The Jacobi – Poisson Method of Building First integrals by Integral Manifolds for Generalized Hamiltonian Systems

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Abstract. In this paper we consider the generalized Hamiltonian differential systems (or the Arzanyh systems). The Jacobi – Poisson method of building first integrals in the form of Poisson brackets by known integral manifolds is developed. The results obtained in this paper can be used in the analytical theory of differential equations and in the analytical mechanics.

Introduction

One of the remarkable properties of the Hamiltonian differential systems

$$\frac{dq_i}{dt} = \partial_{p_i} H(t,q,p), \qquad \frac{dp_i}{dt} = -\partial_{q_i} H(t,q,p), \quad i = 1,\dots,n, \tag{1}$$

is its invariance with respect to canonical (contact) transformations (see, for example [1, pp. 146 – 150] and [2, pp. 285 – 301]). This property of the Hamiltonian differential system (1) is very essential in the construction of its first integrals (or constants of motion), and, in particular, the Hamilton – Jacobi method of integrating the Hamiltonian differential system (1) follows directly from this property.

The most general systems of differential equations that are invariant with respect to the group of canonical transformations are systems of canonical equations of rank $m \geq 0$ or the generalized Hamiltonian systems (the Arzanyh systems) [3, 4]

$$\begin{aligned} \frac{dq_i}{dt} &= \partial_{p_i} H(t,q,p) + \sum_{j=1}^{m} K_j(t,q,p)\,\partial_{p_i} H_j(t,q,p), \\ \frac{dp_i}{dt} &= -\partial_{q_i} H(t,q,p) - \sum_{j=1}^{m} K_j(t,q,p)\,\partial_{q_i} H_j(t,q,p), \quad i = 1,\dots,n, \end{aligned} \tag{2}$$

where $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$ and $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, respectively, $t \in \mathbb{R}$, the functions $H, H_j \in C^2(D), K_j \in C^1(D)$, $j = 1, \ldots, m$, and the domain $D = T \times G,\ T \subset \mathbb{R},\ G \subset \mathbb{R}^{2n}$.

Among the general methods of integration of the Hamiltonian system (1), the Jacobi – Poisson method is of particular importance. It gives the possibility to find the additional (third) first integral of the Hamiltonian system (1) by two known first integrals of the Hamiltonian system (1). And thus, in certain cases, to build an integral basis of the Hamiltonian system (1). Due to this property, the Poisson method is included in almost all monographs and textbooks on analytical mechanics (see, for example, [1, p. 100] and [5, p. 184]) and formulated as

Theorem (the Jacobi – Poisson theorem). *Suppose twice continuously differentiable functions* $\mathrm{g}_1 \colon D' \to \mathbb{R}$ *and* $\mathrm{g}_2 \colon D' \to \mathbb{R}$ *are first integrals on the domain* $D' \subset D$ *of the Hamiltonian system* (1). *Then the Poisson bracket*

$$\mathrm{g}_{12} \colon (t, q, p) \to\ [\mathrm{g}_1(t, q, p), \mathrm{g}_2(t, q, p)] \quad \forall (t, q, p) \in D'$$

of the functions g_1 *and* g_2 *is also a first integral of the Hamiltonian system* (1).

In his *Lectures on Dynamics* [6, pp. 240 – 241], C. Jacobi referred to the Jacobi – Poisson theorem as "one of the most remarkable theorems of the whole of integral calculus. In the particular case when $H = T - U$, it is the fundamental theorem of analytical mechanics. ... After I discovered this theorem I communicated it to the Academies of Berlin and Paris as an entirely new discovery. But I noticed soon after that this theorem had already been discovered and forgotton for 30 years, because one did not appreciate its real meaning, but had only used it as a lemma in a entirely different problem". Of course, the Jacobi – Poisson theorem does not always supply further first integrals. In some cases the result is trivial, the Poisson bracket being a constant. In other cases the first integral obtained is simply a function of the original integrals. If neither of these two possibilities occurs, however, then the Poisson bracket is a further first integral of system (1).

