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НАЦИОНАЛЬНЫЕ
ПРОЕКТЫ
РОССИИ

НАУКА
И УНИВЕРСИТЕТЫ

Nonlinear Formation Flying Control Of Small Spacecraft In The Vicinity Of Asteroid 4179 Toutatis

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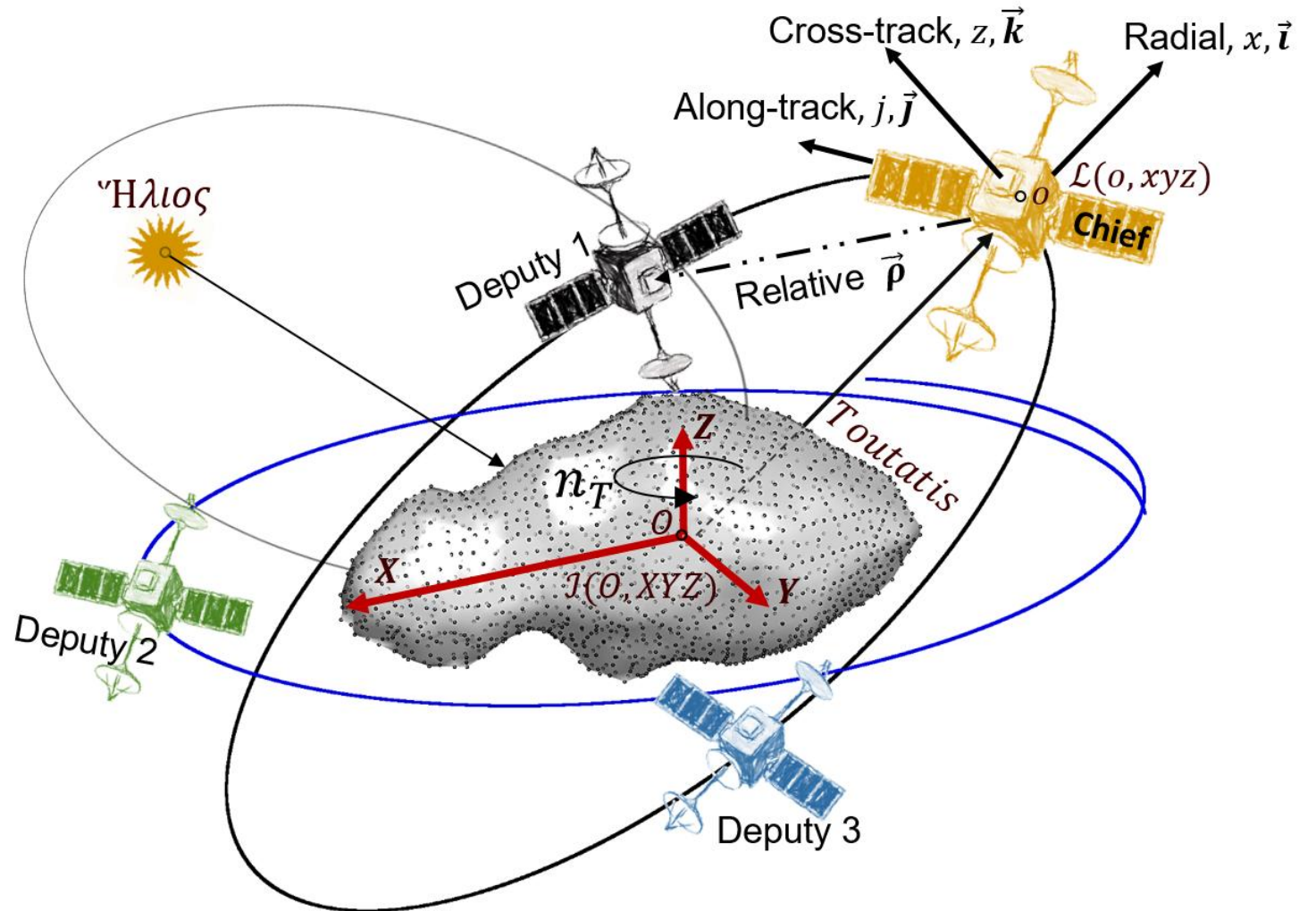
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- Vladislav Viktorovich Sidorenko
- Sergei Victorovich Sokolov
- [Scheeres, D.J., 2016. Orbital motion in strongly perturbed environments: applications to asteroid, comet and planetary satellite orbiters. Springer.]
- [Wang, W., Mengali, G., Quarta, A.A. and Baoyin, H., 2024. Spacecraft relative motion control near an asteroid with uncertainties: a Lyapunov redesign approach. IEEE Transactions on Aerospace and Electronic Systems, 60(4), pp.4507-4517.]

Problem Formulation

- Synthesis of a control algorithm that ensures the motion of satellites within a prescribed bounded region relative to a reference point.
- Development of an algorithm for stabilizing trajectories in the vicinity of an asteroid while accounting for uncertainties in the gravitational field and other perturbation.
- Ensuring the stability of the generated trajectories.



Relative Motion

$${}^I\ddot{\mathbf{p}} = \frac{{}^L d^2 \mathbf{p}}{dt^2} + 2 {}^I \boldsymbol{\omega}^L \times \frac{{}^L d \mathbf{p}}{dt} + \frac{d {}^I \boldsymbol{\omega}^L}{dt} \times \mathbf{p} + {}^I \boldsymbol{\omega}^L \times ({}^I \boldsymbol{\omega}^L \times \mathbf{p}) = \mathbf{u} \quad \Rightarrow \quad \begin{aligned} \ddot{x} - 3\omega^2 x - 2\omega \dot{y} &= u_x \\ \ddot{y} + 2\omega \dot{x} &= u_y \\ \ddot{z} + \omega^2 z &= u_z \end{aligned}$$

$$\begin{aligned} C_1 &= 2x + \frac{\dot{y}}{\omega} \\ C_4 &= y - \frac{2\dot{x}}{\omega} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{C}_1 &= \frac{u_y}{\omega} \\ \dot{C}_4 &= -3\omega C_1 - \frac{2u_x}{\omega} \end{aligned}$$

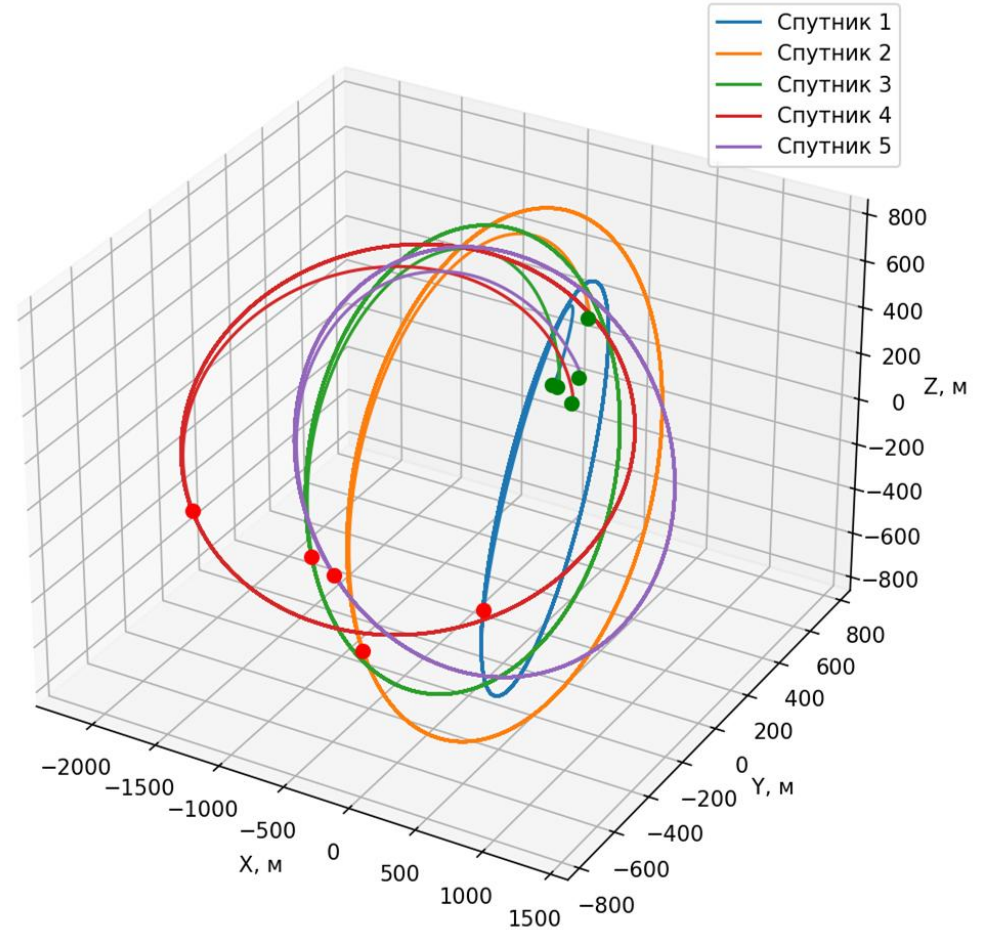
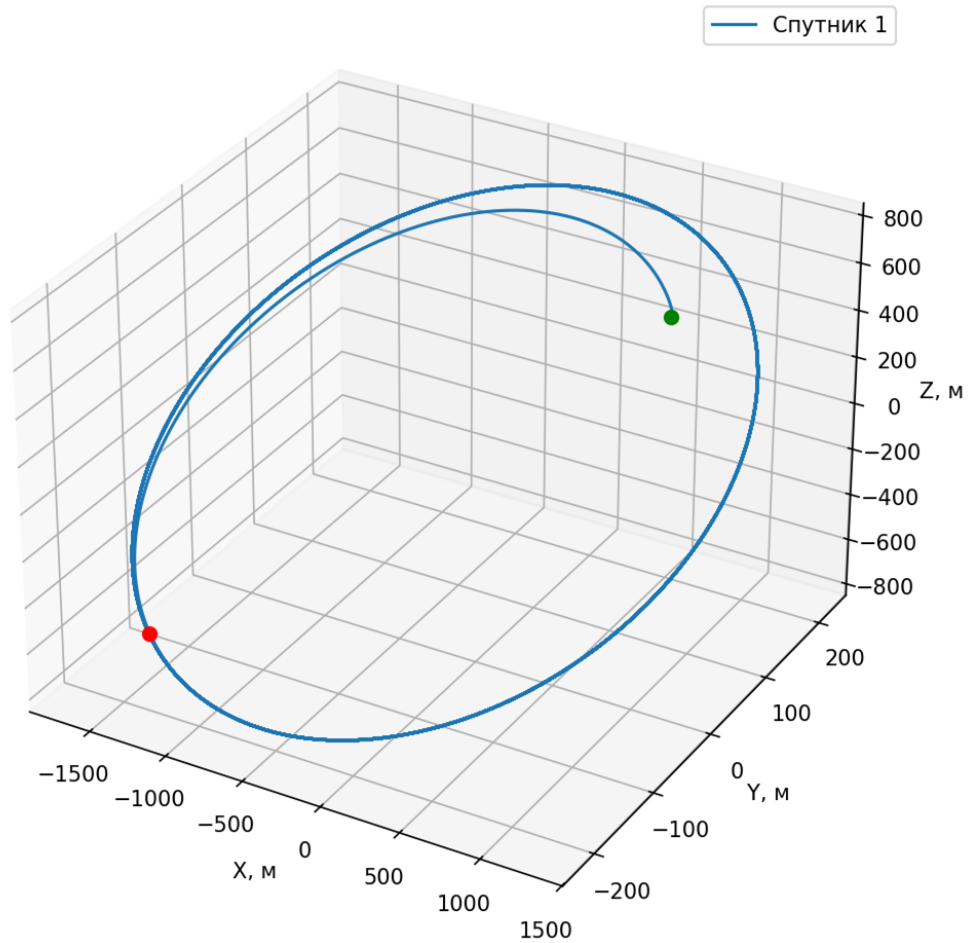
- C1 represents the drift rate
- C4 represents the along-track offset

