

The Jacobi – Poisson Method of Building First integrals by Integral Manifolds for Generalized Hamiltonian Systems

Andrei Pranevich, Mikalai Skabelka

Yanka Kupala State University of Grodno, Belarus

Analytical Methods of Celestial Mechanics 2026

Euler International Mathematical Institute

St. Petersburg, RUSSIA,

August 17-21, 2026

Plan:

- 1 The Jacobi – Poisson method of building first integrals
 - Introduction. Problem statement
 - Integral manifolds
 - Partial integrals
 - Applications for general differential systems

Consider an ordinary Hamiltonian system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(t, q, p), \quad \frac{dp_i}{dt} = -\partial_{q_i} H(t, q, p), \quad i = 1, \dots, n. \quad (1)$$



Arzhanykh Ivan Semenovitch
(1914–1980)

- **Problem:** find systems that are invariant with respect to canonical transformations
- *Bilinear covariant for system of ordinary differential equations of even order // Doklady of the Academy of Sciences of UzbSSR. 1948. No. 1. P. 5-8. (Russ).*
- *On integration of canonical system of equations in exact differentials // Uspekhi Matem. Nauk. 1953. Vol 3(55). P. 99-104.*

- Canonical systems with rank greater than zero

$$\frac{dq_i}{dt} = \partial_{p_i} H(t, q, p) + \sum_{j=1}^k K_j(t, q, p) \partial_{p_i} H_j(t, q, p), \quad (2)$$

$$\frac{dp_i}{dt} = -\partial_{q_i} H(t, q, p) - \sum_{j=1}^k K_j(t, q, p) \partial_{q_i} H_j(t, q, p), \quad i = 1, \dots, n.$$

- If we put $dt = dt_1$, $K_j dt = dt_{1+j}$, $j = 1, \dots, k$, and $m = k + 1$, then we obtain the multidimensional Hamiltonian system

$$dq_i = \partial_{p_i} H_1(t, q, p) dt_1 + \dots + \partial_{p_i} H_m(t, q, p) dt_m,$$

$$dp_i = -\partial_{q_i} H_1(t, q, p) dt_1 - \dots - \partial_{q_i} H_m(t, q, p) dt_m, \quad i = 1, \dots, n,$$

where the generalized coordinates and momenta $q, p \in \mathbb{R}^n$, $t \in \mathbb{R}^m$, and the Hamiltonians $H_j \in C^2(D)$, $j = 1, \dots, m$, $D \subset \mathbb{R}^{m+2n}$.

- Arzhanykh I.S. *On integration of canonical system of equations in exact differentials* // Uspekhi Matem. Nauk. 1953. Vol 3(55). P. 99-104. (Russian).
- Arzhanykh I.S. *Canonical equations of rank greater than zero*, Tashkent, Academy of Sciences of the Uzbek SSR, 1962. (Russian).

The Poisson theorem

Suppose twice continuously differentiable functions $g_1: D \rightarrow \mathbb{R}$ and $g_2: D \rightarrow \mathbb{R}$ are first integrals on the domain $D \subset \mathbb{R}^{1+2n}$ for Hamiltonian system (1). Then the Poisson bracket

$$g_{12}: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] \text{ for all } (t, q, p) \in D$$

of the functions g_1 and g_2 is also a first integral for system (1).

- for any two functions $u, v \in C^1(D)$, the Poisson bracket is defined by

$$[u, v]: (t, q, p) \rightarrow \sum_{i=1}^n (\partial_{q_i} u(t, q, p) \partial_{p_i} v(t, q, p) - \partial_{p_i} u(t, q, p) \partial_{q_i} v(t, q, p)).$$

- *Remark.* Of course, Poisson's theorem does not always supply further first integrals. In some cases the result is trivial, the Poisson bracket being a constant. In other cases the integral obtained is simply a function of the original integrals. If neither of these two possibilities occurs, however, then the Poisson bracket is a further first integral.



