

Figure-eight orbit with unequal masses in the planar three-body problem

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Space of Forms

The transition to the space of forms is performed using two transformations:

1. The Jacobi transformation from the barycentric frame ($m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 = 0$):

$$\begin{cases} \mathbf{Q}_1 = \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{Q}_2 = \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \end{cases}$$

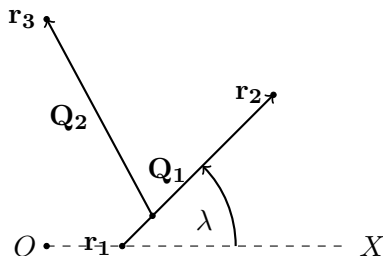
2. Reduction by rotations:

$$\begin{cases} \xi_1 = \mu_1|\mathbf{Q}_1|^2 - \mu_2|\mathbf{Q}_2|^2 \\ \xi_2 = 2\sqrt{\mu_1\mu_2}(\mathbf{Q}_1, \mathbf{Q}_2) \\ \xi_3 = 2\sqrt{\mu_1\mu_2}[\mathbf{Q}_1 \times \mathbf{Q}_2]_z \\ \lambda = \angle(OX, \mathbf{Q}_1) \end{cases}$$

Space of Forms

where

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2}$$
$$\mu_2 = \frac{m_3 (m_1 + m_2)}{m_1 + m_2 + m_3}$$



Finally, we introduce generalized spherical coordinates:

$$\begin{cases} \xi_1 = \rho^2 \cos \varphi \cos \theta \\ \xi_2 = \rho^2 \sin \varphi \cos \theta \\ \xi_3 = \rho^2 \sin \theta \end{cases}$$

Kinetic energy:

$$T = \frac{\dot{\rho}^2}{2} + \frac{\rho^2}{8} \left(\dot{\varphi}^2 \cos^2 \theta + \dot{\theta}^2 \right) - \frac{J^2}{2\rho^2}$$

Potential function:

$$V = \frac{D(\varphi, \theta)}{\rho},$$

where

$$D(\varphi, \theta) = \sqrt{2} \sum_{i < j} \frac{(m_i m_j)^{3/2}}{\sqrt{m_i + m_j} \sqrt{1 + \cos(\varphi - \varphi_{ij}) \cos \theta}}.$$

The critical points of $D(\varphi, 0)$ correspond to Euler configurations.

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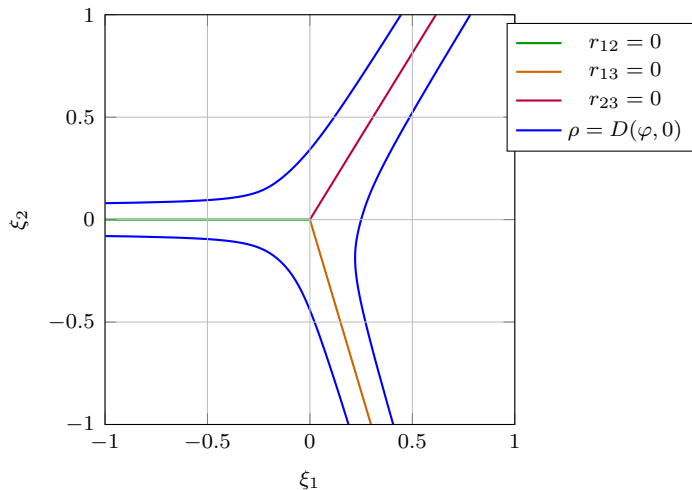


Figure: The triangular potential $D(\varphi, 0)$ for $m_1 = 0.3$, $m_2 = 0.4$, $m_3 = 0.5$ in the plane $\xi_3 = 0$; the rays indicate the directions of binary collisions.

Searching for the Figure-Eight with Unequal Masses

Search for the Figure-Eight

The figure-eight is a solution of the three-body problem for equal masses satisfying the following properties:

- I. It is a periodic solution.
- II. It is a choreography of the three-body problem, i.e., the bodies move along the same trajectory with a time shift equal to one third of the period.
- III. During one period, the bodies pass through Euler configurations six times.

It is of interest to search for solutions with unequal masses satisfying at least some of these properties, possibly in a weakened form.