- Lyapunov based control

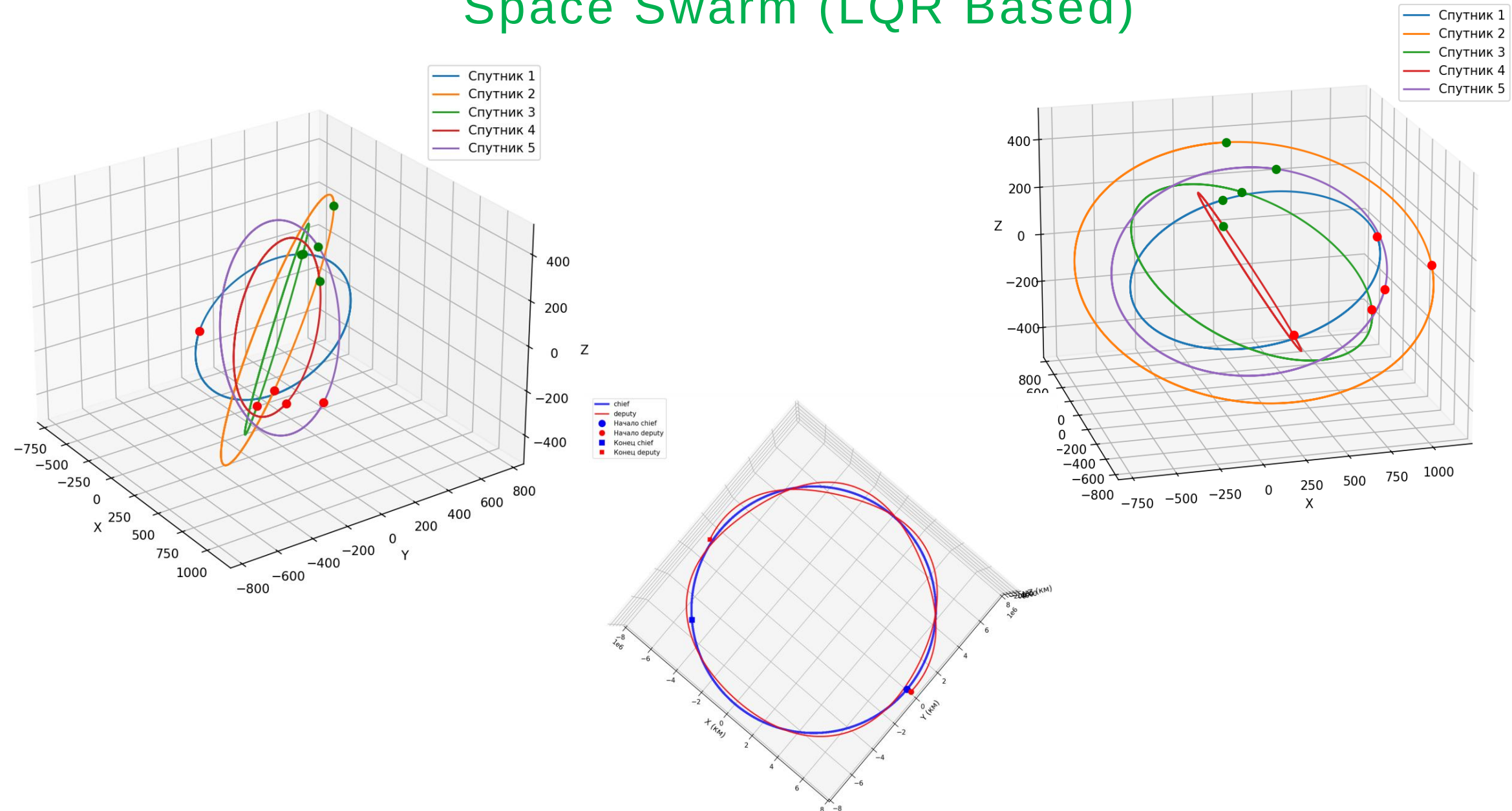
$$V = \frac{1}{2} (\Delta C_1^{ij})^2 + \frac{1}{2} (\Delta C_4^{ij})^2, \quad \mathbf{u}^{ij} = -k\omega \begin{bmatrix} \Delta C_1^{ij} \\ 0 \end{bmatrix} + 3\omega^2 \begin{bmatrix} 0 \\ \Delta C_4^{ij} \end{bmatrix}$$

- LQR control $\mathbf{u}^{ij} = -\mathbf{R}^{-1} \mathbf{B} \mathbf{P} \begin{bmatrix} \Delta C_1^{ij} \\ \Delta C_4^{ij} \end{bmatrix}$

Space Swarm (Lyapunov Based Control)

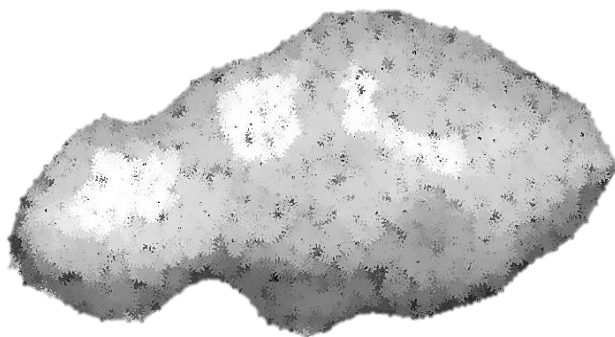


Space Swarm (LQR Based)

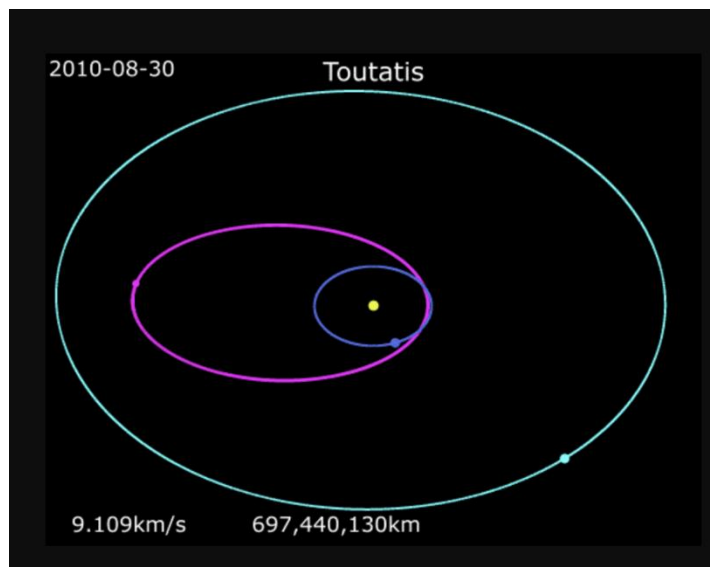


Asteroid Case

now the task is set to develop and comparatively analyze nonlinear adaptive strategies for maintaining the relative formation of small satellites near the asteroid (4179) Toutatis under conditions of high gravitational uncertainty.



First was observed 10 Feb 1934



parameter	value
period	1474 days
radius	1225 m
mass	5e13 kg
density	2.1 g/cm ³
category	Apollo asteroids

<https://ssd.jpl.nasa.gov/horizons/>

Gravitational Field Of Asteroid

$$U = -\frac{\mu}{r} \sum_{l=0}^{\infty} \sum_{m=0}^l \left(\frac{R_0}{r} \right)^l P_{lm}(\sin \varphi) [C_{lm} \cos(m\lambda) + S_{lm} \sin(m\lambda)]$$

$$U \approx \frac{\mu}{r} \left[1 + \left(\frac{R_0}{r} \right)^2 (C_{20} P_{20}(\sin \varphi) + C_{22} P_{22}(\sin \varphi) \cos 2\lambda) \right]$$

$$U \approx -\frac{\mu}{r} \left[1 + \left(\frac{R_0}{r} \right)^2 \left(C_{20} \left(1 - \frac{3}{2} \cos^2 \varphi \right) + 3C_{22} \cos^2 \varphi \cos 2\lambda \right) \right]$$

$$P_{20}(\sin \varphi) = \frac{1}{2} (3 \sin^2 \varphi - 1) = 1 - \frac{3}{2} \cos^2 \varphi$$

$$P_{22}(\sin \varphi) = 3(1 - \sin^2 \varphi) = 3 \cos^2 \varphi$$

φ and λ are latitude and longitude with respect to the first principal axis

$u = \omega + f$ Argument of latitude

$$\tan \lambda = \frac{\sin(\Omega_R) \cos u + \cos(\Omega_R) \sin u \cos i}{\cos(\Omega_R) \cos u - \sin(\Omega_R) \sin u \cos i}, \sin \varphi = \sin i \sin u,$$

$$\Omega_R = \Omega - n_T t, r(f) = \frac{a(1-e^2)}{1+e \cos f}, \frac{dM}{df} = \frac{(1-e^2)^{3/2}}{(1+e \cos f)^2}.$$

- And assume the asteroid to slowly rotate with a rate $n_T \ll n$ all of the orbit elements can be regarded as constant over one orbit period
- We can investigate the evolution of the spacecraft mean orbit elements with Lagrange Planetary Equations.

Gravitational Field Of Asteroid

$$\Pi = -\mu/r + U,$$

$$\bar{U} = \frac{1}{T} \int_0^T U(t) dt = \frac{1}{2\pi} \int_0^{2\pi} U(f) dM.$$

$$\bar{U} = \frac{\mu R_0^2}{2\bar{a}^3 (1-\bar{e}^2)^{3/2}} \left(C_{20} \left(\frac{3}{2} \sin^2 \bar{i} - 1 \right) + C_{22} 3 \sin^2 \bar{i} \sin^2 \bar{\Omega}_R \right)$$

for a very slow rotator such as 4179 Toutatis

$$T = \frac{2\pi}{n_T} = 5.43 \text{ days}$$

And the mean orbital elements can be written as

Lagrange Planetary Equations

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial U}{\partial M}$$

$$f \rightarrow f, U \rightarrow \bar{U}$$

$$\dot{\bar{a}} = 0.$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e} \frac{\partial U}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial U}{\partial \omega}$$

$$a, e, i, \Omega, \omega, M \rightarrow \bar{a}, \bar{e}, \bar{i}, \bar{\Omega}, \bar{\omega}, \bar{M}$$

$$\dot{\bar{e}} = 0.$$

$$\frac{di}{dt} = \frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial U}{\partial \omega} - \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial U}{\partial \Omega}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial U}{\partial i}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial U}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial U}{\partial e}$$

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e} \frac{\partial U}{\partial e} - \frac{2}{na} \frac{\partial U}{\partial a}$$

$$\dot{\bar{i}} = \frac{3\bar{n}C_{22} \sin \bar{i} \sin 2\bar{\Omega}}{(p/R_0)^2}$$

$$\dot{\bar{\Omega}}_R = \frac{3}{2} \frac{\bar{n} \cos \bar{i} (C_{20} + 2C_{22} \cos 2\bar{\Omega}_R)}{(p/R_0)^2} - n_T.$$

$$\dot{\bar{\omega}} = -\frac{3\bar{n}}{8(p/R_0)^2} \left[C_{20} (3 + 5 \cos 2\bar{i}) + 2C_{22} \cos 2\bar{\Omega} (-1 + 5 \cos 2\bar{i}) \right].$$

$$\dot{\bar{M}} = n - \frac{3\bar{n} \sqrt{1-\bar{e}^2}}{8(p/R_0)^2} \left[C_{20} (1 + 3 \cos 2\bar{i}) + 6C_{22} \cos 2\bar{\Omega} (\cos 2\bar{i} - 1) \right].$$

Equation Of Motion

Averaged Gauss equations with vector $[u_x, u_y, u_z]$ can be added to write hand side of averaged Lagrange planetary equation

Gauss Equations

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}} [e \sin f u_x + (1+e \cos f) u_y]$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na} \left[\sin f u_x + \frac{e + 2 \cos f + e \cos^2 f}{1+e \cos f} u_y \right]$$

$$\frac{di}{dt} = \frac{\sqrt{1-e^2} \cos(\omega + f)}{na(1+e \cos f)} u_z$$

$$\frac{d\Omega}{dt} = \frac{\sqrt{1-e^2} \sin(\omega + f)}{na \sin i (1+e \cos f)} u_z$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{nae} \left[-\cos f u_x + \frac{2+e \cos f}{1+e \cos f} \sin f u_y \right] - \cos i \frac{d\Omega}{dt}$$

$$\frac{dM}{dt} = n + \frac{1-e^2}{nae} \left[\left(\cos f - \frac{2er}{p} \right) u_x - \left(1 + \frac{r}{p} \right) \sin f u_y \right]$$



