directed along the coordinate axis (limiting case of the problem of two fixed centers).

Transfer between circular coplanar orbits (analogous to the Hohmann transfer):

$$\Delta V = \left| \sqrt{\frac{\mu}{r_0}} - \sqrt{\frac{\mu}{r_k}} \right|,$$

where μ – gravitational parameter, r_0 – radius of the initial orbit, r_k – radius of the final orbit.

Averaged Equations of Motion

Considering the system in osculating elements as a weakly perturbed system with slow variables a, e, i, Ω, ω and the fast variable E it is expedient, for a qualitative analysis of the orbit evolution, to switch to averaged equations retaining only the secular changes.

Let us consider the increment per revolution of the orbital element x :

$$\Delta x = \int_0^{2\pi} \frac{dx}{dE}(E) dE.$$

To return to time derivatives, we divide the expressions obtained by the orbital period:

$$\dot{\tilde{x}} = \frac{\Delta x}{T}, \quad T = 2\pi \sqrt{\frac{a^3}{\mu}}.$$

Edelbaum's Solution

Transfer between non-coplanar circular orbits.

Let $e \approx 0$ and $f_r = 0$, $f_\theta = f \cos(\beta)$, $f_h = f \sin(\beta)$.

Then the averaged equations of motion can be represented as:

$$\begin{cases} \frac{d\tilde{i}}{dt} = \frac{2f}{\pi V} \sin(\beta), \\ \frac{dV}{dt} = -f \cos(\beta), \end{cases} \Rightarrow \begin{cases} \frac{d\tilde{i}}{dV} = -\frac{2}{\pi V} \operatorname{tg}(\beta), \\ \frac{dt}{dV} = -\frac{1}{f \cos(\beta)}, \end{cases}$$

Let us consider the isoperimetric problem:

$$I = \int_{V_0}^{V_f} \left(\frac{d\tilde{i}}{dV} \right) dV \rightarrow \max, \quad J = \int_{V_0}^{V_f} \left(\frac{dt}{dV} \right) dV = \Delta t,$$

$$L = I + \lambda J \rightarrow \max,$$

$$\frac{\partial L}{\partial \beta} = 0, \quad J = 0.$$

Edelbaum's Solution

From which it follows:

$$\frac{2}{\pi V} \frac{1}{\cos^2(\beta)} + \frac{\lambda}{f} \frac{\sin(\beta)}{\cos^2(\beta)} = 0,$$

$$V \sin(\beta) = -\frac{2f}{\pi\lambda} = \text{const} = V_0 \sin(\beta_0) \quad \text{и} \quad \sin(\beta) = \frac{V_0 \sin(\beta_0)}{V},$$

And we obtain the boundary-value problem:

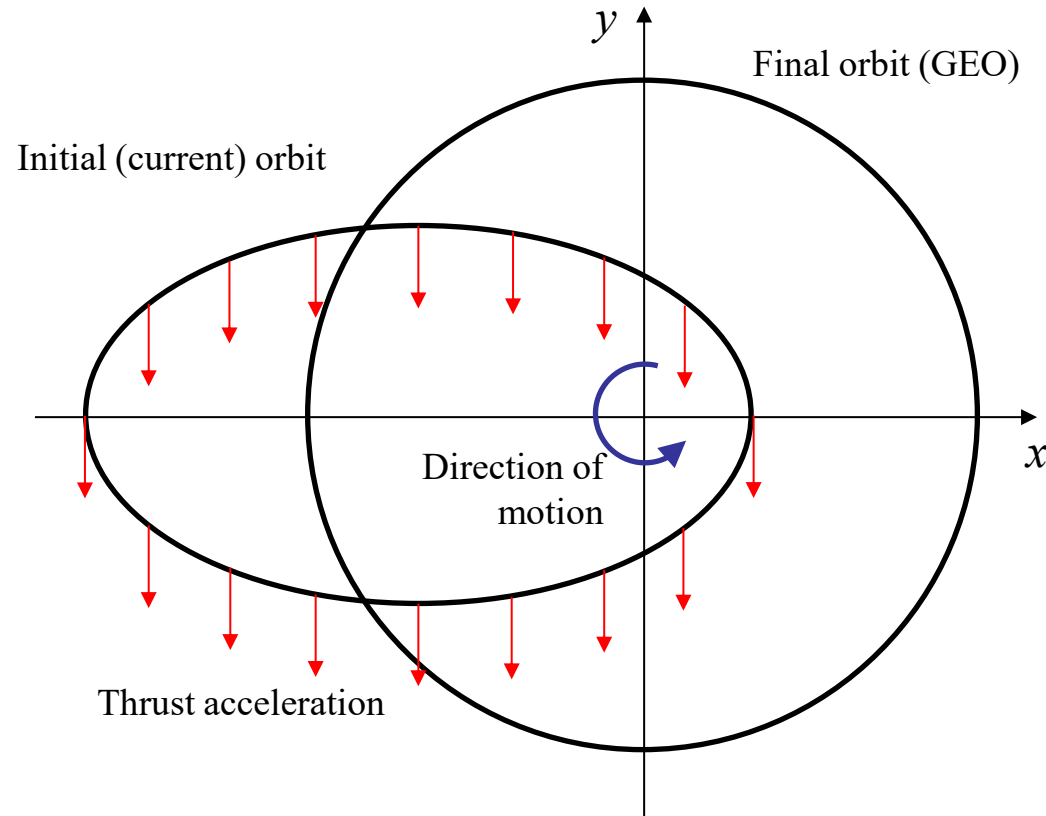
$$\frac{d\tilde{i}}{dV} = -\frac{2}{\pi V} \operatorname{tg}(\beta), \quad V \in [V_0, V_f], \quad \begin{cases} \tilde{i}(V_0) = i_0, \\ \tilde{i}(V_f) = i_f, \end{cases} \quad \beta_0 - ?$$