Siméon Denis Poisson (1781–1840)

Carl G.J. Jacobi (1804–1851)

Jacobi (*Lectures on Dynamics*, 1842-1843) referred to Poisson's theorem as *"one of the most remarkable theorems of the whole of integral calculus. In the particular case when $H = T - U$, it is the fundamental theorem of analytical mechanics"*.

After I (Jacobi) discovered this theorem I communicated it to the Academies of Berlin and Paris as an entirely new discovery. But I noticed soon after that this theorem had already been discovered and forgotten for 30 years, because one did not appreciate its real meaning, but had only used it as a lemma in a entirely different problem.

Problem statement:

Our aim is to develop the Poisson method for integral manifolds and partial integrals (or Darboux polynomials) of the Hamiltonian differential system (1) and the generalized Hamiltonian differential system (2). Also, we focus our attention on generalizations of the Poisson method for general ordinary differential systems.

- ① Pranevich, A. F., Poisson Theorem of Building Autonomous Integrals for Autonomous Systems of Total Differential Equations, *Problems of Physics, Mathematics and Technics*, 2016, no. 3, pp. 52–57.
- ② Pranevich, A. F., On Poisson's Theorem of Building First Integrals for Ordinary Differential Systems, *Russian Journal of Nonlinear Dynamics*, 2019, no. 1, pp. 89–98.
- ③ Pranevich, A. F., Partial integrals of autonomous polynomial Hamiltonian ordinary differential systems, *Differential Equations and Control Processes*, 2022, no. 1, pp. 1–63.
- ④ Pranevich, A. F., The Jacobi - Poisson method of building first integrals for system of ordinary differential equations, *Differential Equations and Control Processes*, 2023, no. 4, pp. 125–141.

Definitions:

- A function $g \in C^1(D')$ is called a *first integral on the domain* if

$$\mathfrak{G}g(t,q,p) = 0 \quad \text{for all } (t,q,p) \in D',$$

where the linear differential operator

$$\mathfrak{G}(t,q,p) = \partial_t + \sum_{i=1}^n (\partial_{p_i} H(t,q,p) \partial_{q_i} - \partial_{q_i} H(t,q,p) \partial_{p_i}).$$

- A smooth manifold $g(t,q,p) = 0$ is said to be an *integral manifold* if

$$\mathfrak{G}g(t,q,p) = \Phi(t,q,p), \quad \Phi(t,q,p)|_{g(t,q,p)=0} = 0 \quad \text{for all } (t,q,p) \in D'.$$

- A function $g \in C^1(D')$ is called a *partial integral* if

$$\mathfrak{G}g(t,q,p) = g(t,q,p) M(t,q,p) \quad \text{for all } (t,q,p) \in D',$$

where the polynomial M has the degree $\deg_{(p,q)} M \leq h-2$.

Moreover, the polynomial M is the *cofactor of the partial integral* g .

- $I_{D'}, M_{D'}, J_{D'}.$
- $(g, \Phi) \in M_{D'}, (g, M) \in J_{D'}.$

1.1. Integral manifolds: Hamiltonian system (1)

Theorem 1 (main result).

Suppose $(g_k, \Phi_k) \in M_D$ and $g_k \in C^2(D)$, $k = 1, 2$. Then the function

$$g: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] - \int_{t_0}^t \varphi(\tau) d\tau \quad \text{for all } (t, q, p) \in D'$$

is a first integral on the domain D' of the Hamiltonian differential system (1) if and only if the following identity holds

$$[g_1(t, q, p), \Phi_2(t, q, p)] - [g_2(t, q, p), \Phi_1(t, q, p)] = \varphi(t), \quad \varphi \in C(D').$$

Idea of the proof

- Since $(g_k, \Phi_k) \in M_D$, $k = 1, 2$, we have

$$\mathfrak{G} g_k(t, q, p) = \Phi_k(t, q, p) \quad \text{for all } (t, q, p) \in D, \quad k = 1, 2.$$

- From these identities it follows that

$$\partial_t g_k(t, q, p) = \Phi_k(t, q, p) - [g_k(t, q, p), H(t, q, p)], \quad k = 1, 2.$$