$f \rightarrow \bar{f}$

$a, e, i, \Omega, \omega, M \rightarrow \bar{a}, \bar{e}, \bar{i}, \bar{\Omega}, \bar{\omega}, \bar{M}$

$$\dot{\bar{a}} = \frac{2\bar{e} \sin \bar{f}}{\bar{n}\sqrt{1-\bar{e}^2}} u_x + \frac{2(1+\bar{e} \cos \bar{f})}{\bar{n}\sqrt{1-\bar{e}^2}} u_y$$

$$\dot{\bar{e}} = \frac{\sqrt{1-\bar{e}^2} \sin \bar{f}}{\bar{n}\bar{a}} u_x + \frac{\sqrt{1-\bar{e}^2} (\bar{e} + 2 \cos \bar{f} + \bar{e} \cos^2 \bar{f})}{\bar{n}\bar{a} (1+\bar{e} \cos \bar{f})} u_y$$

$$\dot{\bar{i}} = \frac{3\bar{n}C_{22} \sin \bar{i} \sin 2\bar{\Omega}}{(p/R_0)^2} + \frac{\sqrt{1-\bar{e}^2} \cos(\bar{f} + \bar{\omega})}{\bar{n}\bar{a} (1+\bar{e} \cos \bar{f})} u_z$$

$$\dot{\bar{\Omega}}_R = \frac{3}{2} \frac{\bar{n} \cos \bar{i} (C_{20} + 2C_{22} \cos 2\bar{\Omega}_R)}{(p/R_0)^2} - n_T + \frac{\sqrt{1-\bar{e}^2} \sin(\bar{f} + \bar{\omega})}{\bar{n}\bar{a} \sin \bar{i} (1+\bar{e} \cos \bar{f})} u_z$$

$$\dot{\bar{\omega}} = \dots$$

$$\dot{\bar{M}} = \dots$$

Equation Of Motion

In general the system of differential equations can be written as:

$$\dot{\bar{\xi}} = \mathbf{f}(\bar{\xi}) + \mathbf{B}(\bar{\xi})\mathbf{u} + \delta, \quad \bar{\xi} = [a, e, i, \Omega]^T$$

$$\mathbf{f}(\bar{\xi}) = \begin{bmatrix} 0 \\ 0 \\ \frac{3\bar{n}C_{22} \sin \bar{i} \sin 2\bar{\Omega}}{(p/R_0)^2} \\ \frac{3}{2} \frac{\bar{n} \cos \bar{i} (C_{20} + 2C_{22} \cos 2\bar{\Omega}_R)}{(p/R_0)^2} - n_T \end{bmatrix}, \mathbf{B}(\bar{\xi}) = \begin{bmatrix} \frac{2\bar{e} \sin f}{\bar{n}\sqrt{1-\bar{e}^2}} & \frac{2(1+\bar{e} \cos f)}{\bar{n}\sqrt{1-\bar{e}^2}} & 0 \\ \frac{\sqrt{1-\bar{e}^2} \sin \nu}{na} & \frac{\sqrt{1-\bar{e}^2} (\bar{e} + 2 \cos f + \bar{e} \cos^2 f)}{\bar{na} (1+\bar{e} \cos f)} & 0 \\ 0 & 0 & \frac{\sqrt{1-\bar{e}^2} \cos(f + \bar{\omega})}{\bar{na} (1+\bar{e} \cos f)} \\ 0 & 0 & \frac{\sqrt{1-\bar{e}^2} \sin(f + \bar{\omega})}{\bar{na} \sin \bar{i} (1+\bar{e} \cos f)} \end{bmatrix}$$

Quasi-frozen Orbit

The frozen orbit is one with constant orbital shape and orientation

$$\dot{a} = 0$$

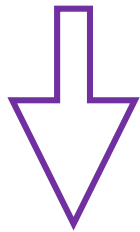
$$\dot{e} = 0$$

$$\dot{i} = \frac{3\bar{n}C_{22} \sin \bar{i} \sin 2\bar{\Omega}}{(p/R_0)^2} = 0$$

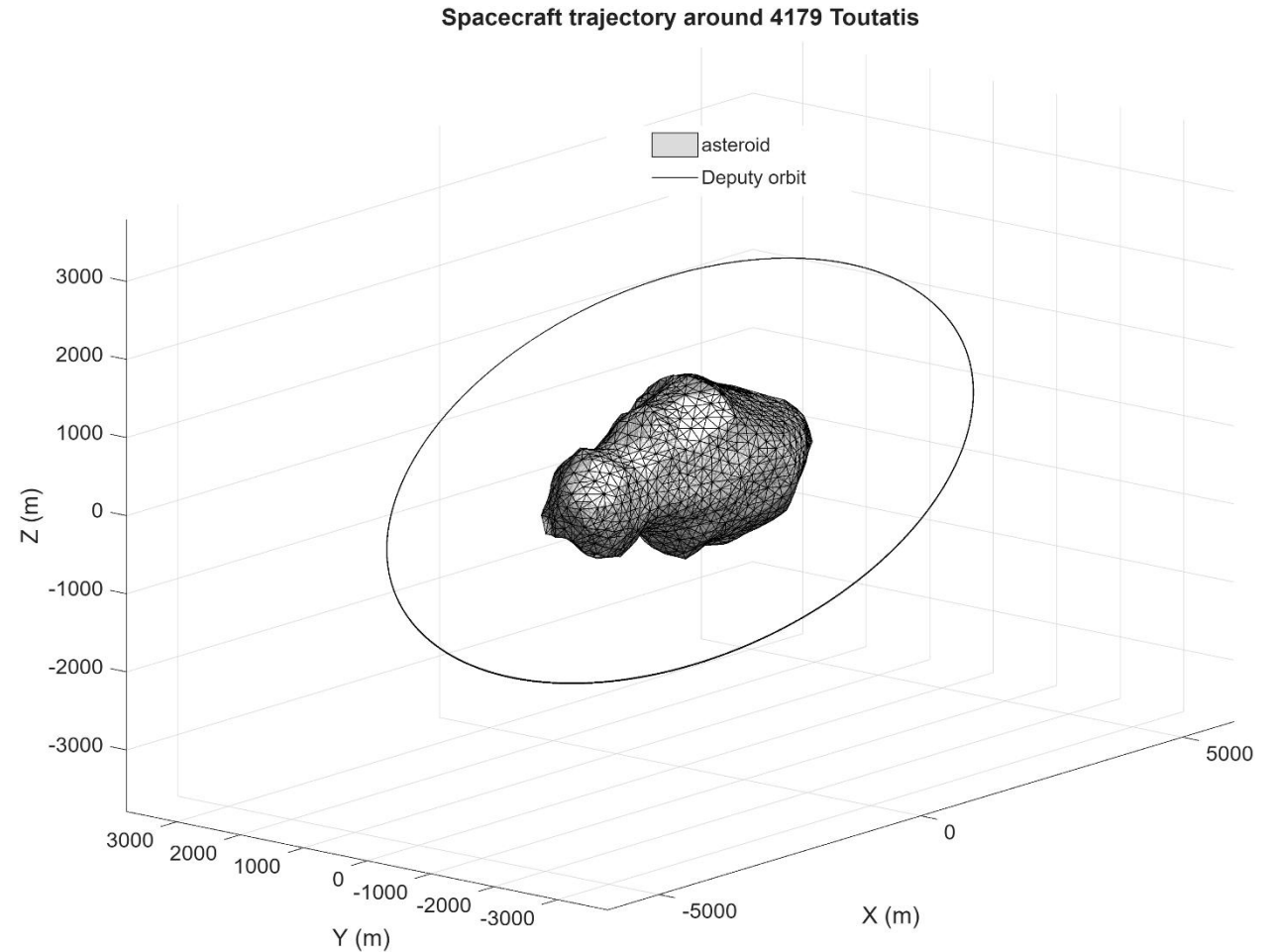
$$\dot{\bar{\Omega}}_R = \frac{3}{2} \frac{\bar{n} \cos \bar{i} (C_{20} + 2C_{22} \cos 2\bar{\Omega}_R)}{(p/R_0)^2} - n_T = 0$$

$$\dot{\bar{\omega}} = \dots$$

$$\dot{\bar{M}} = \dots$$



$$\bar{\Omega}_R = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \rightarrow \bar{i} = \arccos \left(\frac{2n_T \bar{p}^2}{3\bar{n}R_0^2 (C_{20} - 2C_{22})} \right)$$



Relative Motion And Boundedness Condition A

$$\dot{\bar{a}} = 0.$$

$$\dot{\bar{e}} = 0.$$

$$\dot{\bar{i}} = \frac{3\bar{n}C_{22} \sin \bar{i} \sin 2\bar{\Omega}}{(p/R_0)^2}$$

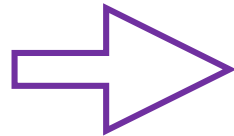
$$\dot{\bar{\Omega}}_R = \frac{3}{2} \frac{\bar{n} \cos \bar{i} (C_{20} + 2C_{22} \cos 2\bar{\Omega}_R)}{(p/R_0)^2} - n_T.$$

$$\dot{\bar{\omega}} = -\frac{3\bar{n}}{8(p/R_0)^2} [C_{20} (3 + 5 \cos 2\bar{i}) + 2C_{22} \cos 2\bar{\Omega} (-1 + 5 \cos 2\bar{i})].$$

$$\dot{\bar{M}} = n - \frac{3\bar{n}\sqrt{1-\bar{e}^2}}{8(p/R_0)^2} [C_{20} (1 + 3 \cos 2\bar{i}) + 6C_{22} \cos 2\bar{\Omega} (\cos 2\bar{i} - 1)].$$

- a straightforward way to avoid the mutual departure of the two spacecraft is to match the secular drift rates of the two spacecraft

$$\boxed{\delta \dot{\bar{i}} = 0 \cap \delta \dot{\bar{\Omega}} = 0 \cap \delta \dot{\bar{\omega}} = 0 \cap \delta \dot{\bar{M}} = 0}$$



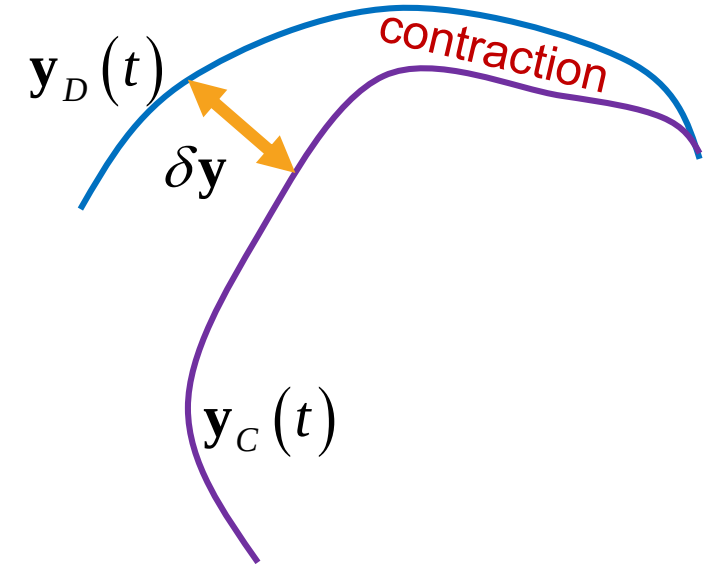
- Relative orbital elements can be defined as (between two trajectories) as variation

$$\delta \mathbf{y} = \mathbf{y}_D - \mathbf{y}_C$$

$$\bar{\mathbf{X}} = [\bar{i}, \bar{\Omega}, \bar{\omega}, \bar{M}]^T$$

$$\bar{\boldsymbol{\zeta}} = [\bar{a}, \bar{e}, \bar{i}, \bar{\Omega}]^T$$

$$\delta \dot{\bar{\mathbf{X}}} = \frac{\partial \dot{\bar{\mathbf{X}}}}{\partial \bar{\boldsymbol{\zeta}}} \delta \bar{\boldsymbol{\zeta}}$$



$$\delta \bar{a} = 0 \cap \delta \bar{e} = 0 \cap \delta \bar{i} = 0 \cap \delta \bar{\Omega} = 0$$