Searching for the Figure-Eight with Unequal Masses

Search for the Figure-Eight

Two interesting approaches to defining a (pseudo) figure-eight can be distinguished:

1. One can try to abandon properties I. and II. and maximize the number of passages through Euler configurations. In this case, at least four passages are achievable for case $m_1 = m_2$.
2. One can preserve the periodicity of the solution and, among such solutions, search for one minimizing the deviation from Euler configurations over one period.

The first approach was studied numerically.

Searching for the Figure-Eight with Unequal Masses

Search for the Figure-Eight

In the case $J = 0$, the Hamiltonian of the system takes the form:

$$\mathcal{H} = \frac{p_\rho^2}{2} + \frac{2p_\varphi^2}{\cos^2 \theta \rho^2} + \frac{2p_\theta^2}{\rho^2} - \frac{D}{\rho}$$

The canonical equations are:

$$\begin{cases} \dot{\rho} = p_\rho, \\ \dot{\varphi} = \frac{4p_\varphi}{\rho^2 \cos^2 \theta}, \\ \dot{\theta} = \frac{4p_\theta}{\rho^2}, \\ \dot{p}_\rho = \frac{4}{\rho^3} \left(\frac{p_\varphi^2}{\cos^2 \theta} + p_\theta^2 \right) - \frac{D}{\rho^2}, \\ \dot{p}_\varphi = \frac{D_\varphi}{\rho}, \\ \dot{p}_\theta = \frac{D_\theta}{\rho} - \frac{4p_\varphi^2 \sin \theta}{\rho^2 \cos^3 \theta}. \end{cases} \quad (1)$$

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Dividing the canonical equations by $\frac{d\varphi}{dt}$, we obtain:

$$\left\{ \begin{array}{l} \frac{d\rho}{d\varphi} = \frac{p_\rho \rho^2 \cos^2 \theta}{4p_\varphi} \\ \frac{d\theta}{d\varphi} = \frac{p_\theta \cos^2 \theta}{p_\varphi} \\ \frac{dp_\rho}{d\varphi} = \frac{\cos^2 \theta}{4p_\varphi} \left[\frac{4}{\rho} \left(\frac{p_\varphi^2}{\cos^2 \theta} + p_\theta^2 \right) - D \right] \\ \frac{dp_\varphi}{d\varphi} = \frac{D_\varphi \rho \cos^2 \theta}{4p_\varphi} \\ \frac{dp_\theta}{d\varphi} = \frac{D_\theta \rho \cos^2 \theta}{4p_\varphi} - \tan \theta \end{array} \right.$$

Searching for the Figure-Eight with Unequal Masses

Search for the Figure-Eight

By dividing the integration domain into three intervals $[\varphi_1, \varphi_2]$, $[\varphi_2, \varphi_3]$, $[\varphi_3, 2\pi + \varphi_1]$, the search for a (pseudo) figure-eight can be reduced to the following boundary value problem:

$$\theta(\varphi_i) = 0,$$

with respect to $p_\rho(\varphi_1)$, $p_\varphi(\varphi_1)$, and $p_\theta(\varphi_1)$, while fixing $\theta(\varphi_1) = 0$ and $\rho(\varphi_1) = \sqrt{2}$, due to scale symmetry.

For integration over the intervals, a collocation-based three-stage Runge–Kutta method on a Gauss grid was used. The boundary value problem was solved using Newton's method. All computations were performed using 100-digit decimal arithmetic. To control summation errors, the solution obtained on a grid with N steps was compared with the solution obtained on a grid with $2N$ steps.

Searching for the Figure-Eight with Unequal Masses

Search for the Figure-Eight

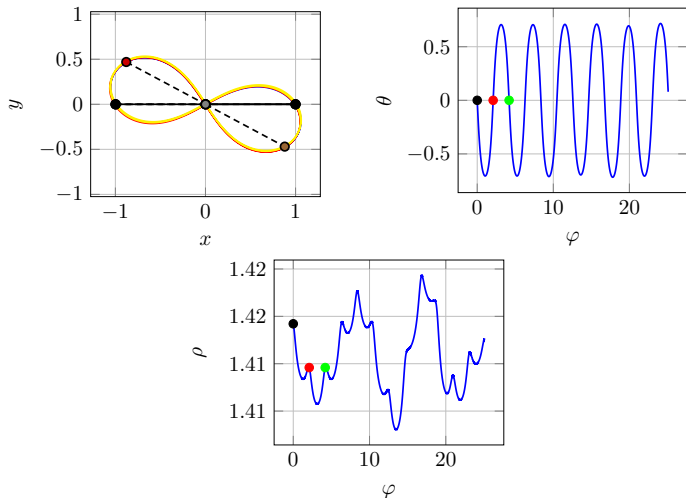
Initial data for the figure-eight:

$$\left\{ \begin{array}{l} \rho = \sqrt{2}, \\ \varphi = 0, \\ \theta = 0, \\ p_\rho = 0, \\ p_\varphi = 0.6012240891875504, \\ p_\theta = -0.922706672027483. \end{array} \right.$$

The momentum values from these initial data were used as the initial approximation in Newton's method.

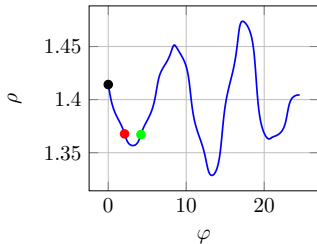
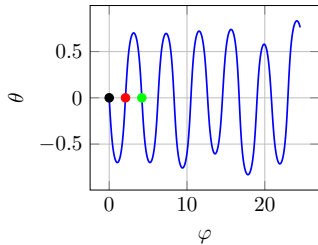
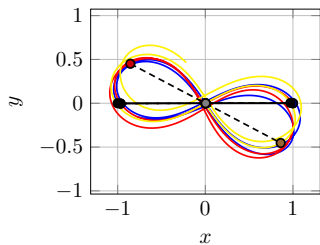
Searching for the Figure-Eight with Unequal Masses

Results. $m_3 = 0.999$



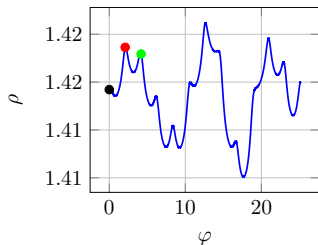
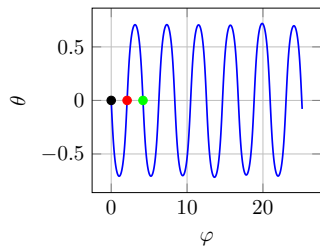
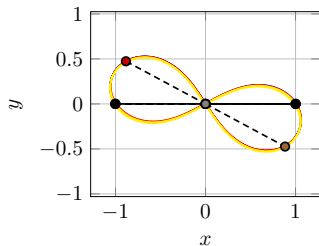
Searching for the Figure-Eight with Unequal Masses

Results. $m_3 = 0.99$



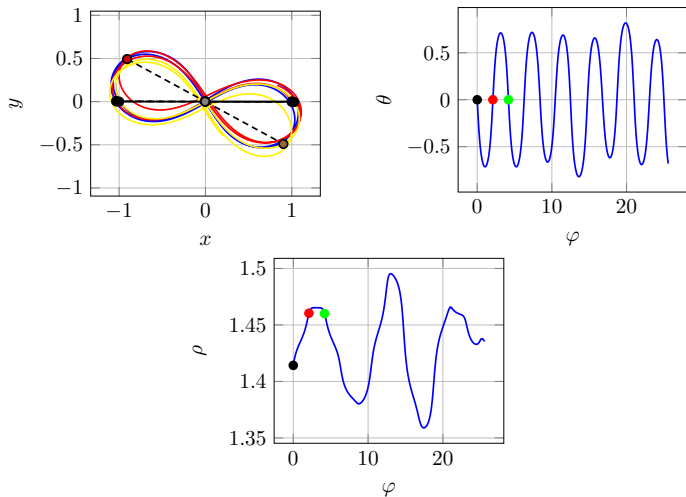
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Results. $m_3 = 1.001$



Searching for the Figure-Eight with Unequal Masses

Results. $m_3 = 1.01$



Searching for the Figure-Eight with Unequal Masses

Results

Initial conditions for (pseudo) figure-eight solutions:

m_3	p_ρ	p_φ	p_θ	ϵ
1.001	5.01722e-3	6.00834e-1	-9.23871e-1	3e-6
1.01	4.68088e-2	6.00971e-1	-9.34137e-1	3e-6
0.99	-4.94253e-2	6.00853e-1	-9.11090e-1	3.5e-6
0.999	-4.81961e-3	6.01225e-1	-9.21555e-1	1e-20