- Using these identities and the properties of Poisson brackets (time derivative, bilinearity, anticommutativity, and Jacobi identity), we obtain the derivative by virtue of system (1):

$$\begin{aligned} \mathfrak{G} [g_1(t, q, p), g_2(t, q, p)] &= \\ &= [g_1(t, q, p), \Phi_2(t, q, p)] - [g_2(t, q, p), \Phi_1(t, q, p)] \quad \text{for all } (t, q, p) \in D. \end{aligned}$$

- We have

$$\mathfrak{G} g(t, q, p) = \mathfrak{G} [g_1(t, q, p), g_2(t, q, p)] - \partial_t \int_{t_0}^t \varphi(\tau) d\tau = 0.$$

Theorem 2 (the generalized Poisson theorem).

Suppose $(g_k, \Phi_k) \in M_D$ and $g_k \in C^2(D)$, $k = 1, 2$. Then the Poisson bracket $[g_1, g_2] \in I_D$ if and only if the following identity holds

$$[g_1(t, q, p), \Phi_2(t, q, p)] = [g_2(t, q, p), \Phi_1(t, q, p)] \text{ for all } (t, q, p) \in D.$$

- If $g_1 \in I_D$ and $(g_2, \Phi_2) \in M_D$, then the Poisson bracket $[g_1, g_2] \in I_D$ if and only if the functions g_1 and Φ_2 are in involution, i.e.

$$[g_1(t, q, p), \Phi_2(t, q, p)] = 0 \text{ for all } (t, q, p) \in D.$$

- If $g_1 \in I_D$ and $g_2 \in I_D$, then from Theorem 2, we obtain the Poisson theorem.

Definitions (for the generalized Hamiltonian system (2)):

- A function $g \in C^1(D')$ is called a *Jacobi last multiplier* if

$$\mathfrak{G} g(t, q, p) = -g(t, q, p) \operatorname{div} \mathfrak{G}(t, q, p) \quad \text{for all } (t, q, p) \in D',$$

where the linear differential operator

$$\begin{aligned} \mathfrak{G}(t, q, p) = & \partial_t + \sum_{i=1}^n \left(\left(\partial_{p_i} H(t, q, p) + \sum_{j=1}^m K_j(t, q, p) \partial_{p_i} H_j(t, q, p) \right) \partial_{q_i} - \right. \\ & \left. - \left(\partial_{q_i} H(t, q, p) + \sum_{j=1}^m K_j(t, q, p) \partial_{q_i} H_j(t, q, p) \right) \partial_{p_i} \right) \quad \forall (t, q, p) \in D. \end{aligned}$$

- The divergence

$$\operatorname{div} \mathfrak{G}(t, q, p) = \sum_{j=1}^m [K_j(t, q, p), H_j(t, q, p)].$$

Integral manifolds: generalized Hamiltonian system (2)

First integrals (main result).

Suppose $(g_k, \Phi_k) \in M_D$ and $g_k \in C^2(D)$, $k = 1, 2$. Then the function

$$g: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] - \int_{t_0}^t \varphi(\tau) d\tau \text{ for all } (t, q, p) \in D'$$

is a first integral on the domain D' of the generalized Hamiltonian differential system (2) if and only if the following identity holds

$$A(t, q, p) + [g_1(t, q, p), \Phi_2(t, q, p)] - [g_2(t, q, p), \Phi_1(t, q, p)] = \varphi(t),$$

where $\varphi \in C(D')$ and the function

$$A(t, q, p) = \sum_{j=1}^m \begin{vmatrix} [g_1(t, q, p), H_j(t, q, p)] & [g_2(t, q, p), H_j(t, q, p)] \\ [g_1(t, q, p), K_j(t, q, p)] & [g_2(t, q, p), K_j(t, q, p)] \end{vmatrix}.$$

Integral manifolds: generalized Hamiltonian system (2)

The Jacobi last multiplier (main result).