As is well known, it admits an analytical solution:

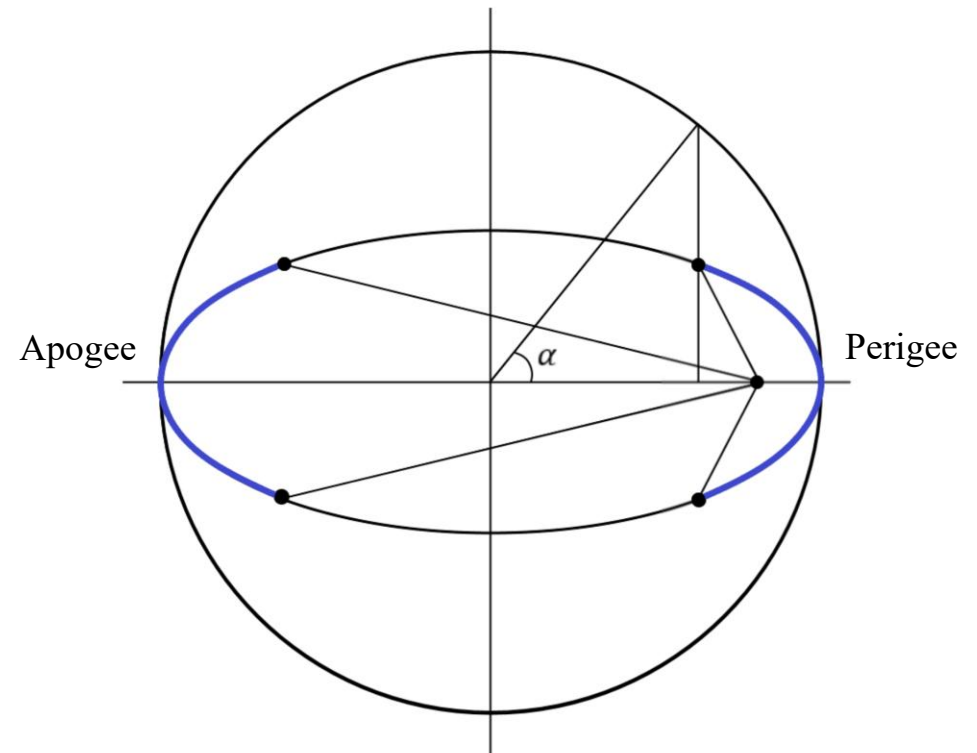
$$\Delta V = \sqrt{V_0^2 + V_f^2 - 2V_0V_f \cos\left(\frac{\pi}{2} \Delta i\right)}, \quad \Delta i = i_f - i_0, \quad \Delta t = \frac{\Delta V}{a_T}$$

Transfer with Constant Orbital Period

The main idea, following from a qualitative analysis of “Spitzer's scheme”:
the semi-major axis (a), the longitude of the ascending node (Ω), and the argument of pericenter (ω) **on average** remain constant.



