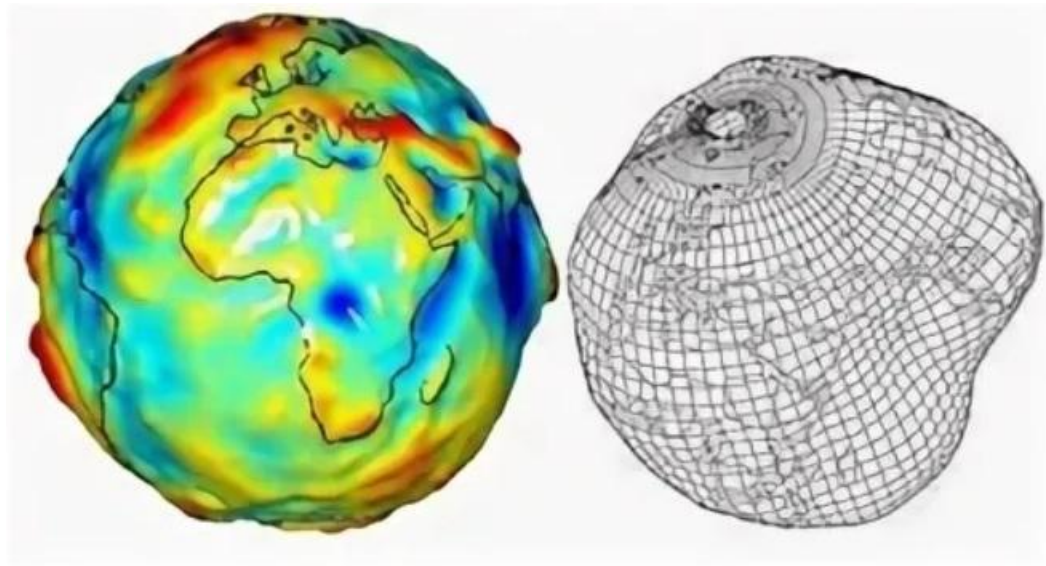


Total Effect of the Earth Gravitational Field on the Tide-Generating Potential

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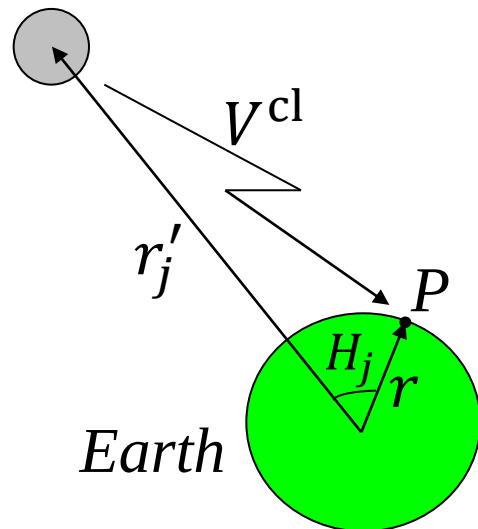
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Earth tide-generating potential (TGP)

The “classical” representation of the TGP at a point $P(r, \phi, \lambda)$ located on (or above) the Earth’s surface is

$$V^{\text{cl}}(r, \phi, \lambda) = \sum_j \mu'_j \sum_{l=2}^{\infty} \frac{r^l}{r_j'^{l+1}} P_l(\cos H_j),$$

*External body j
(e.g., the Moon)*



where V^{cl} – the value of classical TGP at P ,
 r – geocentric distance of P ,
 ϕ – geocentric latitude of P ,
 λ – longitude of P east of Greenwich,
 μ'_j – gravitational parameter of body j ,
 r'_j – geocentric distance of body j ,
 H_j – the angle: P – geocenter – body j ,
 P_l – Legendre polynomial of degree l .

Both Earth and body j are assumed to be spheres of uniform density.

§ 1.32. Спутниковый вариант задачи трех тел

Effect of the “third bodies” on a planet’s satellite motion

Рассмотрим движение спутника P под действием притяжения центральной планеты S с массой m_0 и некоторого внешнего тела с массой m' . Будем предполагать, что масса спутника пренебрежимо мала по сравнению с массами m_0 и m' .