Relative Motion And Boundedness Condition B

$$\dot{\bar{a}} = 0.$$

$$\dot{\bar{e}} = 0.$$

$$\dot{\bar{i}} = \frac{3\bar{n}C_{22} \sin \bar{i} \sin 2\bar{\Omega}}{(p/R_0)^2}$$

$$\dot{\bar{\Omega}}_R = \frac{3}{2} \frac{\bar{n} \cos \bar{i} (C_{20} + 2C_{22} \cos 2\bar{\Omega}_R)}{(p/R_0)^2} - n_T.$$

$$\dot{\bar{\omega}} = -\frac{3\bar{n}}{8(p/R_0)^2} [C_{20} (3 + 5 \cos 2\bar{i}) + 2C_{22} \cos 2\bar{\Omega} (-1 + 5 \cos 2\bar{i})].$$

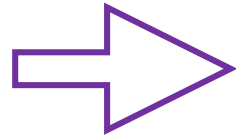
$$\dot{\bar{M}} = n - \frac{3\bar{n}\sqrt{1-\bar{e}^2}}{8(p/R_0)^2} [C_{20} (1 + 3 \cos 2\bar{i}) + 6C_{22} \cos 2\bar{\Omega} (\cos 2\bar{i} - 1)].$$

The first variational rate of **the mean argument of latitude**,

$$\delta \dot{\bar{\theta}} = \delta \dot{\bar{\omega}} + \kappa \delta \dot{\bar{M}}$$

$$\delta \dot{\bar{i}} = 0 \cap \boxed{\delta \dot{\bar{\Omega}} = 0 \cap \delta \dot{\bar{\omega}} + \kappa \delta \dot{\bar{M}} = 0}$$

$$e \ll 1 \Rightarrow \kappa \approx 1$$



$$\bar{\mathbf{X}} = [\bar{a}, \bar{e}, \bar{i}, \bar{\Omega}, \bar{\omega}, \bar{M}]^T$$

$$\delta \dot{\bar{\mathbf{X}}} = \frac{\partial \dot{\bar{\mathbf{X}}}}{\partial \bar{\mathbf{X}}} \delta \bar{\mathbf{X}}$$

$$\kappa_1 = \frac{\delta a}{\delta e} = \frac{7R_0^2 a e (C_{20} - 6C_{22} + (5C_{20} - 10C_{22}) \cos^2(\text{inc}))}{2a^2 + 7R_0^2 (C_{20} - 6C_{22}) + 28R_0^2 (C_{20} - 2C_{22}) \cos^2(\text{inc})}$$

$$\kappa_2 = \frac{\delta \bar{i}}{\delta \bar{e}} = \frac{e \cos(i) [16a^2 + 7R_0^2 (C_{20} - 6C_{22}) - 21R_0^2 (C_{20} - 2C_{22}) \cos^2(i)]}{2 \sin(i) [2a^2 + 7R_0^2 (5C_{20} - 14C_{22}) - 28R_0^2 (C_{20} - 2C_{22}) \sin^2(i)]}$$

$$\delta \bar{a} = \kappa_1 \delta \bar{e}$$

$$\delta \bar{i} = \kappa_2 \delta \bar{e}$$

$$\delta \bar{\Omega} = 0$$

$$\varepsilon_a = \bar{a}_D - \bar{a}_C - \delta \bar{a}^*$$

$$\varepsilon_i = \bar{a}_D - \bar{a}_C - \delta \bar{i}^*$$

$$\varepsilon_\Omega = \bar{\Omega}_D - \bar{\Omega}_C$$

Relative Motion And Boundedness Condition C

the along-track boundedness condition with a single constraint

$$\begin{aligned}\bar{\rho}_x &= \frac{2+\bar{e}^2}{2}\delta\bar{a} + \bar{a}\bar{e}\delta\bar{e} \\ \bar{\rho}_y &= \bar{a}\left(\sqrt{1-\bar{e}^2}\delta\bar{M} + \frac{2+\bar{e}^2}{2}\delta\bar{\omega} + \frac{2+\bar{e}^2}{2}\delta\bar{\Omega}_R \cos\bar{i}\right) \\ \bar{\rho}_z &= \frac{3}{2}\bar{a}\bar{e}\left(\delta\bar{\Omega}_R \cos\bar{\omega} \sin\bar{i} - \delta\bar{i} \sin\bar{\omega}\right)\end{aligned}$$

$\Omega_R = \frac{\pi}{2}, \bar{e}^2 \approx 0$

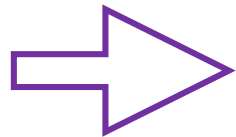
$$\begin{aligned}\dot{\bar{\rho}}_x &= 0 \\ \dot{\bar{\rho}}_y &= \bar{a}\left(\delta\dot{\bar{M}} + \delta\dot{\bar{\omega}} + \delta\dot{\bar{\Omega}}_R \cos\bar{i}\right) \rightarrow \boxed{\delta\dot{\bar{M}} + \delta\dot{\bar{\omega}} + \delta\dot{\bar{\Omega}}_R \cos\bar{i} = 0} \\ \dot{\bar{\rho}}_z &= \frac{3}{2}\bar{a}\bar{e}\left[-\left(\delta\bar{\Omega}_R \sin\bar{i} \sin\bar{\omega} + \delta\bar{i} \cos\bar{\omega}\right)\dot{\bar{\omega}} - \left[-\delta\dot{\bar{i}} \sin\bar{\omega} + \delta\dot{\bar{\Omega}}_R \sin\bar{i} \cos\bar{\omega}\right]\right]\end{aligned}$$

$$\delta\dot{\bar{M}} + \delta\dot{\bar{\omega}} + \delta\dot{\bar{\Omega}}_R \cos\bar{i} = 0$$

$$\bar{\mathbf{X}} = \begin{bmatrix} \bar{i} & \bar{\Omega} & \bar{\omega} & \bar{M} \end{bmatrix}^T$$

$$\bar{\boldsymbol{\zeta}} = \begin{bmatrix} \bar{a} & \bar{e} & \bar{i} & \bar{\Omega} \end{bmatrix}^T$$

$$\delta\dot{\bar{\mathbf{X}}} = \frac{\partial \dot{\bar{\mathbf{X}}}}{\partial \bar{\boldsymbol{\zeta}}} \delta\bar{\boldsymbol{\zeta}}$$



$$\delta\bar{a}^* = \kappa_3\delta\bar{e} + \kappa_4\delta\bar{i}$$

$$\kappa_3 = \frac{7e\left[C_{20}(1+3\cos(2\bar{i})) - 6C_{22}(1-\cos(2\bar{i}))\right]}{4\left(\frac{\bar{a}}{R_0}\right)^2 - 7C_{20}(1+3\cos(2\bar{i})) + 42C_{22}(1-\cos(2\bar{i}))}$$

$$\kappa_4 = \frac{-14\sin(2\bar{i})(C_{20} - 2C_{22})}{4\left(\frac{\bar{a}}{R_0}\right)^2 - 7C_{20}(1+3\cos(2\bar{i})) + 42C_{22}(1-\cos(2\bar{i}))}$$

$$\delta\bar{a}^* = \kappa_3\delta\bar{e} + \kappa_4\delta\bar{i}$$

$$\varepsilon_a = \bar{a}_D - \bar{a}_C - \delta\bar{a}^*$$

Perturbed system under disturbances

- The perturbed system

$$\dot{\bar{\zeta}}_D = \left(\mathbf{f}(\bar{\zeta}_D) + \Delta \mathbf{f}(\bar{\zeta}_D) \right) + \left(\mathbf{B}(\bar{\zeta}_D) + \Delta \mathbf{B}(\bar{\zeta}_D) \right) \mathbf{u}_D$$

- be adjusted accordingly. Since the entries $\Delta \mathbf{f}(\bar{\zeta}_D)$ and $\Delta \mathbf{B}(\bar{\zeta}_D)$ are bounded, it is reasonable to assume the uncertainty satisfies a so-called matching condition

$$\Delta \mathbf{f}(\bar{\zeta}_D) = \mathbf{B}(\bar{\zeta}_D) \Delta \tilde{\mathbf{f}}(\bar{\zeta}_D)$$

$$\Delta \mathbf{B}(\bar{\zeta}_D) = \mathbf{B}(\bar{\zeta}_D) \Delta \tilde{\mathbf{B}}(\bar{\zeta}_D)$$

$$\dot{\bar{\zeta}}_D = \mathbf{f}(\bar{\zeta}_D) + \mathbf{B}(\bar{\zeta}_D) \left[\mathbf{u}_D + \boldsymbol{\tau}(\bar{\zeta}_D, \mathbf{u}_D) \right]$$

where $\Delta \tilde{\mathbf{f}}(\bar{\zeta}_D), \Delta \tilde{\mathbf{B}}(\bar{\zeta}_D)$ are smooth matrix perturbations,

$$\boldsymbol{\tau}(\bar{\zeta}_D, \mathbf{u}_D) = \Delta \tilde{\mathbf{f}}(\bar{\zeta}_D) + \Delta \tilde{\mathbf{B}}(\bar{\zeta}_D) \mathbf{u}_D$$

Control Lyapunov Function

The Lyapunov function $V(\mathbf{e}_D) = \frac{1}{2} \mathbf{e}_D^T \mathbf{K} \mathbf{e}_D$ $\dot{V} < 0$ $\dot{\bar{\boldsymbol{\zeta}}}_D = \mathbf{f}(\bar{\boldsymbol{\zeta}}_D) + \mathbf{B}(\bar{\boldsymbol{\zeta}}_D) [\mathbf{u}_D + \boldsymbol{\tau}(\bar{\boldsymbol{\zeta}}_D, \mathbf{u}_D)]$

Nominal control $\mathbf{u}_{D1} = -\mathbf{B}^T(\bar{\boldsymbol{\zeta}}_D^T) \mathbf{K} \mathbf{e}_D$ Robust part $\mathbf{u}_{D2} = -\eta(t, \bar{\boldsymbol{\zeta}}_D^T) \frac{\mathbf{B}^T(\bar{\boldsymbol{\zeta}}_D^T) \mathbf{K} \mathbf{e}_D}{\|\mathbf{B}^T(\bar{\boldsymbol{\zeta}}_D^T) \mathbf{K} \mathbf{e}_D\|}$

Advantages:

- Guaranteed stability (when properly designed).
- Low computational cost – easy to implement onboard.

Disadvantages:

- The resulting control law may be far from optimal in terms of fuel consumption or time.
- Often leads to unnecessarily aggressive control.

SDRE Control

- The system $\dot{\bar{\zeta}}_D = \mathbf{f}(\bar{\zeta}_D) + \mathbf{B}(\bar{\zeta}_D) \left[\mathbf{u}_D + \boldsymbol{\tau}(\bar{\zeta}_D, \mathbf{u}_D) \right]$ $\mathbf{A} = \frac{f(\bar{\zeta}_D) \bar{\zeta}_D^T}{\bar{\zeta}_D^T \bar{\zeta}_D}$ $\dot{\bar{\zeta}}_D = \mathbf{A} \bar{\zeta}_D + \mathbf{B}(\bar{\zeta}_D) \left[\mathbf{u}_D + \boldsymbol{\tau}(\bar{\zeta}_D, \mathbf{u}_D) \right]$

- State dependent System factorization:

- Nominal control: $\mathbf{u}_{D1} = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P} \mathbf{e}_D$

- adaptive control? $\mathbf{u}_{D2} = -\eta(t, \bar{\zeta}_D) \frac{\mathbf{B}^T (\bar{\zeta}_D^T) \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P} \mathbf{e}_D}{\left\| \mathbf{B}^T (\bar{\zeta}_D^T) \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P} \mathbf{e}_D \right\|}$

Advantages:

- Provides suboptimal control with respect to the selected quadratic performance criterion (e.g., minimum fuel/energy).
- Naturally accounts for nonlinearities and coupling between the channels.

Disadvantages:

- High computational load (the Riccati equation must be solved at each step).
- Requires careful selection of the weighting matrices Q and R.

