The Euler-Lambert problem of determining the orbit of the body based on its two positions

Alexey Ivanyukhin and Vyacheslav Ivashkin

Abstract. The Euler–Lambert problem is a classical problem in celestial mechanics concerned with determining the orbits of celestial bodies. A wide range of solution methods have been developed. Most of these methods rely on the fact that the trajectories connecting two points in space form a one-parameter family of orbits that link the specified points. This paper presents a concise overview of approaches to solving the problem, spanning from the 18th century — when the problem was first systematically studied — to the present day. Particular emphasis is placed on the description of the Okhotsimsky–Egorov method, which was developed by the authors on the basis of Okhotsimsky's formula for solving the Euler–Lambert problem as an inverse dynamics problem. A comparative analysis of the computational efficiency of several widely used methods is also provided.

Introduction

The Euler–Lambert problem is a classical problem in celestial mechanics concerned with determining the orbits of celestial bodies. Traditionally, this problem is formulated as a two-point boundary value problem within the two-body model, in which the position vectors at two distinct time instants are known.

In practice, solving the Lambert problem does not require solving the boundary value problem directly, since a complete set of first integrals for this problem is known. These integrals enable the reduction of the solution to a single nonlinear equation. The form of this equation depends on the formulation of Kepler's equation relating the orbital position and time.

Initially, the objective of solving this problem was to determine the orbits of solar system bodies from two observations. In practical astronautics, the problem has acquired new significance and is often used as a basis for designing spacecraft transfer trajectories. In this context, it is assumed that the orbit connecting two given positions represents the transfer trajectory.

Currently, the problem is frequently employed at the initial stage of mission design to identify suitable launch dates and transfer durations. Moreover, it serves as a foundation for developing more sophisticated methods for solving trajectory problems involving n-impulse maneuvers, multiple gravity-assist fly-bys, and visits to several targets. Consequently, developing an efficient solution algorithm with an explicit determination of the velocity along the transfer orbit remains a relevant task.

1. Review of solution methods

Over the long history of the problem, several dozen solution methods have been proposed. The majority of these methods are based on the fact that trajectories between two points in space form a one-parameter family of orbits connecting the given points.

The earliest algebraic-geometric methods date back to the works of Newton, Euler, Lambert, Lagrange, and Gauss [**?**, **?**]. They are distinguished by their clarity and transparency of approach. However, they exhibit high computational complexity and significant numerical error. Arguably, the first computationally efficient method of its time was Gauss's method, as it spawned an entire direction of development that remains active to this day.

Modern methods for solving the Lambert problem can be classified according to the type of nonlinear equation employed, the techniques used for its solution, and the rule for selecting the initial guess. Competition among approaches now occurs primarily at the level of expression compactness, computational complexity, and ease of use in applied problems.

Broadly, the main groups of methods can be divided into the following categories:

- Methods based on the classical form of Kepler's equation, employing iterations in one of the Keplerian elements (typically semi-major axis, eccentricity, focal parameter, or anomaly) [**?**, **?**, **?**, **?**];
- Methods based on the universal form of Kepler's equation (with a universal variable) [**?**, **?**];
- Methods using regularized variables in the Kustaanheimo–Stiefel (K–S) formulation [**?**];
- Methods based on the geometry of the eccentricity vector [**?**];
- Methods based on the inverse ballistic problem [**?**, **?**, **?**, **?**].

2. Okhotsimsky–Egorov Method

In particular, the one-parameter family of orbits connecting two given points is generated by varying the flight path angle (the angle between the velocity vector and the local horizontal) at the initial point. This ballistic formulation without constraints on transfer time was thoroughly examined by D.E. Okhotsimsky [**?**].

For each specified value of the initial flight path angle, the velocity—and consequently all orbital parameters—is determined from Okhotsimsky's equation [**?**]. Thus, the problem of determining the transfer orbit for a given time reduces to determining the initial flight path angle. This scheme by Yegorov for finding a solution based on Okhotsimsky's equation defines the general idea of the Okhotsimsky–Egorov method.

The authors have considered the solution of the Euler–Lambert problem based on the ballistic approach of Okhotsimsky–Egorov as an inverse ballistic problem [**?**]. An efficient algorithm for solving the problem based on Lagrange's time equation [**?**] has been proposed. Convergence rate estimates demonstrate that when using Halley's method for determining the solution, the typical number of iterations is approximately 5–7 for all solution types, which represents a very favorable performance for Euler–Lambert problem algorithms.

The advantages of this approach include the bounded and well-structured solution domain, the overall transparency of the method due to the use of physically intuitive trajectory parameters, and the explicit dependence of the solution on the initial velocity vector. These features enable effective integration of this solution scheme into the optimization of impulsive trajectories and interplanetary mission design.

Conclusion

This work continues the development of the Okhotsimsky–Egorov method. An attempt is made to determine the place of the solution technique based on the inverse ballistic problem within the general structure of methods for solving the Euler–Lambert problem.

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References

[1] A.N. Krylov, *The Fate of a Famous Theorem*, Archive of the Institute for the History of Science and Technology, Series 1, Issue 8, Publishing House of the Academy of Sciences of the USSR, Moscow-Leningrad, 1936, pp. 281-299.

[2] M.F. Subbotin, *A Course in Celestial Mechanics*, Vol. 1, Gostekhizdat, 1933, 321 p.

[3] A.D. Dubyago, *The Determination of Orbits*, translated from the Russian by R.D. Burke et al., The Rand Corporation, Macmillan, New York, 1961, 434 p.

[4] P.R. Escobal, *Methods of Orbit Determination*, John Wiley and Sons, New York-London-Sydney, 1965, 463 p.

[5] D. Torre Sangra, E. Fantino, *Review of Lambert's problem*, 25th International Symposium on Space Flight Dynamics (ISSFD 2015), 2015, pp. 1-15.

[6] A.A. Sukhanov, *Universal solution of Lambert's problem*, Cosmic Research, 1989, Vol. 26, No. 4, pp. 415-423.

[7] D.E. Okhotsimsky, *Dynamics of Space Flights*, Lecture Notes, Publishing House of Moscow University, Moscow, 1968, 158 p.

[8] V.V. Ivashkin, *On the application of the Okhotsimsky–Egorov method for solving the Euler–Lambert problem*, Doklady Rossiiskoi Akademii Nauk. Fizika. Tekhnicheskie Nauki, 2024, Vol. 514, No. 1, pp. 58-62.

[9] A.V. Ivanyukhin, V.V. Ivashkin, *Solution of the Euler–Lambert Problem Based on the Okhotsimsky–Egorov Ballistic Approach*, Solar System Research, 2024, Vol. 58, No. 6, pp. 769-779.

[10] P.V. Arlulkar, S.D. Naik, *Solution Based on Dynamical Approach for Multiple-Revolution Lambert Problem*, Journal of Guidance, Control, and Dynamics, 2011, Vol. 34, No. 3, pp. 920-923.

Alexey Ivanyukhin
Research Institute of Applied Mechanics and Electrodynamics of Moscow Aviation institute
Peoples Friendship University of Russia (RUDN University)
Moscow, Russia
e-mail: ivanyukhin.a@yandex.ru

Vyacheslav Ivashkin
Keldysh Institute of Applied Mathematics, Russian Academy of Sciences
Research Institute of Applied Mechanics and Electrodynamics of Moscow Aviation institute
Moscow, Russia