Возьмем прямоугольную систему координат $Sxyz$ с началом в центре масс тела S и фиксированным направлением осей. Тогда дифференциальные уравнения движения

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Подставляя полученное выражение в (1.32.3) и отбрасывая при этом член, не зависящий от координат тела P , находим

Perturbation function

$$R = \frac{fm'}{r'} \sum_{n=2}^{\infty} \left(\frac{r}{r'} \right)^n P_n(\cos H). \quad (1.32.5)$$

Разложение (1.32.5) сходится для всех $r < r'$. Во мно-

The Earth TGP is one of the most important potentials of celestial mechanics. It is used, e.g., in theories of Earth nutation, developments of solid Earth and ocean tides, analysis of geodetic and gravimetric observations, etc.



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Software and Tools

Tidal Analysis and Prediction Software

- **ETERNA 3.40** (Wenzel, 1996)
- **ETERNA 3.40 updates**, to include ksm03 tidal potential (Kudryavtsev, 2004)
- The latest version of "**etddt.dat**" and "**etpolut1.dat**" files to be put in the "commdat" directory can be found [here](#).

The history of the Earth TGP developments.

I. Classical approach

- The conception of the tide-generating potential was introduced by P.-S. Laplace in 1790, he also made the first developments of the Earth TGP in harmonic series;
- Thompson (1868);
- Ferrel (1874);
- Darwin (1883);
- Doodson (1921);
- Cartwright & Tayler (1971); Cartwright & Edden (1973);
- Büllsfeld (1985);
- Xi (1987, 1989);
- Tamura (1987, 1995);

Then the following question was raised...

**А ЕСЛИ ЗЕМЛЯ
НА САМОМ ДЕЛЕ
НЕ КРУГЛАЯ?**

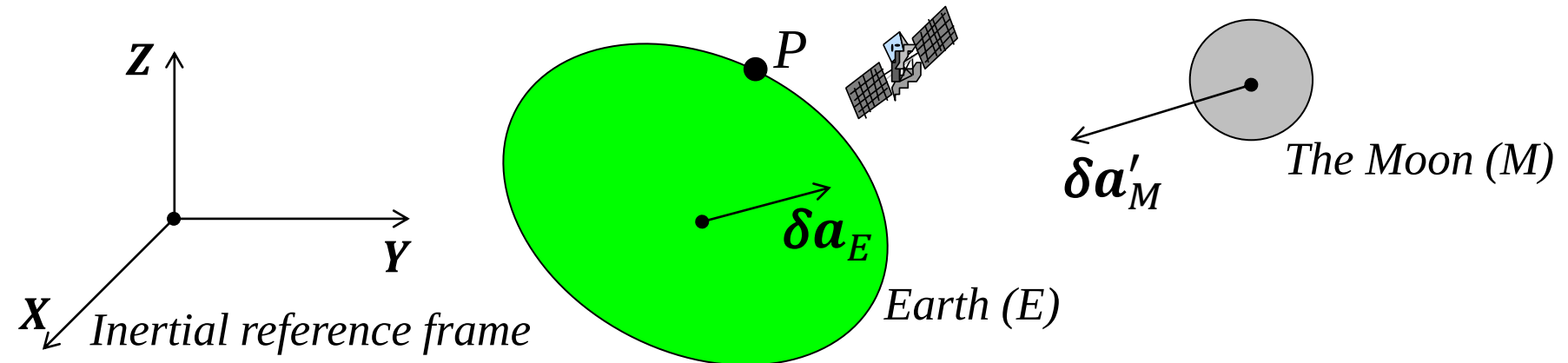


Is there any effect of the Earth's figure on the TGP?

Yes, it is! Why it is?

Let us consider the real Earth's figure (“geoid”) and a point-mass Moon:

- The oblate part of the Earth causes an additional acceleration of the Moon in an inertial reference frame: $\delta \mathbf{a}'_M$;
- According to Newton's third law, the geocenter gets an additional acceleration as well: $\delta \mathbf{a}_E = -\frac{\mu'_M}{\mu_E} \delta \mathbf{a}'_M$;
- As a result, there is an *inertial* acceleration of any point in space *relative to the moving geocenter*, e.g., for a satellite (it is taken into account in motion theories) or point P on the Earth surface (“tide”).



The history of the Earth TGP developments.