Suppose $(g_k, \Phi_k) \in M_D$ and $g_k \in C^2(D)$, $k = 1, 2$. Then the function

$$g: (t, q, p) \rightarrow \exp \left([g_1(t, q, p), g_2(t, q, p)] - \int_{t_0}^t \varphi(\tau) d\tau \right) \quad \text{for all } (t, q, p) \in D'$$

is a Jacobi last multiplier on the domain D' of the generalized Hamiltonian differential system (2) if and only if the following identity holds

$$\begin{aligned} & A(t, q, p) + [g_1(t, q, p), \Phi_2(t, q, p)] - [g_2(t, q, p), \Phi_1(t, q, p)] = \\ & = - \sum_{j=1}^m [K_j(t, q, p), H_j(t, q, p)] + \varphi(t) \quad \forall (t, q, p) \in D', \quad \varphi \in C(D'). \end{aligned}$$

1.2. Partial integrals

In this section, we assume that the Hamiltonian system (1) is polynomial.

Proposition 1.

Let $(g_k, M_k) \in J_D$, $k = 1, 2$. Then the Poisson bracket

$$([g_1, g_2], M_1 + M_2) \in J_D$$

if and only if the following identity holds

$$\frac{[g_1(t, q, p), M_2(t, q, p)]}{[g_2(t, q, p), M_1(t, q, p)]} = \frac{g_1(t, q, p)}{g_2(t, q, p)} \quad \text{for all } (t, q, p) \in D_0 \subset D.$$

Theorem 3.

Suppose the Hamiltonian system (1) has partial integrals

$$(g_k, M_k) \in J_D, \quad k = 1, 2,$$

such that the sum of cofactors

$$M_1(t, q, p) + M_2(t, q, p) = \varphi(t) \quad \text{for all } (t, q, p) \in D, \quad \varphi \in C(T),$$

and the following identity holds

$$\frac{[g_1(t, q, p), M_1(t, q, p)]}{[g_2(t, q, p), M_1(t, q, p)]} = -\frac{g_1(t, q, p)}{g_2(t, q, p)} \quad \text{for all } (t, q, p) \in D_0 \subset D.$$

Then the Hamiltonian system (1) has the first integral

$$g: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] \exp\left(-\int_{t_0}^t \varphi(\tau) d\tau\right) \quad \text{for all } (t, q, p) \in D,$$

where t_0 is arbitrary fixed point from the interval $T \subset \mathbb{R}$.

$$g: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] e^{-\lambda t} \quad \text{for all } (t, q, p) \in D'.$$

Theorem 4.

If the functions $g_1 \in I_{D'}$, $(g_2, M_2) \in J_{D'}$, $g_k \in C^2(D')$, $k=1,2$, such that

$$[g_1(t,q,p), M_2(t,q,p)] = \lambda \quad \forall (t,q,p) \in D', \quad \lambda \in \mathbb{R},$$

then a first integral of the Hamiltonian system (1) is the function

$$g: (t,q,p) \rightarrow \frac{[g_1(t,q,p), g_2(t,q,p)]}{g_2(t,q,p)} - \lambda t \quad \text{for all } (t,q,p) \in D_0.$$

Theorem 5.

If the functions $g_1 \in I_{D'}$, $(g_2, M_2) \in J_{D'}$, $g_k \in C^2(D')$, $k=1,2$, such that

$$[g_1(t,q,p), M_2(t,q,p)] = \alpha M_2(t,q,p) \quad \forall (t,q,p) \in D', \quad \alpha \in \mathbb{R},$$

then a first integral of the Hamiltonian system (1) is the function

$$g: (t,q,p) \rightarrow g_2^\alpha(t,q,p) \exp\left(-\frac{[g_1(t,q,p), g_2(t,q,p)]}{g_2(t,q,p)}\right) \quad \text{for all } (t,q,p) \in D_0.$$

Example.