(Spitzer A., 1995)



(Pollard J.E., 1997)

*Near optimal transfer orbit trajectory using electric propulsion (Spitzer A. – 1995),
Simplified approach for assessment of low-thrust elliptical orbit transfers (Pollard J.E. – 1997)*

Transfer with Constant Orbital Period

Transition to the pericentric coordinate system

$$f_r = \frac{\cos E - e}{1 - e \cos E} f_x + \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E} f_y, \quad f_\theta = -\frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E} f_x + \frac{\cos E - e}{1 - e \cos E} f_y, \quad f_h = f_h.$$

The averaged system with symmetric active arcs and $f_x = 0$ takes the form:

- general case:

$$\dot{a} = 0,$$

$$\dot{e} = \frac{\beta}{2\pi} \sqrt{\frac{a}{\mu}} f_y \cos \omega (6\alpha + \sin 2\alpha),$$

$$\dot{i} = \frac{2(1 + e^2)}{\pi\beta} \sqrt{\frac{a}{\mu}} f_h \cos \omega \sin \alpha,$$

$$\dot{\Omega} = \frac{2(1 + e^2)}{\pi\beta \sin i} \sqrt{\frac{a}{\mu}} f_h \sin \omega \sin \alpha,$$

$$\dot{\omega} = -\frac{\beta}{2\pi e} \sqrt{\frac{a}{\mu}} f_y \sin \omega (6\alpha - \sin 2\alpha) - \cos i \dot{\Omega},$$

- if $\omega = 0$:

$$\dot{a} = 0,$$

$$\dot{e} = \frac{\beta}{2\pi} \sqrt{\frac{a}{\mu}} f_y (6\alpha + \sin 2\alpha),$$

$$\dot{i} = \frac{2(1 + e^2)}{\pi\beta} \sqrt{\frac{a}{\mu}} f_h \sin \alpha,$$

$$\dot{\Omega} = 0,$$

$$\dot{\omega} = 0.$$

Transfer with Constant Orbital Period: Problem Statement

Let us switch to e as the independent variable and augment the system with an equation for the velocity expenditure

$$\begin{aligned}\frac{di}{de} &= \frac{4(1+e^2)\sin\alpha}{(1-e^2)(6\alpha+\sin 2\alpha)} \tan\beta, \\ \frac{dt}{de} &= \frac{2\pi}{f \cos\beta(6\alpha+\sin 2\alpha)} \sqrt{\frac{\mu}{a}} \frac{1}{\sqrt{1-e^2}}, \\ \frac{d(\Delta V)}{de} &= \frac{4\alpha}{\cos\beta(6\alpha+\sin 2\alpha)} \sqrt{\frac{\mu}{a}} \frac{1}{\sqrt{1-e^2}}.\end{aligned}$$

$$e \in [e_0, e_f], \quad i(e_0) = i_0, \quad i(e_f) = i_f, \quad t(e_0) = 0, \quad t(e_f) = \Delta t, \quad \Delta V(e_0) = 0, \quad \Delta V(e_f) \rightarrow \min.$$

The control variables are

$$0 < \alpha_{\min} \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq \beta \leq \frac{\pi}{2}.$$

Transfer with Constant Orbital Period: Constant Control

An analytical solution is readily obtained for constant control

$$\begin{aligned}\Delta i &= \frac{4 \sin \alpha}{6\alpha + \sin 2\alpha} \tan \beta S_i, \\ \Delta t &= \frac{2\pi}{f \cos \beta (6\alpha + \sin 2\alpha)} \sqrt{\frac{\mu}{a}} S_e, \\ \Delta V &= \frac{4\alpha}{\cos \beta (6\alpha + \sin 2\alpha)} \sqrt{\frac{\mu}{a}} S_e,\end{aligned}$$

where

$$S_i = \ln \frac{(1+e_f)(1-e_0)}{(1-e_f)(1+e_0)} - (e_f - e_0), \quad S_e = \arcsin e_f - \arcsin e_0.$$

In this case, the first two expressions, together with the boundary conditions, yield the required constant control, while the last expression gives the corresponding value of the velocity expenditure.

(Optimization with respect to duration only.)

Transfer with Constant Orbital Period: Optimal Control

Let us consider the isoperimetric problem, analogous to the Edelbaum problem:

$$I = \sqrt{\frac{\mu}{a}} \int_{e_0}^{e_f} \frac{4\alpha}{\cos \beta (6\alpha + \sin 2\alpha)} \frac{de}{\sqrt{1-e^2}} \rightarrow \min,$$

$$J_1 = \sqrt{\frac{\mu}{a}} \int_{e_0}^{e_f} \frac{2\pi}{f \cos \beta (6\alpha + \sin 2\alpha)} \frac{de}{\sqrt{1-e^2}} = \Delta t,$$

$$J_2 = \int_{e_0}^{e_f} \frac{4(1+e^2) \sin \alpha}{(1-e^2)(6\alpha + \sin 2\alpha)} \tan \beta de = \Delta i.$$