II. Taking into account the geopotential

a) The effect of the Earth's *polar flattening* (or zonal harmonic C_{20} in the geopotential) was studied in an analytical solution (Wilhelm 1983). The relevant terms were calculated and included in the TGP catalogs:

- Hartmann & Wenzel (1994, 1995): *HW95*;
- Roosbeek (1996): *RATGP95*;
- Kudryavtsev (2004, 2007a): *KSM03*.

b) The effect of the Earth's both *polar and equatorial flattening* (or harmonics C_{20} and C_{22}/S_{22} of the geopotential expansion) was considered in an analytical solution and the TGP catalog by:

- Kudryavtsev (2025a, 2025b): *KSM03ext*

Here we present an analytical solution for the case of *arbitrary* geopotential coefficients C_{nm}/S_{nm} and find the relevant TGP terms.

The TGP part due to the complete geopotential (1/4)

1. The acceleration of the Moon $\ddot{X}'$ along X -axis of an inertial reference frame (IRF), which is caused by the complete geopotential, can be represented by the following standard expression:

$$\ddot{X}' = -\mu_E \frac{x'}{r'^3} + \frac{\partial}{\partial x'} \left\{ \frac{\mu_E}{r'} \sum_{n=2}^{\infty} \sum_{m=0}^n \left(\frac{R_{eq}}{r'} \right)^n P_{nm}(\sin \varphi') [\textcolor{red}{c}_{nm} \cos m\lambda' + \textcolor{red}{s}_{nm} \sin m\lambda'] \right\},$$

where μ_E is the gravitational parameter of Earth, r' , φ' , λ' - are, respectively, the geocentric distance, latitude and east longitude of the Moon in the ITRS; R_{eq} is the mean Earth's equatorial radius; P_{nm} are associated Legendre functions of degree n and order m ;

$$x' \equiv X' - X_E,$$

where X' , X_E are, respectively, X -coordinates of the Moon and Earth in the IRF; and $\textcolor{red}{c}_{nm}$, $\textcolor{red}{s}_{nm}$ are coefficients of the geopotential expansion in spherical harmonic functions, also determined in the ITRS.

The TGP part due to the complete geopotential (2/4)

2. The total acceleration of Earth along X -axis of the IRF is:

$$\ddot{X}_E = -\frac{\mu'}{\mu_E} \ddot{X}' = \mu' \frac{x'}{r'^3} - \ddot{X}_E^{\text{gp}}$$

the Earth's acceleration due to the interaction of the Moon with the non-spherical part of the geopotential

where $\ddot{X}_E^{\text{gp}} \equiv \frac{\partial}{\partial x'} \left\{ \frac{\mu'}{r'} \sum_{n=2}^{\infty} \sum_{m=0}^n \left(\frac{R_{\text{eq}}}{r'} \right)^n P_{nm}(\sin \varphi') [c_{nm} \cos m\lambda' + s_{nm} \sin m\lambda'] \right\}.$

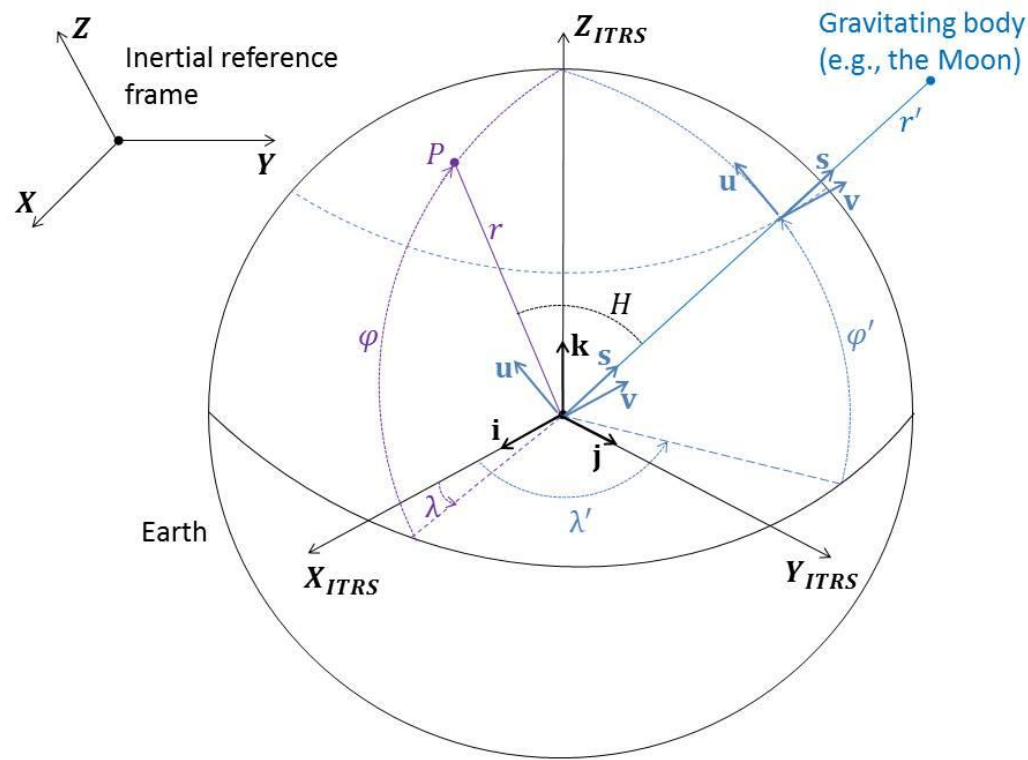
3. The acceleration of P along X -axis of the IRF relative to the geocenter is:

