

Figure-eight orbit with unequal masses in the planar three-body problem

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Abstract. The generalization of the figure-eight orbit to the unequal-mass case in the planar three-body problem is studied within the space-of-forms framework. A high-precision shooting method yields solutions that pass through Eulerian configurations exactly four times for small deviations of the third mass from unity. These trajectories are shown to lose periodicity and choreographic property; for 10% mass deviation, no such solutions are found. Topological constraints on periodic solutions are also derived: the absence of 2π -periodic-in- φ orbits at zero angular momentum is proved, and an integral condition necessary for 4π -periodic solutions is obtained. The results confirm the space of forms as a useful tool for analysing the existence and properties of periodic regimes in the three-body problem.

Introduction

The figure-eight orbit [1] of the planar three-body problem exists for equal masses and zero angular momentum. Whether similar orbits exist for unequal masses is an open question. This question is investigated using the space-of-forms formalism [2, 5], which factors out rotations and provides a geometric description of the dynamics.

1. Space of forms

Following [5], the space of forms is obtained by factoring the Jacobi coordinates over rotations. The shape coordinates are

$$\xi_1 = \frac{1}{2}(Q_1^2 - Q_2^2), \quad \xi_2 = Q_1 \cdot Q_2, \quad \xi_3 = Q_1 \times Q_2,$$

with $\xi_1^2 + \xi_2^2 + \xi_3^2 = I^2$. Generalised spherical coordinates are

$$\xi_1 = \rho^2 \cos \varphi \cos \theta, \quad \xi_2 = \rho^2 \sin \varphi \cos \theta, \quad \xi_3 = \rho^2 \sin \theta,$$

where ρ is the scale factor, φ is the equatorial angle, and θ is the latitude. Collinear configurations correspond to $\theta = 0$. The Eulerian collinear points are the minima of the triangular potential

$$D(\varphi, \theta) = \sum_{i < j} \frac{(m_i m_j)^{3/2}}{\sqrt{m_i + m_j}} \frac{1}{\sqrt{1 + \cos(\varphi - \varphi_{ij}) \cos \theta}}. \quad (1)$$

The Hamiltonian system (for zero angular momentum) is

$$\begin{cases} \dot{\rho} = p_\rho, \\ \dot{\varphi} = \frac{4p_\varphi}{\rho^2 \cos^2 \theta}, \\ \dot{\theta} = \frac{4p_\theta}{\rho^2}, \\ \dot{p}_\rho = \frac{4}{\rho^3} \left(\frac{p_\varphi^2}{\cos^2 \theta} + p_\theta^2 \right) - \frac{D}{\rho^2}, \\ \dot{p}_\varphi = \frac{D_\varphi}{\rho}, \\ \dot{p}_\theta = \frac{D_\theta}{\rho} - \frac{4p_\varphi^2 \sin \theta}{\rho^2 \cos^3 \theta}. \end{cases} \quad (2)$$

The equal-mass figure-eight has initial conditions

$$\begin{cases} \rho(0) = \sqrt{2}, \quad \varphi(0) = 0, \quad \theta(0) = 0, \\ p_\rho(0) = 0, \quad p_\varphi(0) = 0.6012240891875504, \quad p_\theta(0) = 0.922706672027483. \end{cases} \quad (3)$$

2. Numerical search

Two approaches to defining a (pseudo) figure-eight for unequal masses are considered. The first preserves the maximal number of passages through Eulerian configurations. The second minimises a cost function measuring deviation from such passages. Since the shooting equations (first approach) are locally non-degenerate, any trajectory with many exact crossings coincides with the first approach's solution. Thus the first approach is used.

For given m_3 ($m_1 = m_2 = 1$), the initial point is fixed at an Eulerian configuration: $\varphi = 0, \theta = 0, \rho = \sqrt{2}$. The remaining unknowns are found by requiring $\theta = 0$ at the next three Eulerian angles $\varphi_1, \varphi_2, \varphi_3$ (minima of $D(\varphi, 0)$):

$$F(p_\rho, p_\varphi, p_\theta; \Delta m_3) = \begin{pmatrix} \theta(\varphi_1) \\ \theta(\varphi_2) \\ \theta(\varphi_3) \end{pmatrix} = 0. \quad (4)$$

The system is solved by Newton's method; equations 2equation.2 is integrated with a 3-stage collocation Runge–Kutta method using 100-digit arithmetic.

Table 1table.1 lists the initial data for several m_3 . ϵ is the precision.