The corresponding Lagrangian and the necessary optimality conditions:

$$L(\alpha, \beta, \lambda_1, \lambda_2) = I + \lambda_1 J_1 + \lambda_2 J_2 \rightarrow \min,$$

$$\frac{\partial L}{\partial \alpha} = 0, \quad \frac{\partial L}{\partial \beta} = 0, \quad J_1 = \Delta t, \quad J_2 = \Delta i.$$

Transfer with Constant Orbital Period: Optimal Control

The system of nonlinear equations for determining $\alpha(e), \beta(e)$:

$$\sin \beta = C \frac{1+e^2}{\sqrt{1-e^2}} \frac{\sin \alpha}{A(\alpha, \lambda)},$$
$$4B(\alpha) = A(\alpha, \lambda) \left[B'(\alpha) \cos^2 \beta + B(\alpha) \operatorname{ctg} \alpha \sin^2 \beta \right],$$

where

$$A(\alpha, \lambda) = 4\alpha + \frac{2\pi\lambda}{f},$$

$$B(\alpha) = 6\alpha + \sin 2\alpha,$$

$$B'(\alpha) = 6 + 2 \cos 2\alpha.$$

The constants C and λ are determined by the required Δt and Δi or by the initial values α_0, β_0 .

Transfer with Constant Orbital Period: Optimal Control

Calculation procedure:

1. Choose α_0, β_0 .
2. Determine the corresponding constants

$$A_0 = \frac{4B(\alpha_0)}{B'(\alpha_0)\cos^2 \beta_0 + B(\alpha_0)\operatorname{ctg} \alpha_0 \sin^2 \beta_0}, \quad C = \frac{\sin \beta_0 \sqrt{1 - e_0^2} A_0}{(1 + e_0^2) \sin \alpha_0}, \quad \lambda = \frac{f(A_0 - 4\alpha_0)}{2\pi}.$$

3. Divide the interval $[e_0, e_f]$ into N parts, and for each e_i determine the corresponding α_i, β_i from the system of conditions defining the control.
4. Using the analytical formulas with constants α_i, β_i for each segment we determine $\Delta t_i, \Delta i_i, \Delta V_i$ and the corresponding

$$\Delta i(\alpha_0, \beta_0) = \sum_i \Delta i_i, \quad \Delta t(\alpha_0, \beta_0) = \sum_i \Delta t_i, \quad \Delta V(\alpha_0, \beta_0) = \sum_i \Delta V_i.$$

5. The required initial angles are determined from the condition

$$\Delta i(\alpha_0^*, \beta_0^*) = \Delta i^*, \quad \Delta t(\alpha_0^*, \beta_0^*) = \Delta t^*.$$

Transfer with Constant Orbital Period: Special Cases

$\alpha = \text{constant}$:

$$I = \frac{4\alpha}{(6\alpha + \sin 2\alpha)} \sqrt{\frac{\mu}{a}} \int_{e_0}^{e_f} \frac{1}{\cos \beta} \frac{de}{\sqrt{1-e^2}} \rightarrow \min, \quad J = \frac{4 \sin \alpha}{(6\alpha + \sin 2\alpha)} \int_{e_0}^{e_f} \frac{1+e^2}{1-e^2} \tan \beta de = \Delta i.$$

$$\frac{\Delta V}{\Delta t} = \frac{2\alpha}{\pi} f,$$

The variation law of β retains its form, while Δt , Δi , ΔV are expressed through Legendre integrals.