- Case A $\delta \bar{a} = 0 \cap \delta \bar{e} = 0 \cap \delta \bar{i} = 0 \cap \delta \bar{\Omega} = 0$

$$\mathbf{u}_{D1} = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}_A \begin{bmatrix} \varepsilon_a = \bar{a}_D - a_C \\ \varepsilon_e = \bar{e}_D - \bar{e}_C \\ \varepsilon_i = \bar{i}_D - \bar{i}_C \\ \varepsilon_\Omega = \bar{\Omega}_D - \bar{\Omega}_C \end{bmatrix}$$

- Case B $\delta \bar{a}^* = \kappa_1 \delta \bar{e}$
 $\delta \bar{i}^* = \kappa_2 \delta \bar{e}$

$$\delta \bar{\Omega} = 0$$

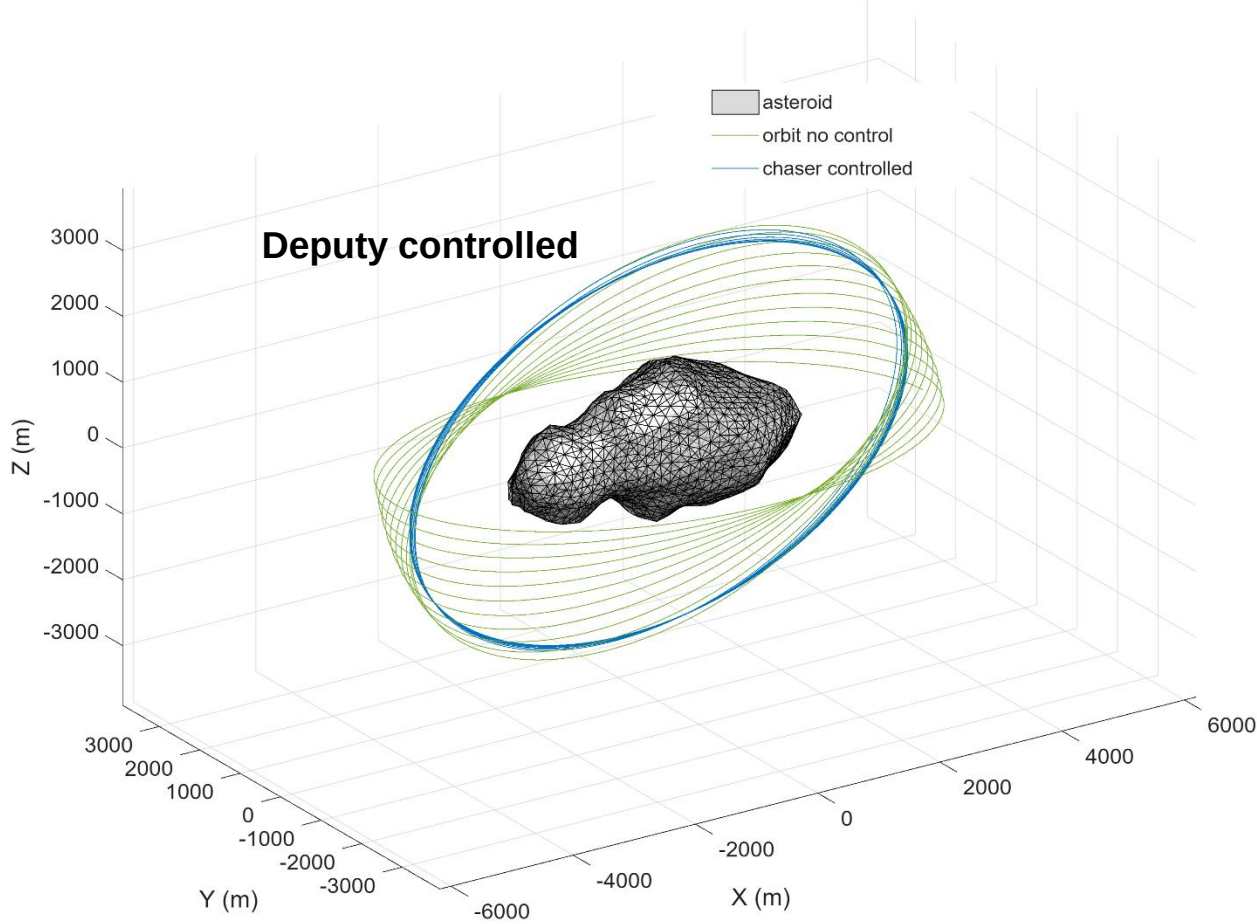
$$\mathbf{u}_{D1} = -\mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}_B \begin{bmatrix} \varepsilon_a = \bar{a}_D - \bar{a}_C - \delta \bar{a}^* \\ \varepsilon_i = \bar{i}_D - \bar{i}_C - \delta \bar{i}^* \\ \varepsilon_\Omega = \bar{\Omega}_D - \bar{\Omega}_C \end{bmatrix}$$

- Case C $\delta a^* = \kappa_1 \delta e + \kappa_2 \delta i$

$$\mathbf{u}_{D13 \times 1} = -\mathbf{R}^{-1} \mathbf{B}_{3 \times 1}^T P_c \begin{bmatrix} \varepsilon_a = \bar{a}_D - \bar{a}_C - \delta \bar{a}^* \end{bmatrix}$$

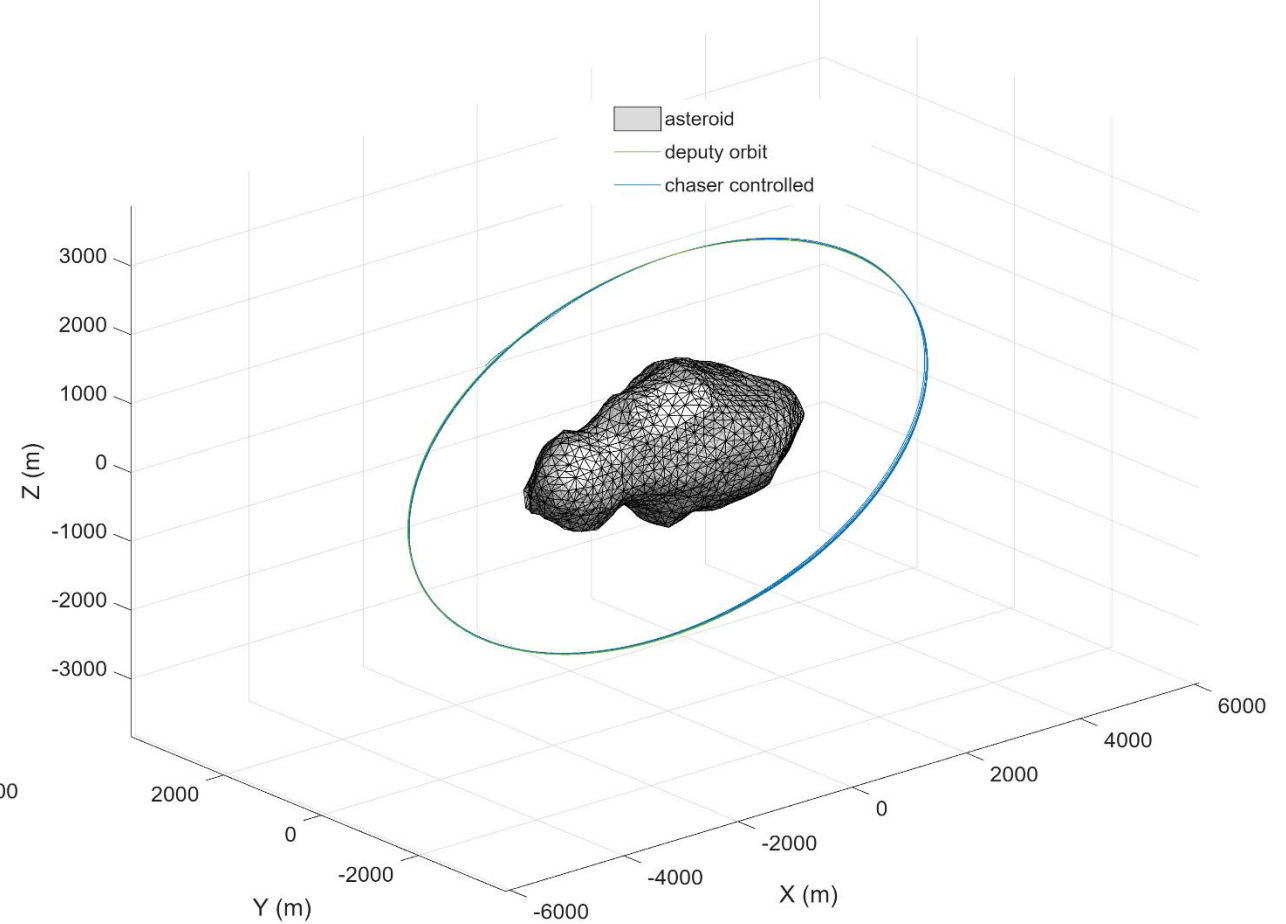
SDRE Control

Spacecraft trajectory around 4179 Toutatis



$$\delta \bar{a} = 0 \cap \delta \bar{e} = 0 \cap \delta \bar{i} = 0 \cap \delta \bar{\Omega} = 0$$

Spacecraft trajectory around 4179 Toutatis



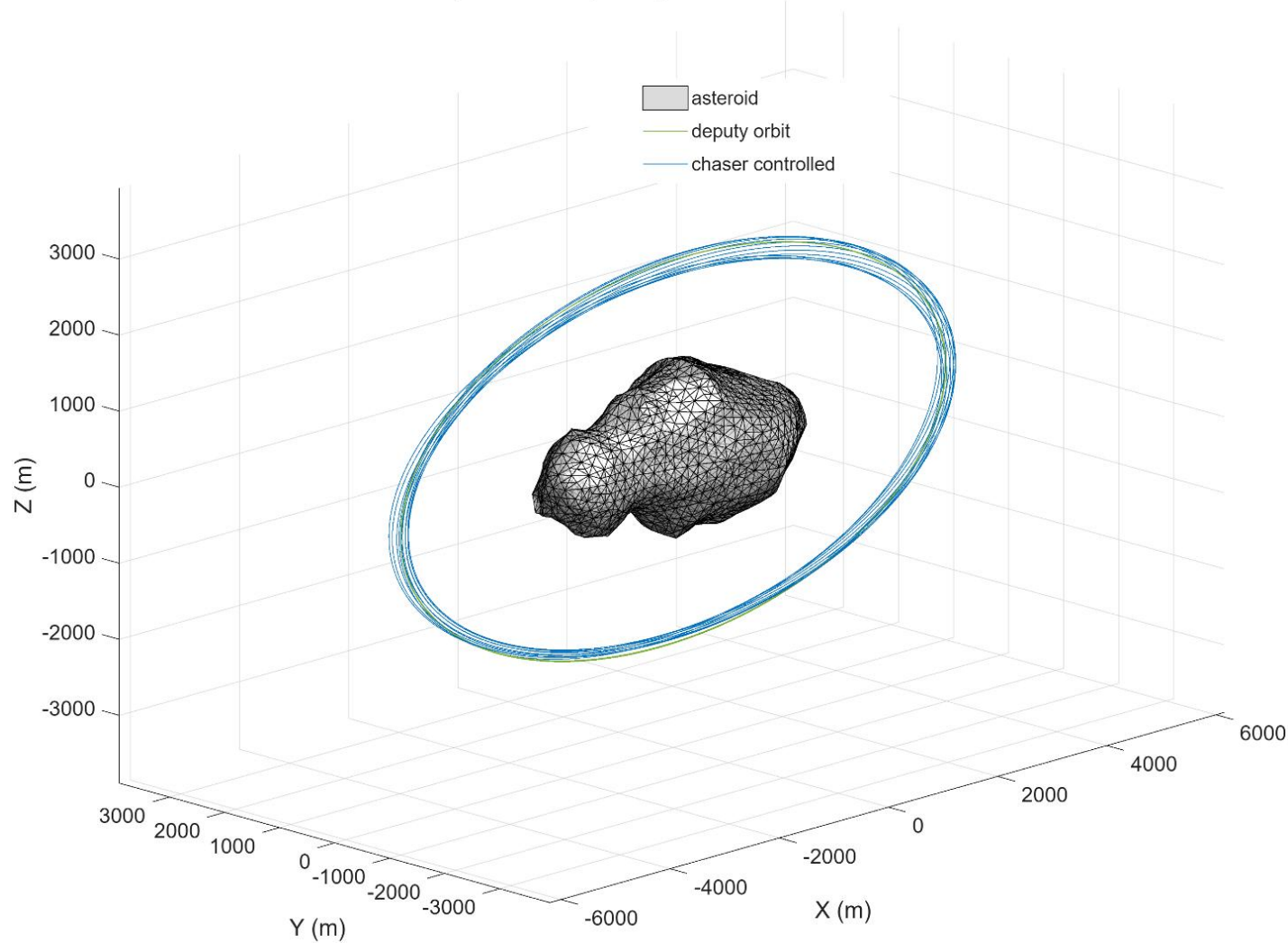
$$\delta \bar{a} = \kappa_1 \delta \bar{e}$$