- The system (1) with the Hamiltonian

$$H: (q,p) \rightarrow q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2 \quad \text{for all } (q,p) \in \mathbb{R}^4, \quad a, b \in \mathbb{R},$$

has the first integral

$$g_1: (q,p) \rightarrow \frac{p_1 - a q_1}{q_2} \quad \text{for all } (q,p) \in D \subset \{(q,p): q_2 \neq 0\}$$

and the partial integral

$$g_2: (q,p) \rightarrow p_2 - b q_2 \quad \text{with the cofactor } M_2(q,p) = 1.$$

- Using Theorem 6 under $\alpha = 0$, we can build the additional first integral of the Hamiltonian system

$$g(q,p) = -\frac{[g_1(q,p), g_2(q,p)]}{g_2(q,p)} = -\frac{\left[\frac{p_1 - a q_1}{q_2}, p_2 - b q_2\right]}{p_2 - b q_2} = \frac{p_1 - a q_1}{q_2^2 (p_2 - b q_2)}$$

where the domain $D_0 \subset \{(q,p): q_2(p_2 - b q_2) \neq 0\}$.

1.3. Applications for general ordinary differential systems

- Consider an n -th order ordinary differential systems

$$\frac{dq_i}{dt} = X_i(t, q), \quad i = 1, \dots, n, \quad (3)$$

where X_i are continuously differentiable on the domain $\mathcal{X} \subset \mathbb{R}^{1+n}$.

- applications of Poisson's theorem to the integration of systems (3) were considered in the works of A. Boole, P. Appel, and M.F. Shulgin.

The Boole Theorem

Suppose $F_1 \in I_{\mathcal{X}'}$ and there are scalar functions $f_i: \mathcal{X}' \rightarrow \mathbb{R}$, $i = 1, \dots, n$, such that the following identities hold

$$\sum_{i=1}^n X_i(q) \partial_{q_i} f_j(q) = \sum_{i=1}^n f_i(q) \partial_{q_i} X_j(q), \quad j = 1, \dots, n.$$

Then the autonomous system (3) has the additional first integral

$$F_2: q \rightarrow \sum_{i=1}^n f_i(q) \partial_{q_i} F_1(q) \quad \text{for all } q \in \mathcal{X}'.$$

Analytical relations between systems

- By Liouville's method, we can build the augmented Hamiltonian system

$$\frac{dq_i}{dt} = \partial_{p_i} H(t, q, p), \quad \frac{dp_i}{dt} = -\partial_{q_i} H(t, q, p), \quad i = 1, \dots, n, \quad (4)$$

with the Hamiltonian

$$H: (t, q, p) \rightarrow \sum_{i=1}^n X_i(t, q) p_i \quad \text{for all } (t, q, p) \in X \times \mathbb{R}^n;$$

- The Hamiltonian system such that the first n equations are the system

$$\frac{dq_i}{dt} = X_i(t, q), \quad i = 1, \dots, n;$$

and the second n equations are the auxiliary system for defining redundant variables $p_1, \dots, p_n$:

$$\frac{dp_i}{dt} = -\sum_{j=1}^n \partial_{q_i} X_j(t, q) p_j, \quad i = 1, \dots, n. \quad (5)$$

Lemma 1.

A continuously differentiable function

$$\widetilde{g}: (t,q,p) \rightarrow g(t,q) \text{ for all } (t,q,p) \in \mathcal{X}' \times \mathbb{R}^n$$

defines an n -cylindrical integral manifold of the augmented Hamiltonian differential system (4) if and only if the scalar function

$$g: (t,q) \rightarrow g(t,q) \text{ for all } (t,q) \in \mathcal{X}'$$

defines an integral manifold of the ordinary differential system (3).

Lemma 2.

Suppose $g \in I_{\mathcal{X}'}^{(3)}$, $(g_1, \Phi_1) \in M_{D'}^{(4)}$, where the domain $D' = \mathcal{X}' \times P$, $P \subset \mathbb{R}^n$. Then, if the function

$$\widetilde{g}_1: (t,q) \rightarrow g_1(t,q,p) \Big|_{p=\partial_q g(t,q)} \text{ for all } (t,q) \in \mathcal{X}'$$

defines a manifold, then this manifold is an integral manifold of the ordinary differential system (3).