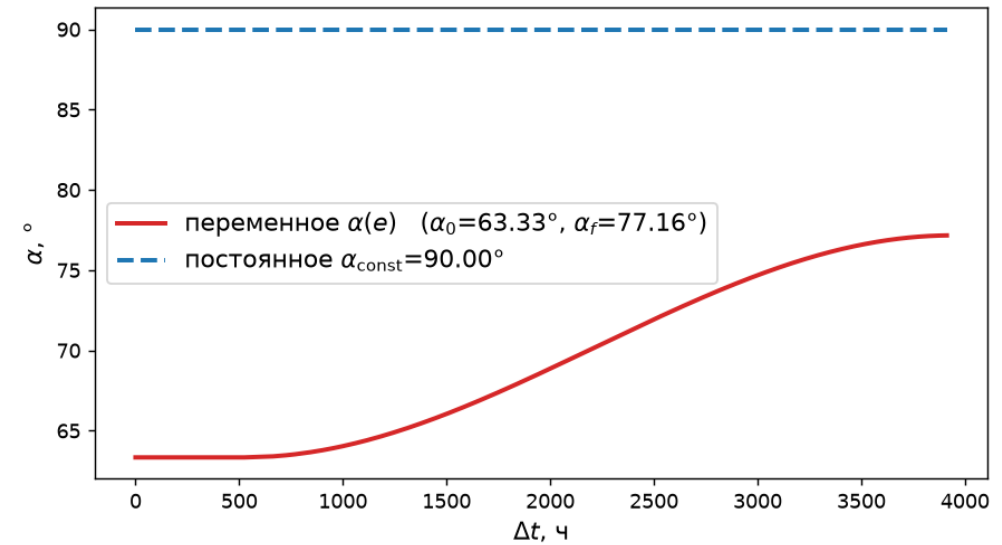
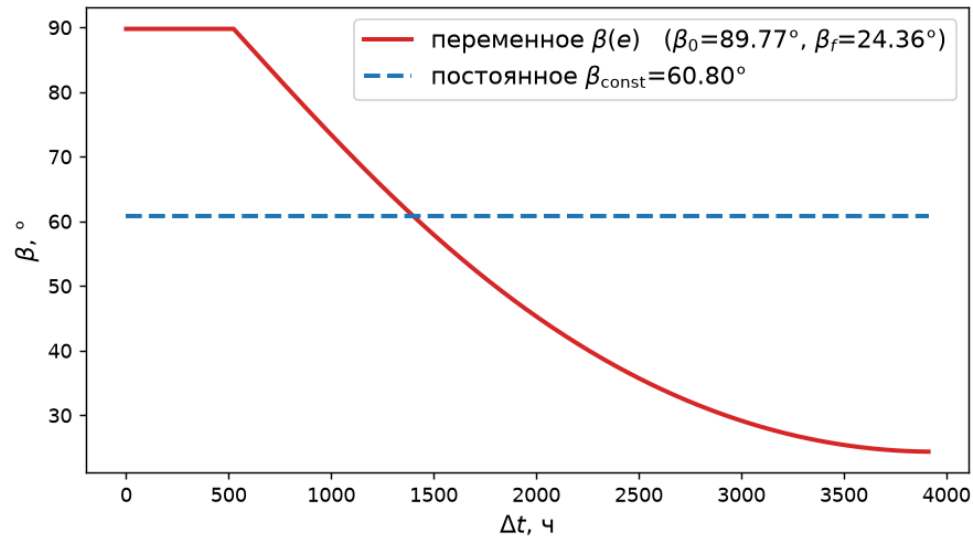