$$\delta \bar{i} = \kappa_2 \delta \bar{e}$$

$$\delta \bar{\Omega} = 0$$

SDRE Control

Spacecraft trajectory around 4179 Toutatis

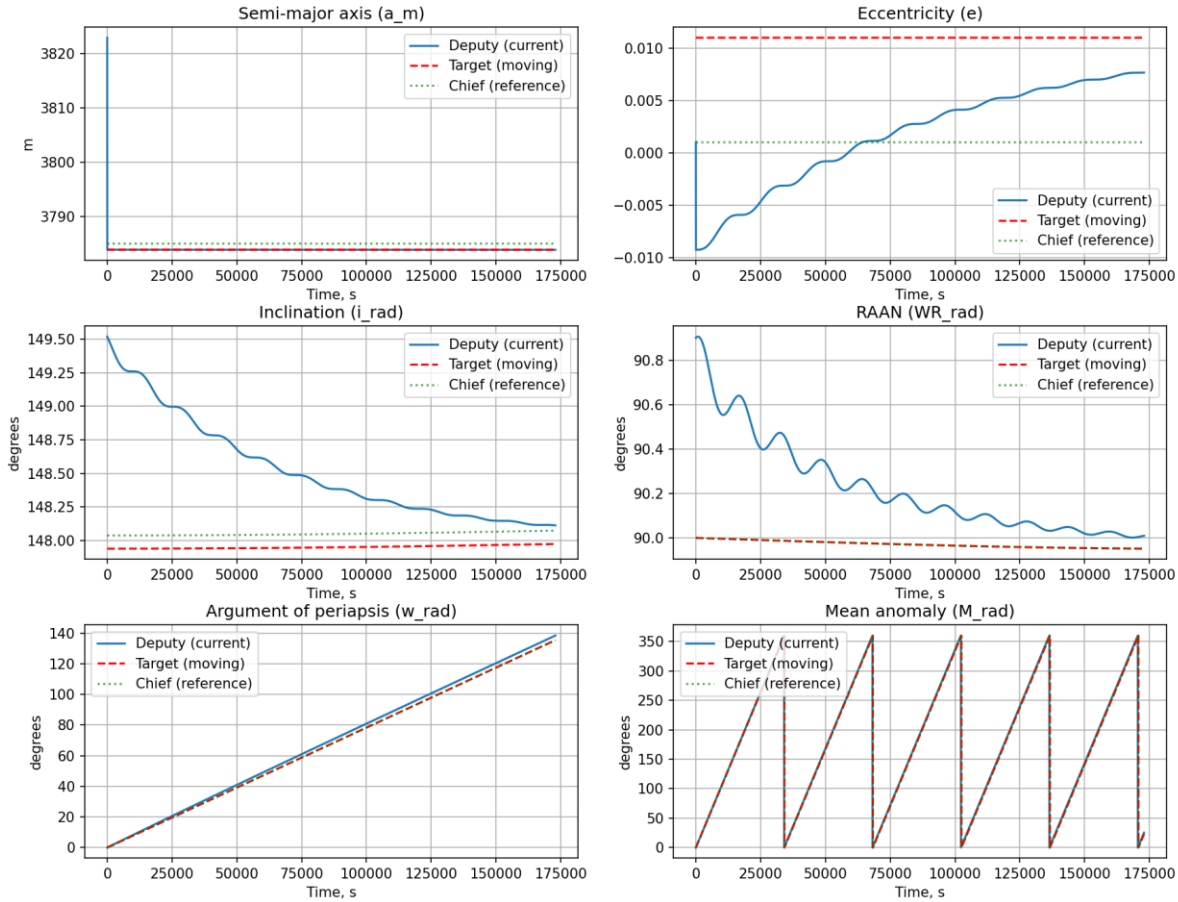


$$\delta \bar{a}^* = \kappa_3 \delta \bar{e} + \kappa_4 \delta \bar{i}$$

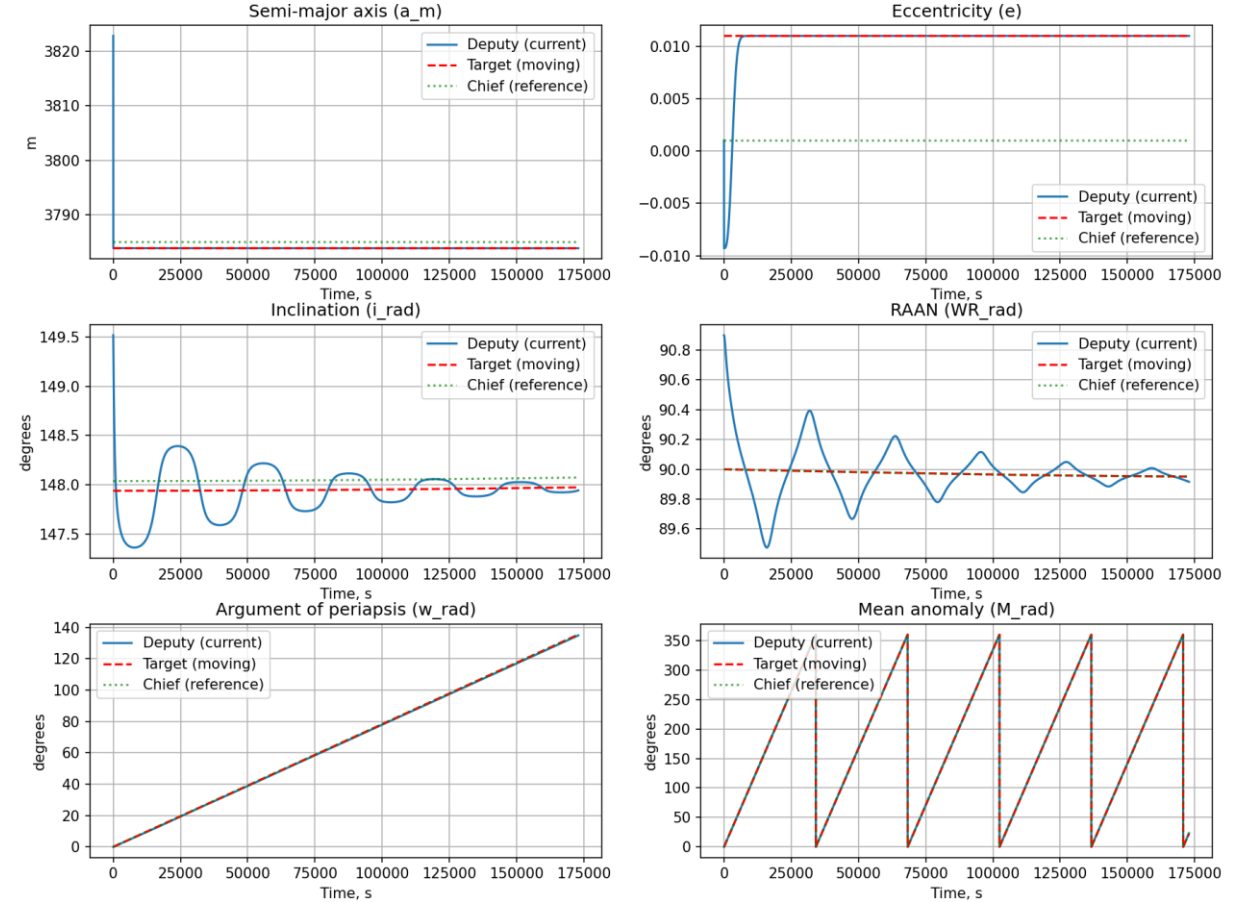
$$\varepsilon_a = \bar{a}_D - \bar{a}_C - \delta \bar{a}^*$$

Control Lyapunov Function

Evolution of Deputy Parameters (Lyapunov Control, Noise: True, u_D2: True)

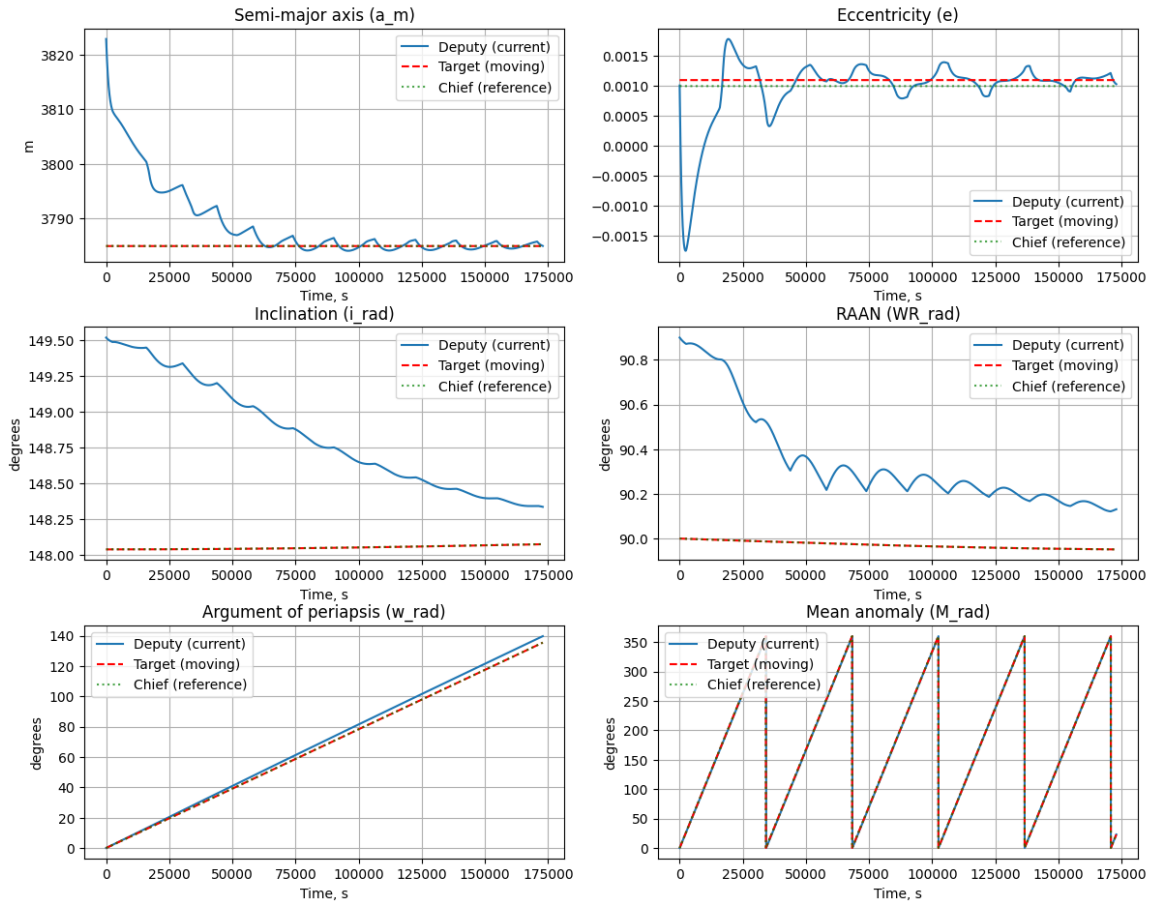


Evolution of Deputy Parameters (Lyapunov Control, Noise: True, u_D2: True)