Theorem 6 (Integral manifolds).

Suppose $g \in I_{\mathcal{X}'}^{(3)}$ and $(g_k, \Phi_k) \in M_{D'}^{(4)}$, $g_k \in C^2(D')$, $k = 1, 2$, such that the following identity holds

$$[g_1(t, q, p), \Phi_2(t, q, p)] = [g_2(t, q, p), \Phi_1(t, q, p)] \quad \text{for all } (t, q, p) \in D'.$$

Then the Poisson bracket

$$\tilde{g}_{12}: (t, q) \rightarrow [g_1(t, q, p), g_2(t, q, p)]|_{p=\partial_q g(t, q)} \quad \text{for all } (t, q) \in \mathcal{X}'$$

is a first integral of the ordinary differential system (3).

- Suppose g_1 and g_2 are infinitely continuously differentiable functions with respect to the variable q . Then, by Theorem 6, the functions

$$\tilde{g}_{12,k}: (t, q) \rightarrow [g_1(t, q, p), g_2(t, q, p)]|_{p=\partial_q \tilde{g}_{12,k-1}(t, q)} \quad \text{for all } (t, q) \in \mathcal{X}',$$

$$k = 1, 2, \dots, \quad (\tilde{g}_{12,0} = g),$$

are also first integrals of the ordinary differential systems (3).

Theorem 7.

Suppose $g_1 \in I_{\mathcal{X}'}^{(3)}$, $g_1 \in C^2(\mathcal{X}')$, and $(g_2, \Phi_2) \in M_{D'}^{(4)}$, $g_2 \in C^2(D')$, such that the following identity holds

$$[g_1(t, q), \Phi_2(t, q, p)] = 0 \quad \text{for all } (t, q, p) \in D'.$$

Then a first integral of the differential system (3) is the Poisson bracket

$$g_{12}^*: (t, q) \rightarrow [g_1(t, q), g_2(t, q, p)]|_{p=\partial_q g_1(t, q)} \quad \text{for all } (t, q) \in \mathcal{X}'. \quad (5)$$

Theorem 8 (the generalized Poisson theorem), Shulgin (1958).

If $g_1 \in I_{\mathcal{X}'}^{(3)}$, $g_1 \in C^2(\mathcal{X}')$, and $g_2 \in I_{D'}^{(4)}$, $g_2 \in C^2(D')$, then the Poisson bracket (5) is a first integral of system (3).

Shulgin M. F. *On some Differential Equations of Analytical Dynamics and their Integration*, Tashkent: Middle Asian State University (SAGU), 1958.

Theorem 9 (Partial integrals).

If $g \in I_{\mathcal{X}'}^{(3)}$ and $(g_k, M_k) \in J_{D'}^{(4)}$, $g_k \in C^2(D')$, $k = 1, 2$, such that

$$M_1(t, q, p) + M_2(t, q, p) = \varphi(t) \quad \text{for all } (t, q, p) \in D, \quad \varphi \in C(T),$$

and

$$\frac{[g_1(t, q, p), M_1(t, q, p)]}{[g_2(t, q, p), M_1(t, q, p)]} = - \frac{g_1(t, q, p)}{g_2(t, q, p)} \quad \text{for all } (t, q, p) \in D_0 \subset D,$$

then a first integral of the polynomial differential system (3) is the function

$$\tilde{g}_{12}: (t, q) \rightarrow [g_1(t, q, p), g_2(t, q, p)]|_{p=\partial_q g(t, q)} \exp\left(- \int_{t_0}^t \varphi(\tau) d\tau\right).$$

- $\tilde{g}_{12}: (t, q) \rightarrow [g_1(t, q, p), g_2(t, q, p)]|_{p=\partial_q g(t, q)} e^{-\lambda t}$ for all $(t, q) \in \mathcal{X}'$.
- $\tilde{g}_{12}: (t, q) \rightarrow [g_1(t, q, p), g_2(t, q, p)]|_{p=\partial_q g(t, q)}$ for all $(t, q) \in \mathcal{X}'$.