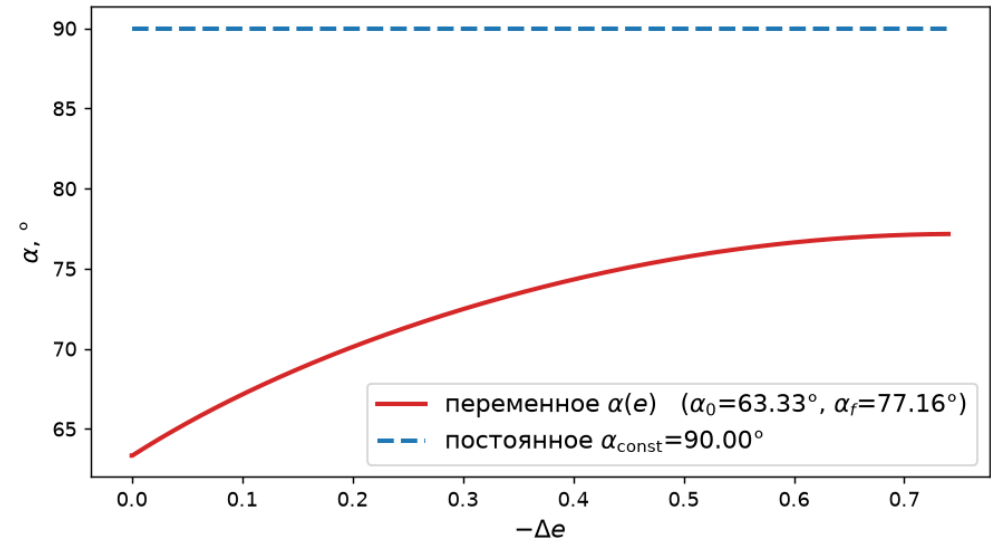
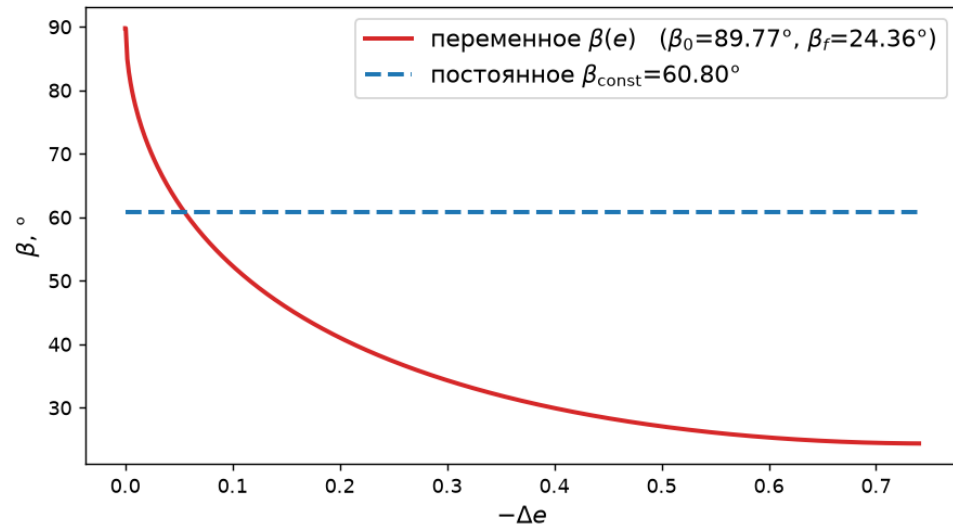
$\beta = \text{constant}$:

$$I = \frac{1}{\cos \beta} \sqrt{\frac{\mu}{a}} \int_{e_0}^{e_f} \frac{4\alpha}{(6\alpha + \sin 2\alpha)} \frac{de}{\sqrt{1-e^2}} \rightarrow \min, \quad J = \frac{1}{f \cos \beta} \sqrt{\frac{\mu}{a}} \int_{e_0}^{e_f} \frac{2\pi}{(6\alpha + \sin 2\alpha)} \frac{de}{\sqrt{1-e^2}} = \Delta t,$$

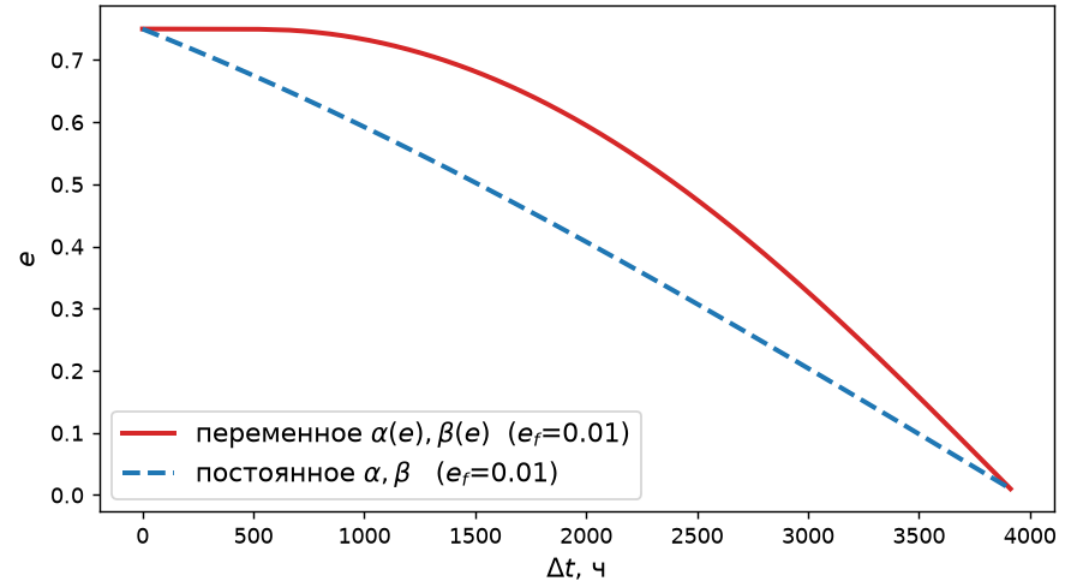
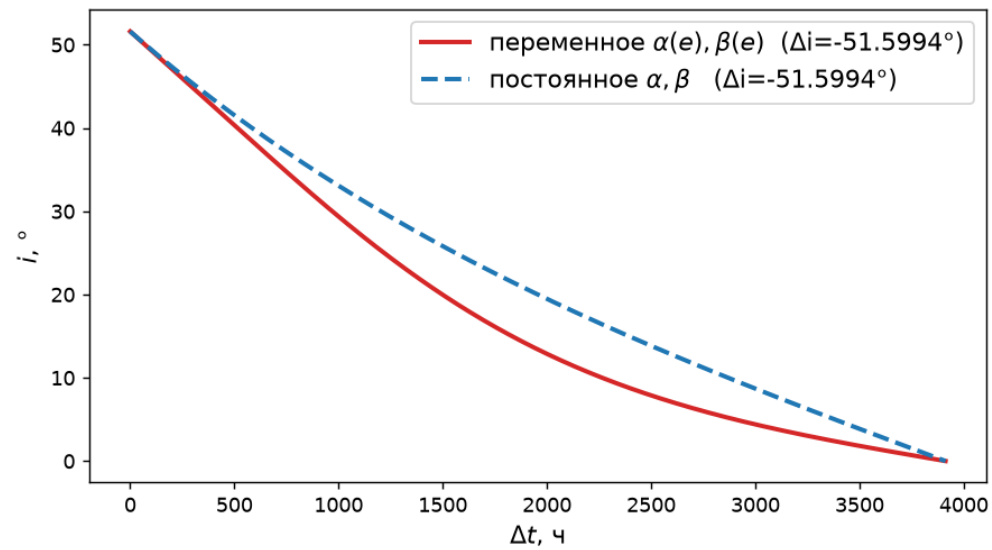
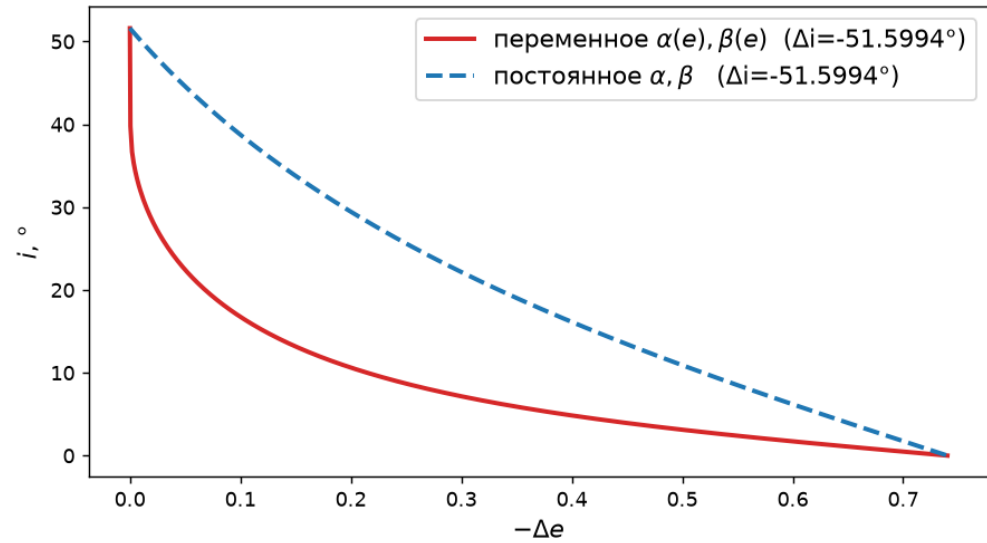
$$\Delta i = \tan \beta \int_{e_0}^{e_f} \frac{4(1+e^2) \sin \alpha}{(1-e^2)(6\alpha + \sin 2\alpha)} de.$$

The condition on α does not depend on e . The optimal solution is constant control.