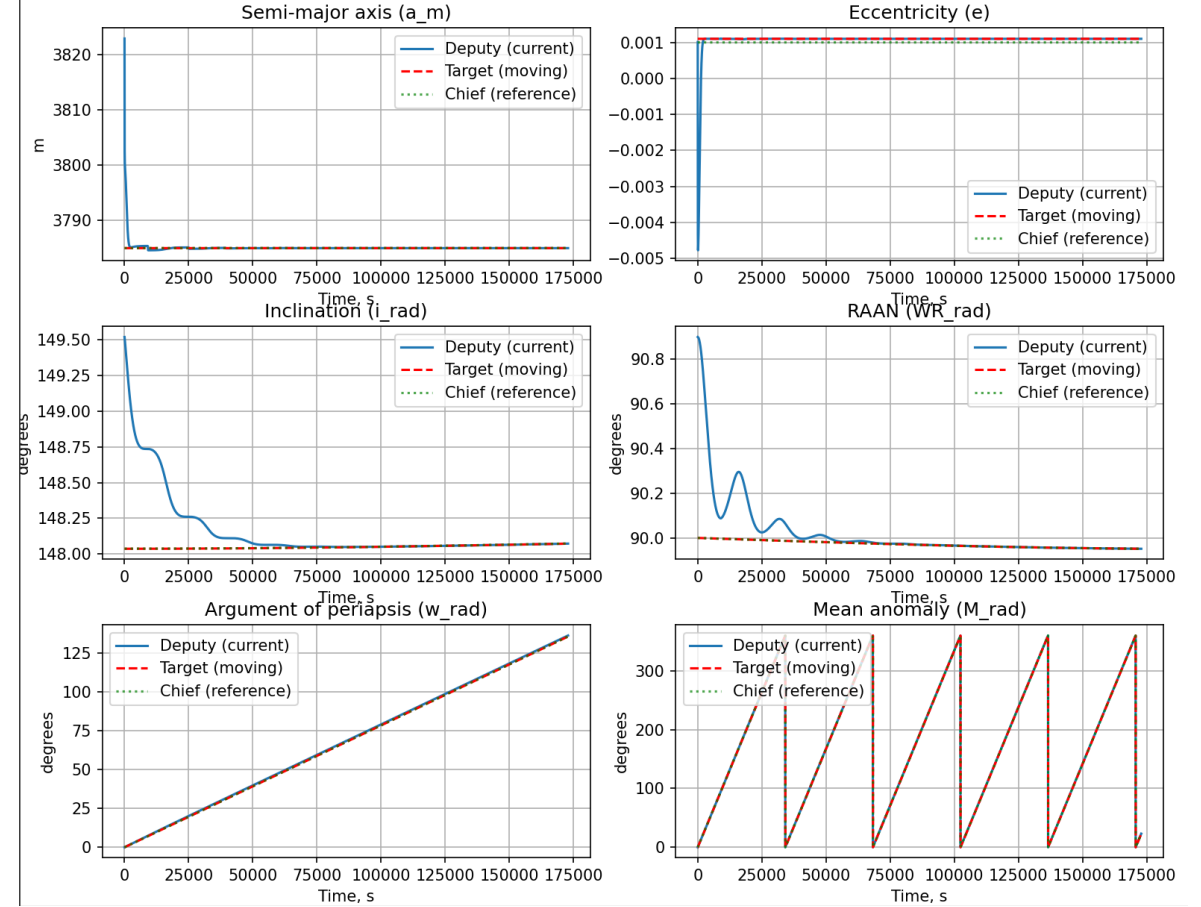


SDRE Control

SDRE Control (Noise: True, u_D2: True)



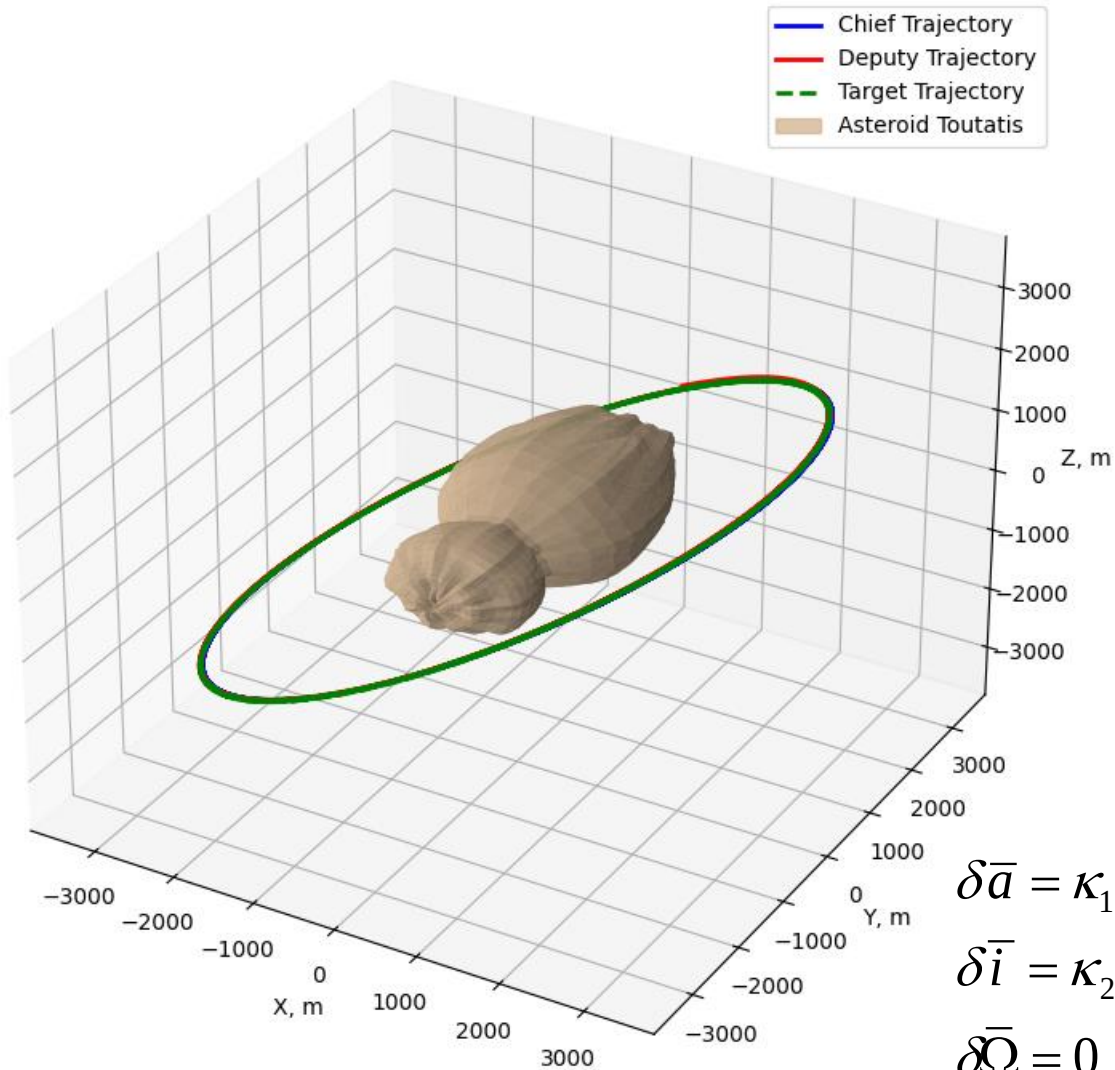
SDRE Control (Noise: True, u_D2: True)



	cost u_{D1}	cost u_{D2}	Metric G	Convergence time	Computation cost (time)
Lyapunov	7.578×10^{-1}	6.492×10^{-15}	3.811×10^{-2}	121567	5.2
	7.577×10^{-4}	3.538×10^{-16}	1.332×10^{-4}	66938	5.0
	7.577×10^{-3}	6.232×10^{-17}	3.939×10^{-4}	37785	4.2
SDRE	2.117×10^{-6}	2.117×10^{-22}	2.131×10^{-7}	76127	19.2
	3.908×10^{-6}	3.908×10^{-22}	3.858×10^{-7}	61944	19.6
	1.201×10^{-5}	1.201×10^{-22}	1.203×10^{-6}	39369	12.7

Control Lyapunov Function

3D Trajectories (Lyapunov Control)