Our aim in this paper is to developed the Jacobi – Poisson method for integral manifolds of the generalized Hamiltonian system (2). This article continues the authors' research [7, 8, 9, 10, 11, 12] on the integral characteristics of ordinary and multidimensional Hamiltonian differential systems.

Main Results

To avoid ambiguity, we stipulate the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(D)$ is the function

$$[u, v] \colon (t, q, p) \to \sum_{i=1}^{n} \left(\partial_{q_i} u(t, q, p)\, \partial_{p_i} v(t, q, p) - \partial_{p_i} u(t, q, p)\, \partial_{q_i} v(t, q, p) \right) \forall (t, q, p) \in D.$$

A function $\mathrm{g} \in C^1(D')$ is called a *first integral on the domain* $D' \subset D$ of the generalized Hamiltonian system (2) if

$$\mathfrak{G}\,\mathrm{g}(t,q,p)=0 \quad \forall (t,q,p)\in D',$$

where the linear differential operator

$$\mathfrak{G}(t,q,p)=\partial_{t}+\sum_{i=1}^{n}\Bigg(\Big(\partial_{p_i}H(t,q,p)+\sum_{j=1}^{m}K_j(t,q,p)\,\partial_{p_i}H_j(t,q,p)\Big)\partial_{q_i}\;-$$

$$-\;\Big(\partial_{q_i}H(t,q,p)+\sum_{j=1}^{m}K_j(t,q,p)\,\partial_{q_i}H_j(t,q,p)\Big)\partial_{p_i}\Bigg)\quad\forall (t,q,p)\in D.$$

A smooth manifold $\mathrm{g}(t,q,p)=0$ is said to be an *integral manifold* of the generalized Hamiltonian system (2) if the derivative of the function $\mathrm{g}\in C^1(D')$ by virtue of the generalized Hamiltonian system (2) is the identically zero on the manifold $\mathrm{g}(t,q,p)=0$, i.e.,

$$\mathfrak{G}\,\mathrm{g}(t,q,p)=\Phi(t,q,p),\quad \Phi(t,q,p)\big|_{\mathrm{g}(t,q,p)=0}=0\quad\forall (t,q,p)\in D'.$$

By $\mathrm{I}_{D'}$ and $\mathrm{M}_{D'}$ we denote the set of all first integrals and integral manifolds on the domain D' of the generalized Hamiltonian system (2), respectively. The phrase "the function g is an integral manifold with function Φ on the domain D' of the generalized Hamiltonian system (2)" is denoted by $(\mathrm{g},\Phi)\in\mathrm{M}_{D'}$.

The main result of this article is the following statement.

Theorem 1. *Suppose* $(\mathrm{g}_k,\Phi_k)\in\mathrm{M}_{D'}$, $\mathrm{g}_k\in C^2(D')$, $k=1,2$. *Then the function*

$$\mathrm{g}\colon (t,q,p)\to\ [\mathrm{g}_1(t,q,p),\mathrm{g}_2(t,q,p)]-\int\limits_{t_0}^{t}\varphi(\tau)\,d\tau\quad\forall (t,q,p)\in D',\quad t_0\in T,$$

is a first integral of the generalized Hamiltonian differential system (2) *if and only if the following identity holds*

$$A(t,q,p)+\big[\mathrm{g}_1(t,q,p),\,\Phi_2(t,q,p)\big]-\big[\mathrm{g}_2(t,q,p),\,\Phi_1(t,q,p)\big]=\varphi(t)\quad\forall (t,q,p)\in D',$$

where $\varphi\in C(D')$ *and the function*

$$A(t,q,p)=\sum_{j=1}^{m}\begin{vmatrix}\big[\mathrm{g}_1(t,q,p),\,H_j(t,q,p)\big] & \big[\mathrm{g}_2(t,q,p),\,H_j(t,q,p)\big]\\ \big[\mathrm{g}_1(t,q,p),\,K_j(t,q,p)\big] & \big[\mathrm{g}_2(t,q,p),\,K_j(t,q,p)\big]\end{vmatrix}\quad\forall (t,q,p)\in D'.$$

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