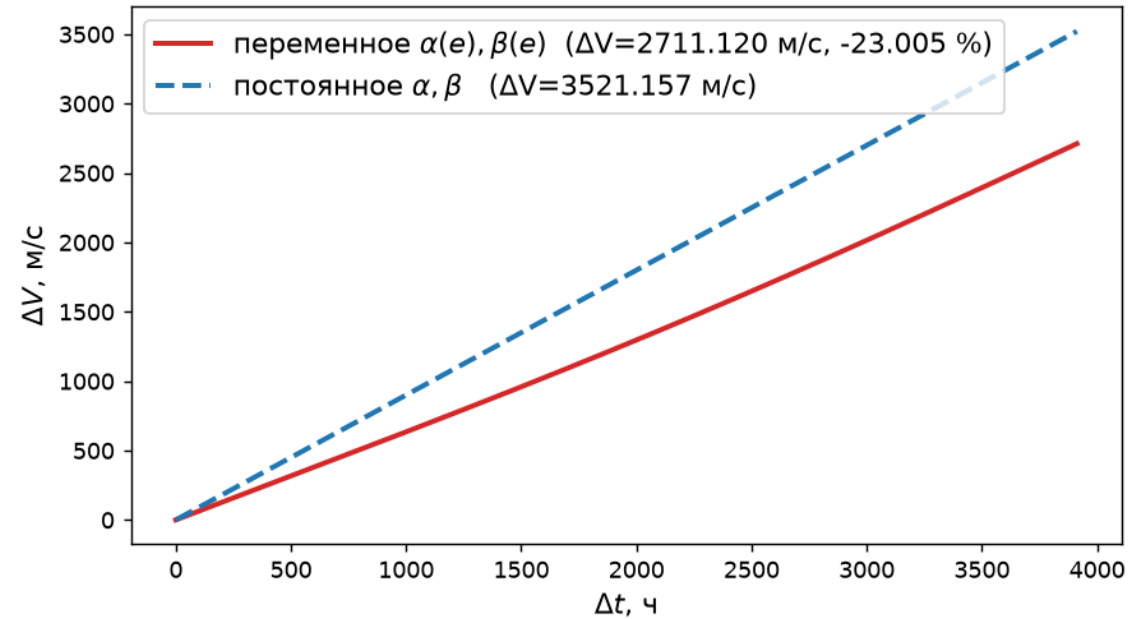
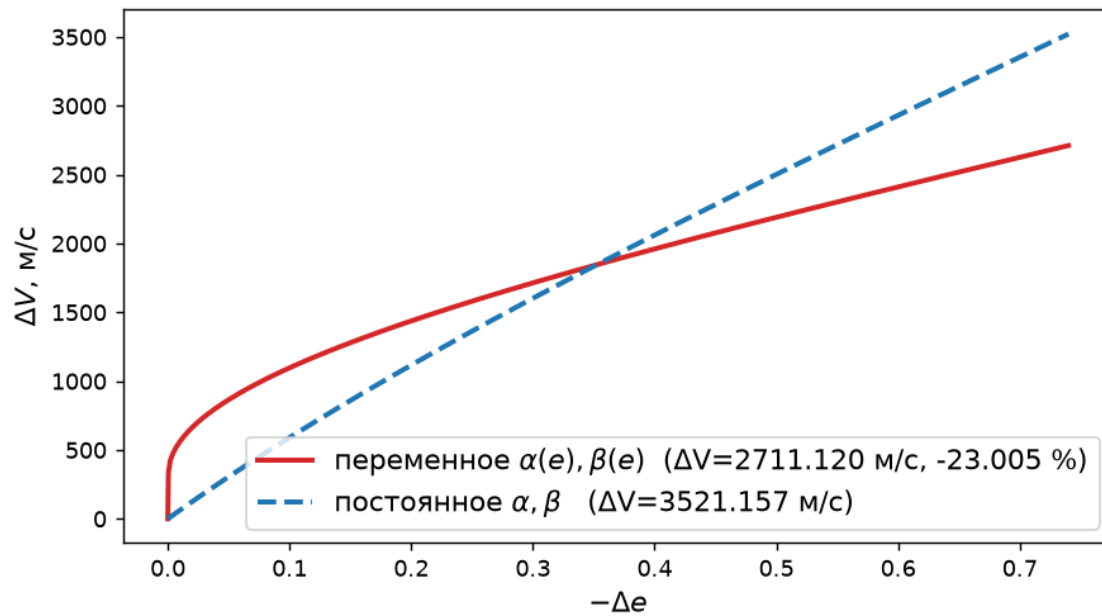
Example. Transfer to GEO.



Example. Transfer to GEO.



Example 1. GEO.



Initial orbit:

semi-major axis, a	42 164	km
eccentricity, e	0.75	
inclination, i	51.6	deg

GEO:

semi-major axis, a	42 164	km
eccentricity, e	0	
inclination, i	0	deg

Spacecraft:

thrust	0.25	N
specific impulse	1740	s
Initial mass	1000	kg

Conclusion

The averaged equations of perturbed motion (with low thrust) in osculating elements have been analyzed. We have investigated **rules for orienting the thrust acceleration and placing the active arcs** in order to maintain the semi-major axis, the longitude of the ascending node, and the argument of pericenter.

We have considered **the problem of an optimal transfer** between **elliptical** (isoparametric) non-coplanar orbits (**analogous to the Edelbaum problem**).

An analysis of GEO orbit-raising schemes has been carried out, using the low-thrust control laws considered. The existence of a **saturation zone** for the optimal yaw control (β).

It is shown that using the derived control laws in a combined GEO orbit-raising scheme can noticeably increase efficiency (by up to 23% in propellant consumption) and reduce the transfer time.

The study was supported by the Russian Science Foundation grant No 25-79-30009,

<https://rscf.ru/project/25-79-30009/>.



Thank you for your attention ...