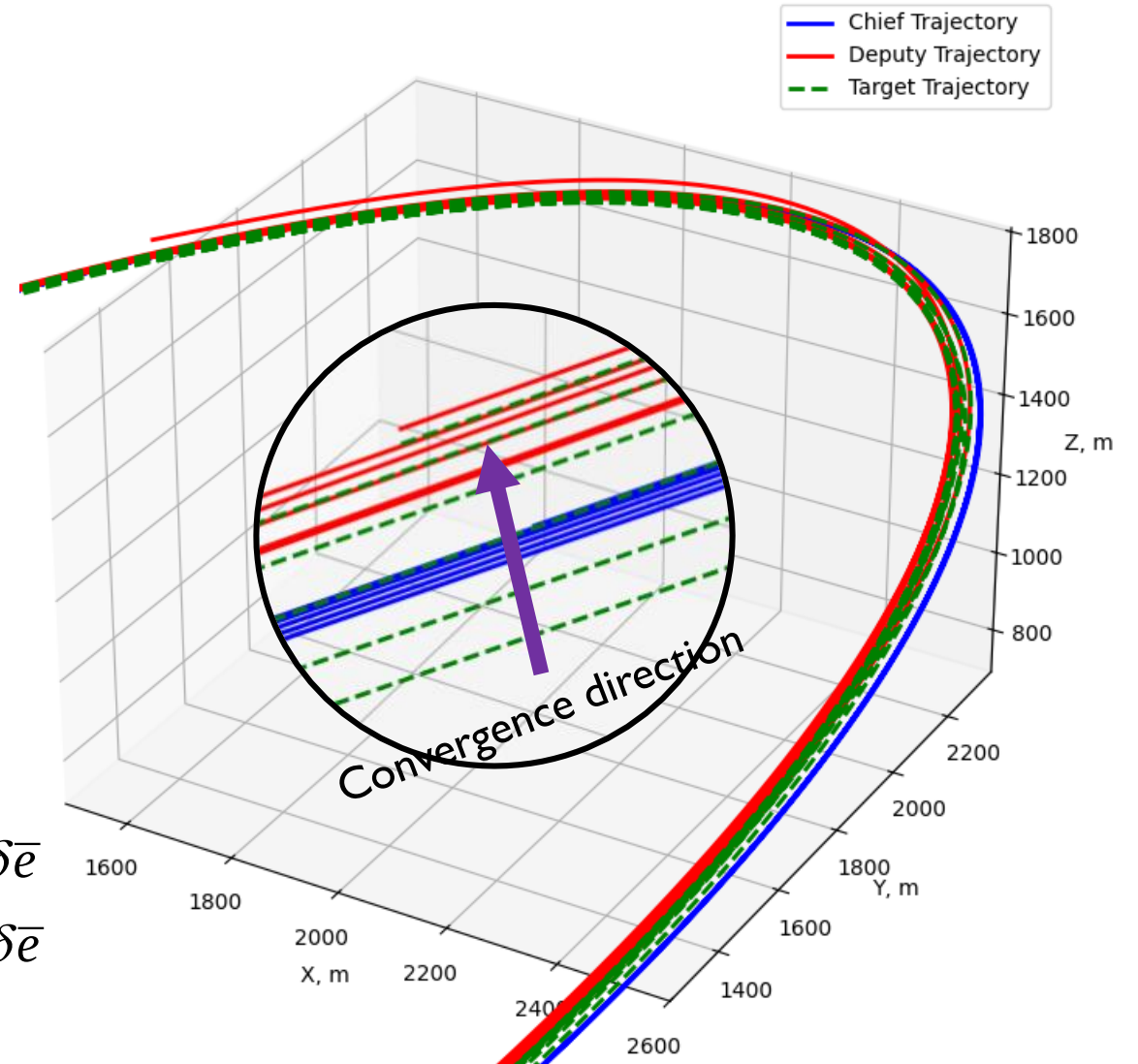


$$\delta \bar{a} = \kappa_1 \delta \bar{e}$$

$$\delta \bar{i} = \kappa_2 \delta \bar{e}$$

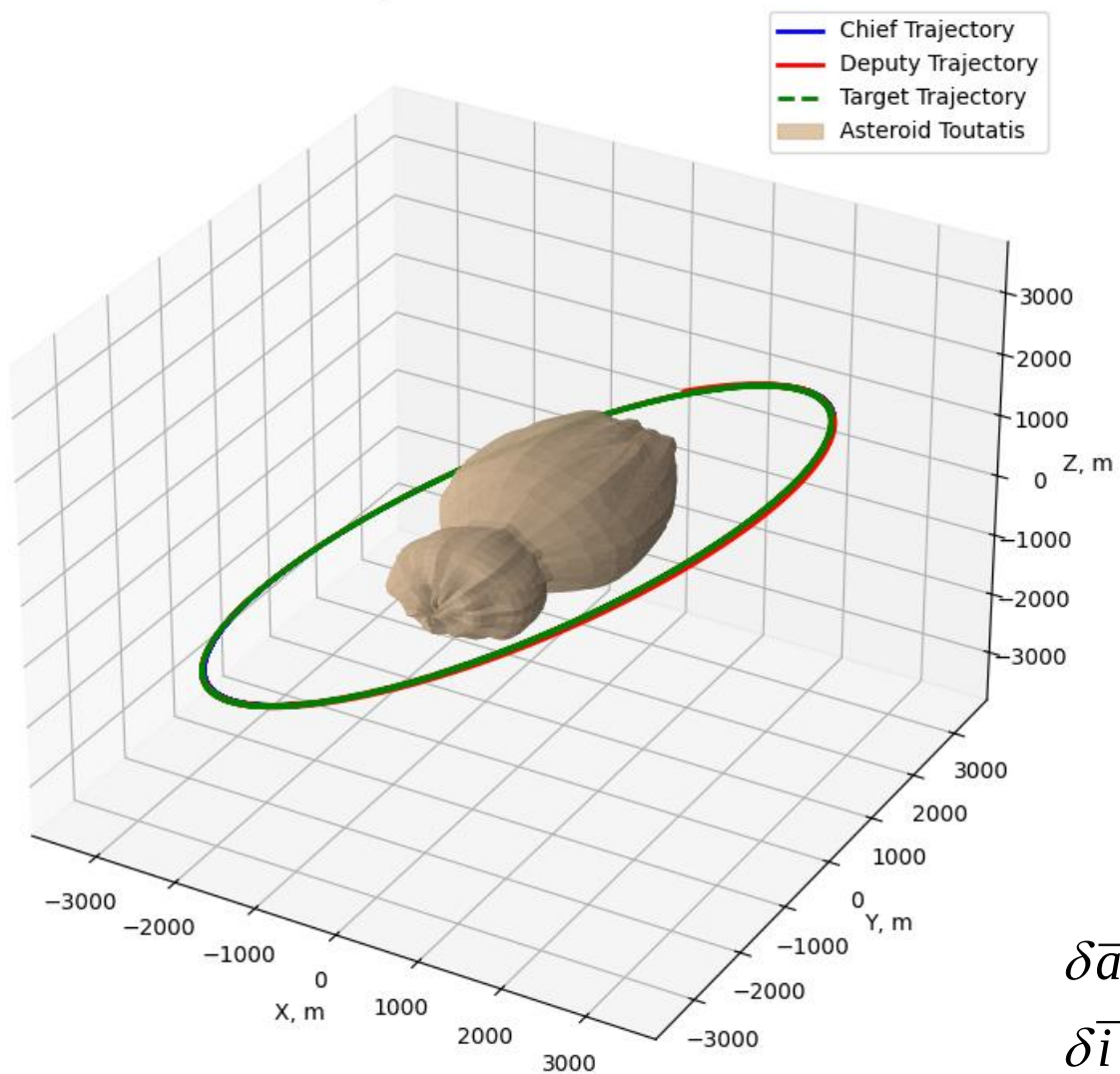
$$\delta \bar{\Omega} = 0$$

3D Trajectories (Lyapunov Control)

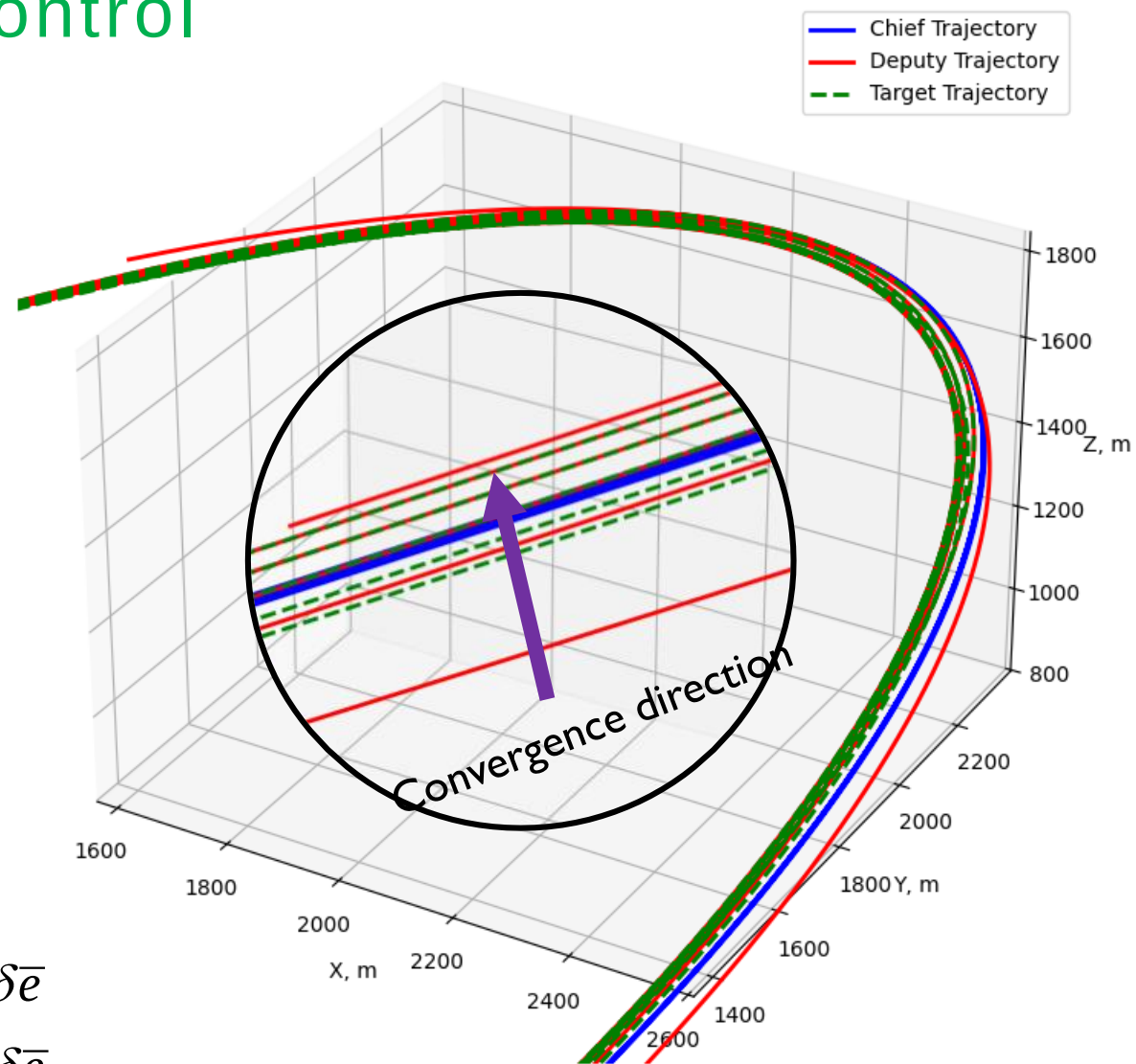


SDRE Control

3D Trajectories (SDRE Control)



3D Trajectories (SDRE Control)



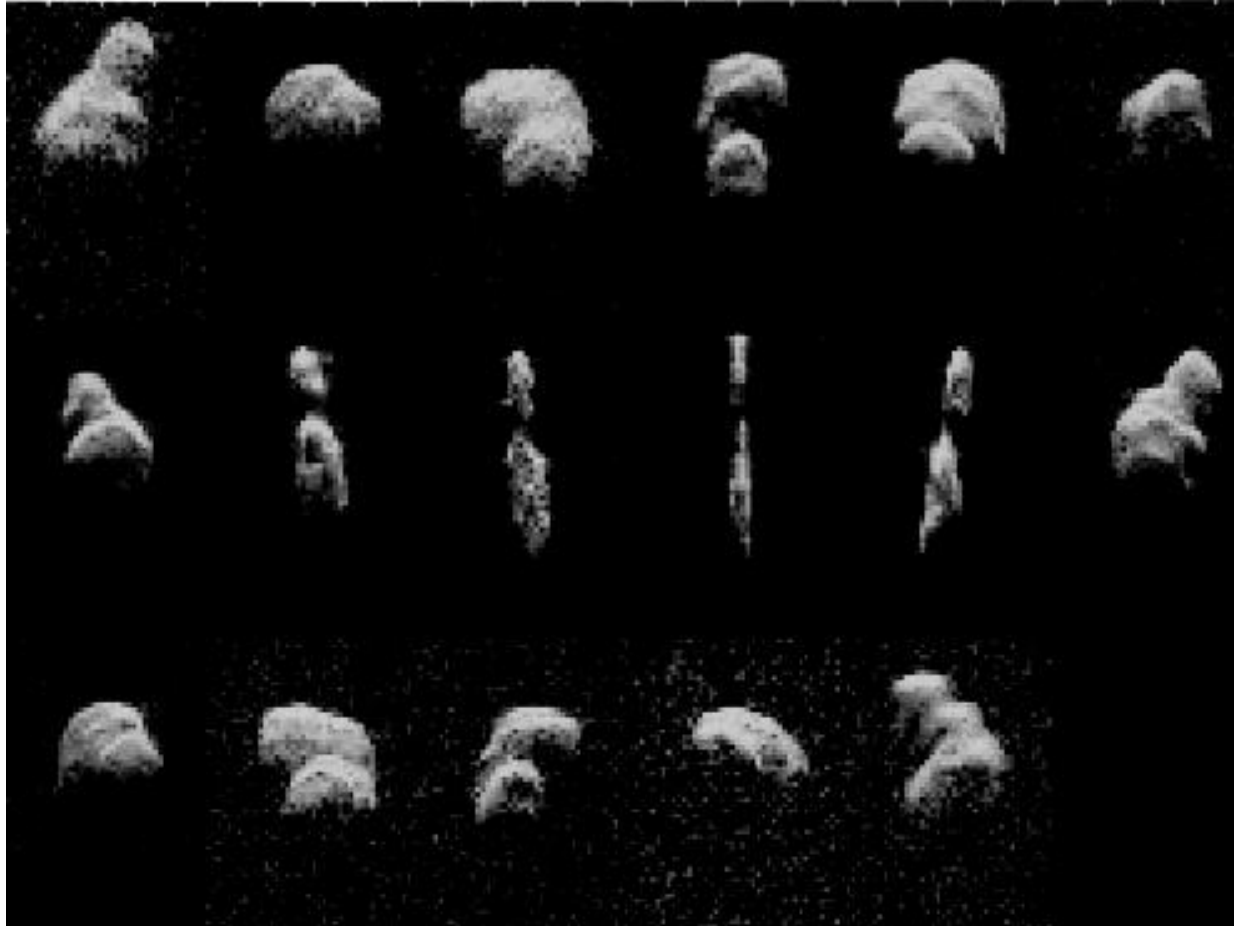
$$\delta \bar{a} = \kappa_1 \delta \bar{e}$$

$$\delta \bar{i} = \kappa_2 \delta \bar{e}$$

$$\delta \bar{\Omega} = 0$$

Conclusion

- The relative motion of a group of spacecraft in the vicinity of Earth and an asteroid was modeled, taking into account uncertainties in the gravitational field.
- For motion around Earth, suboptimal control (LQR) and Lyapunov-function-based control were synthesized for formation keeping; in the vicinity of the asteroid, SDRE-based control and Lyapunov-function-based control were synthesized, respectively.
- A robust control law was developed to compensate for disturbances and model uncertainties.
- A comparative analysis of the two approaches was conducted.
- The use of the SDRE controller reduces control effort and convergence time compared with the Lyapunov-function-based control synthesis. However, the computational time increases several-fold.



Thanks For Your Attention