Theorem 10.

Suppose $g_1 \in I_{\mathcal{X}'}^{(3)}$ and $(g_2, M_2) \in J_{D'}^{(4)}$ such that

$$[g_1(t, q, p), M_2(t, q, p)] = \lambda \quad \forall (t, q, p) \in D', \quad \lambda \in \mathbb{R}.$$

Then a first integral of the polynomial differential system (3) is the function

$$\tilde{g}_{12}: (t, q) \rightarrow \frac{[g_1(t, q), g_2(t, q, p)]}{g_2(t, q, p)} \Big|_{p=\partial_q g_1(t, q)} - \lambda t \quad \text{for all } (t, q) \in \mathcal{X}_0.$$

Theorem 11.

Suppose $g_1 \in I_{\mathcal{X}'}^{(3)}$ and $(g_2, M_2) \in J_{D'}^{(4)}$ such that

$$[g_1(t, q), M_2(t, q, p)] = \alpha M_2(t, q, p) \quad \forall (t, q, p) \in D', \quad \alpha \in \mathbb{R},$$

Then a first integral of the polynomial differential system (3) is the function

$$g: (t, q, p) \rightarrow g_2^\alpha(t, q, p) \exp\left(-\frac{[g_1(t, q), g_2(t, q, p)]}{g_2(t, q, p)}\right) \quad \text{for all } (t, q, p) \in D_0.$$

Multidimensional Hamiltonian systems

$$dq_i = \sum_{j=1}^m \partial_{p_i} H_j(t,q,p) dt_j, \quad dp_i = - \sum_{j=1}^m \partial_{q_i} H_j(t,q,p) dt_j, \quad i = 1, \dots, n;$$

- $\mathfrak{G}_j g(t,q,p) = \Phi_j(t,q,p), \quad \Phi_j(t,q,p)|_{g(t,q,p)=0} = 0, \quad j = 1, \dots, m.$

Theorem 12 (main result).

Suppose $(g_k, \Phi^k) \in M_D$, where the vector-functions $\Phi^k = (\Phi_1^k, \dots, \Phi_m^k)$, $k = 1, 2$. Then the Poisson bracket

$$g: (t,q,p) \rightarrow [g_1(t,q,p), g_2(t,q,p)] \quad \text{for all } (t,q,p) \in D'$$

is a first integral on the domain D' of the multidimensional Hamiltonian system if and only if the following identity holds

$$[g_1(t,q,p), \Phi_j^2(t,q,p)] = [g_2(t,q,p), \Phi_j^1(t,q,p)], \quad j = 1, \dots, m.$$

Completely Integrable Multidimensional Hamiltonian systems

- The linear differential operators

$$\mathfrak{G}_j(t, q, p) = \partial_{t_j} + \sum_{i=1}^n (\partial_{p_i} H_j(t, q, p) \partial_{q_i} - \partial_{q_i} H_j(t, q, p) \partial_{p_i}), \quad j = 1, \dots, m;$$

- the Frobenius conditions of completely integrable systems

$$[\mathfrak{G}_j(t, q, p), \mathfrak{G}_\xi(t, q, p)] = 0, \quad j, \xi = 1, \dots, m.$$

Theorem 13 (main result).

Suppose $(g_k, \Phi^k) \in M_D$, where the vector-functions $\Phi^k = (\Phi_1^k, \dots, \Phi_m^k)$, $k = 1, 2$. Then the function

$$g: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] - \int \sum_{j=1}^m \varphi_j(t) dt_j \quad \text{for all } (t, q, p) \in D'$$

is a first integral of the completely integrable multidimensional Hamiltonian system if and only if the following identity holds

$$[g_1(t, q, p), \Phi_j^2(t, q, p)] - [g_2(t, q, p), \Phi_j^1(t, q, p)] = \varphi_j(t), \quad j = 1, \dots, m.$$

THANK YOU FOR YOUR ATTENTION!