Searching for the Figure-Eight with Unequal Masses

Results. Stability

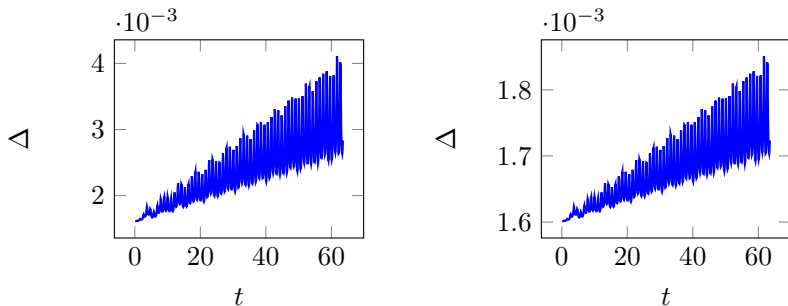


Figure: Mean norm of the difference for $m_3 = 1.001$, for a ball of radius 10^{-5} on the left and a ball of radius 10^{-6} on the right

Searching for the Figure-Eight with Unequal Masses

Results. Stability

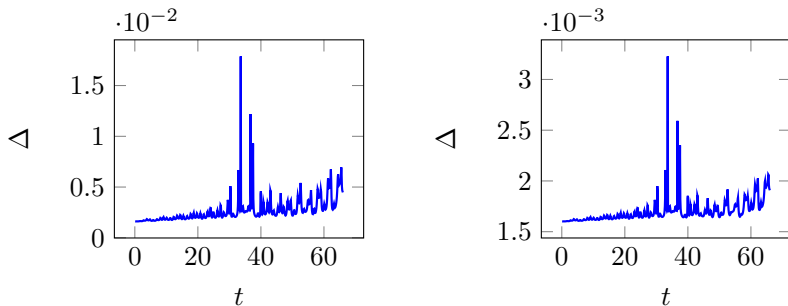


Figure: Mean norm of the difference for $m_3 = 1.01$, for a ball of radius 10^{-5} on the left and a ball of radius 10^{-6} on the right

Searching for the Figure-Eight with Unequal Masses

Results. Stability

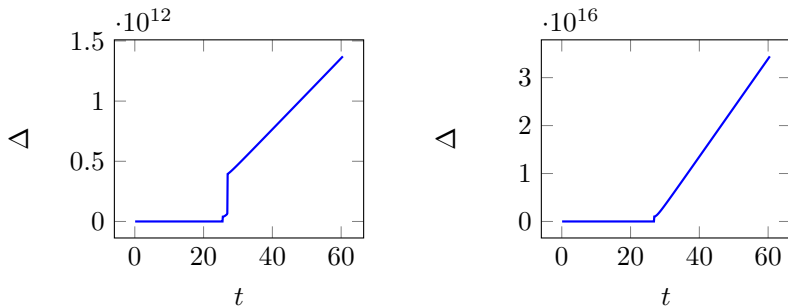


Figure: Mean norm of the difference for $m_3 = 0.99$, for a ball of radius 10^{-5} on the left and a ball of radius 10^{-6} on the right

Searching for the Figure-Eight with Unequal Masses

Results. Stability

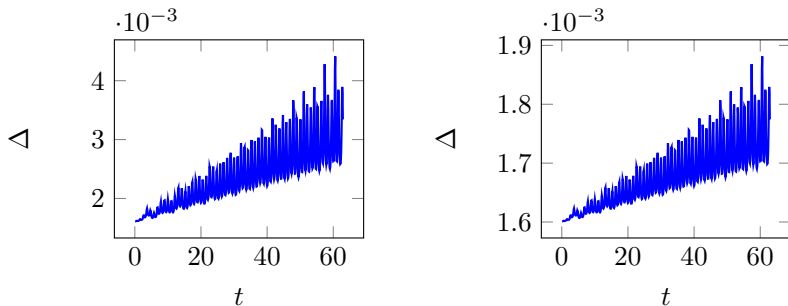


Figure: Mean norm of the difference for $m_3 = 0.999$, for a ball of radius 10^{-5} on the left and a ball of radius 10^{-6} on the right