The corresponding trajectories are shown in figure 1figure.1. Each solution passes through $\theta = 0$ four times with residual $\sim 10^{-23}$. The trajectories lose

TABLE 1. Initial conditions for (pseudo) figure-eight solutions.

m_3	p_ρ	p_φ	p_θ	ϵ
1.001	5.01722e-3	6.00834e-1	-9.23871e-1	3e-6
1.01	4.68088e-2	6.00971e-1	-9.34137e-1	3e-6
0.99	-4.94253e-2	6.00853e-1	-9.11090e-1	3.5e-6
0.999	-4.81961e-3	6.01225e-1	-9.21555e-1	1e-20

periodicity and choreographic property. For $m_3 = 1.1$ and 0.9 no solution was found.

Since the obtained solutions are not periodic, the question of whether periodic (pseudo) figure-eights exist remains. This motivates the topological analysis below, which provides necessary conditions for periodicity.

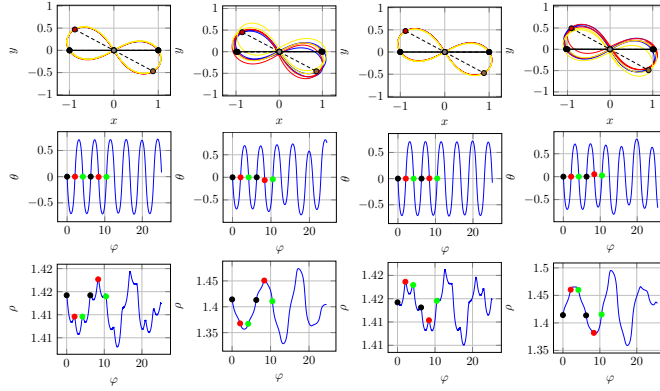


FIGURE 1. Trajectories for $m_3 = 0.999, 0.99, 1.001, 1.01$ (left to right). Eulerian configurations marked with dots.

3. Topological constraints on periodic orbits

For a periodic solution, let J be the angular momentum. The identity

$$\frac{J}{\rho^2} = \dot{\lambda} + \frac{\dot{\theta} \sin \varphi + \dot{\varphi} \sin \theta}{2(\cos \varphi \cos \theta + 1)} - \frac{1}{2} \dot{\varphi} \sin \theta \quad (5)$$

holds, where λ is the rotation angle. For a periodic orbit of period T , $\lambda(T) - \lambda(0) = 2\pi k$. Integrating and using the closedness of

$$\omega = \frac{d\theta \sin \varphi + d\varphi \sin \theta}{2(\cos \varphi \cos \theta + 1)} \quad (6)$$

on $\Pi = \mathbb{R}/2\pi\mathbb{Z} \times (-\pi/2, \pi/2) \setminus \{(\pi, 0)\}$ gives:

Lemma 1. *For any closed curve $C \subset \Pi$:*

1. $\oint_C \omega = \pi l$, $l \in \mathbb{Z}$, if C is transversal to φ ;
2. $\oint_C \omega = 2\pi l$, $l \in \mathbb{Z}$, otherwise.

Hence any periodic solution satisfies

$$\int_0^T \left(\frac{1}{2} \dot{\varphi} \sin \theta + \frac{J}{\rho^2} \right) dt = \begin{cases} \pi l, & \text{if transversal in } \varphi, \\ 2\pi l, & \text{otherwise.} \end{cases} \quad (7)$$

Two corollaries follow for $J = 0$:

1. No 2π -periodic-in- φ solution exists, since $\left| \int_0^{2\pi} \frac{1}{2} \dot{\varphi} \sin \theta dt \right| < \pi$.
2. Any 4π -periodic-in- φ solution must satisfy

$$\int_0^T \dot{\varphi} \sin \theta dt = 0, \quad (8)$$

which implies crossing $\theta = 0$ at least once.

Thus, any periodic generalization of the figure-eight must be at least 4π -periodic-in- φ and must cross the collinear plane.

Conclusion

The numerical search found (pseudo) figure-eight solutions for masses close to equal that pass through Eulerian configurations four times but are not periodic. Topological analysis shows that any periodic generalization must be 4π -periodic in φ and cross $\theta = 0$. The space of forms proves useful for both numerical and analytical studies of the three-body problem.

References

- [1] A. Chenciner and R. Montgomery, *A Remarkable Periodic Solution of the Three-Body Problem in the Case of Equal Masses*, Ann. Math. **152**, 881 (2000).
- [2] G. Lemaître, *Regularization of the three-body problem*, Vistas in Astronomy **1**, 207 (1955).
- [3] C. Simó, *Dynamical properties of the figure eight solution of the three-body problem*, Contemp. Math. **292**, 209 (2002).
- [4] M. Šuvakov, *Numerical search for periodic solutions in the vicinity of the figure-eight orbit: slaloming around singularities on the shape sphere*, Celest. Mech. Dyn. Astr. **119**, 369 (2014).
- [5] V. B. Titov, *The Three-Body Problem in the Space of Forms*, Astron. Vestn. **59**, 281 (2025).

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