Topological constraints on periodic orbits

For the angular momentum integral, we have:

$$J/\rho^2 = \dot{\lambda} + \frac{\dot{\theta} \sin \varphi - \dot{\varphi} \cos \varphi \cos \theta \sin \theta}{2 (\cos \varphi \cos \theta + 1)}$$

Equivalently,

$$J/\rho^2 = \dot{\lambda} + \frac{\dot{\theta} \sin \varphi + \dot{\varphi} \sin \theta}{2 (\cos \varphi \cos \theta + 1)} - \frac{\dot{\varphi} \sin \theta}{2}$$

Topological constraints on periodic orbits

For any periodic solution with period T ,

$$\lambda(T) - \lambda(0) = 2\pi k, \quad k \in \mathbb{Z}.$$

Integrating over one period gives

$$0 = 2\pi k + \oint_C d\omega - \int_0^T \left[\frac{\dot{\varphi} \sin \theta}{2} + \frac{J}{\rho^2} \right] dt$$

where

$$\omega = \frac{d\theta \sin \varphi + d\varphi \sin \theta}{2(\cos \varphi \cos \theta + 1)}$$

and C is a closed curve on the cylinder

$$\Pi = \mathbb{R}/2\pi\mathbb{Z} \times (-\pi/2, \pi/2) \setminus \{(\pi, 0)\}.$$

Topological constraints on periodic orbits

The following lemma holds.

Lemma

The following statements hold:

- 1. The differential form ω is closed on Π ;*
- 2. If the closed curve C is transverse to φ (i.e., it admits a parametrization with $\dot{\varphi} \neq 0$), then*

$$\oint_C d\omega = \pi l, \quad l \in \mathbb{Z};$$

- 3. If the closed curve C is not transverse to φ , then*

$$\oint_C d\omega = 2\pi l, \quad l \in \mathbb{Z}.$$

Topological constraints on periodic orbits

The lemma has two immediate corollaries.

Corollary

If $J = 0$, then there are no periodic solutions that are 2π -periodic in φ .

Corollary

If $J = 0$, then every periodic solution that is 4π -periodic in φ satisfies

$$\int_0^T \dot{\varphi} \sin \theta \, dt = 0.$$

Thus, any periodic generalization of the figure-eight must be at least 4π -periodic-in- φ .

Conclusions

1. Numerically, it was shown that for the mass of the third body in the interval 1 ± 0.01 , a solution passing through four consecutive Euler configurations exists.
2. The stability of the obtained solutions was studied numerically.
3. It was shown that in a neighborhood of the original figure-eight orbit, there are no orbits passing through Euler configurations more than four times exactly.
4. It was proved that for $J = 0$ there are no periodic solutions that are 2π -periodic in φ .
5. It was proved that for $J = 0$, every periodic solution that is 4π -periodic in φ satisfies

$$\int_0^T \dot{\varphi} \sin \theta \, dt = 0.$$

1. Alain Chenciner and Richard Montgomery. *A Remarkable Periodic Solution of the Three-Body Problem in the Case of Equal Masses*. 2000. URL: <http://www.jstor.org/stable/2661357>.
2. G. Lemaître. Regularization of the three-body problem. *Vistas in Astronomy* **1**, 207–215 (1955). ISSN: 0083-6656. DOI: [https://doi.org/10.1016/0083-6656\(55\)90028-3](https://doi.org/10.1016/0083-6656(55)90028-3).
3. Carles Simó. Dynamical properties of the figure eight solution of the three-body problem. *Contemp. Math.* **-1**, 209 (2002). DOI: [10.1090/conm/292/04926](https://doi.org/10.1090/conm/292/04926).

4. V. B. Titov. The Three-Body Problem in the Space of Forms. *Astronomicheskiy vestnik* **59** (3), 281–288 (2025). DOI: 10.31857/S0320930X25030096.
5. I. V. Dmitrašinović. Three Classes of Newtonian Three-Body Planar Periodic Orbits. *Phys. Rev. Lett.* **110**, 114301 (2013). DOI: 10.1103/PhysRevLett.110.114301.
6. Milovan Šuvakov. Numerical search for periodic solutions in the vicinity of the figure-eight orbit: slaloming around singularities on the shape sphere. *Celestial Mechanics and Dynamical Astronomy* **119** (3), 369–377 (2014). ISSN: 1572-9478. DOI: 10.1007/s10569-014-9572-5.

Thanks for your attention!