$$\ddot{x} \equiv \ddot{X}_P - \ddot{X}_E = \mu' \left(\frac{x' - x}{\Delta^3} - \frac{x'}{r'^3} \right) + \ddot{X}_E^{\text{gt}} = \frac{\partial}{\partial x} V^{\text{cl}} + \ddot{X}_E^{\text{gt}}, \quad \text{where } x \equiv X_P - X_E,$$

$$V^{\text{cl}} = \frac{\mu'}{r'} \sum_{l=2}^{\infty} \left(\frac{r}{r'} \right)^l P_l(\cos H) \quad \text{is the “classical” representation of the TGP,}$$

Δ is the distance from P to the Moon, H is the angle “ P – Earth – the Moon”.

The TGP part due to the complete geopotential (3/4)



4. The same for other two coordinates, so $\ddot{\mathbf{R}}_E^{\text{gp}} \equiv (\ddot{X}_E^{\text{gp}}, \ddot{Y}_E^{\text{gp}}, \ddot{Z}_E^{\text{gp}})$, $\mathbf{r} = (x, y, z)$, and $\ddot{\mathbf{r}} = \nabla_{\mathbf{r}} V^{\text{cl}} + \nabla_{\mathbf{r}} (\mathbf{r} \ddot{\mathbf{R}}_E^{\text{gt}}) = \nabla_{\mathbf{r}} V^{\text{cl}} + \nabla_{\mathbf{r}} V^{\text{gt}}$, where $V^{\text{gt}} \equiv \mathbf{r} \ddot{\mathbf{R}}_E^{\text{gt}}$ is the required TGP.

5. Next, the $\ddot{\mathbf{R}}_E^{\text{gp}}$ vector is calculated in a geocentric coordinate frame defined by three unit vectors ($\mathbf{s}$, $\mathbf{u}$, $\mathbf{v}$). Then, $\mathbf{s}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{r}$ are expressed in the ITRS (defined by the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$), and a scalar product $\mathbf{r}_{\text{ITRS}} \ddot{\mathbf{R}}_{E,\text{ITRS}}^{\text{gt}}$ is found.

The TGP part due to the complete geopotential (4/4)

6. Therefore, the general form of the TGP part caused by the complete geopotential is as follows:

$$V^{\text{gp}}(r, \varphi, \lambda) = -\frac{\mu'}{R_{\text{eq}}^2} \sum_{n=2}^{\infty} \left(\frac{R_{\text{eq}}}{r'} \right)^{n+2} \sum_{m=0}^n \left[\begin{aligned} & (n+1)P_{nm}(\sin \varphi') C_{nm} \mathbf{r}_{\text{ITRS}} \mathbf{S}_{\text{ITRS}} \\ & - \frac{\partial P_{nm}(\sin \varphi')}{\partial \varphi'} C_{nm} \mathbf{r}_{\text{ITRS}} \mathbf{u}_{\text{ITRS}} \\ & + \frac{m}{\cos \varphi'} P_{nm}(\sin \varphi') S_{nm} \mathbf{r}_{\text{ITRS}} \mathbf{v}_{\text{ITRS}} \end{aligned} \right],$$

where

$$C_{nm} \equiv c_{nm} \cos m\lambda' + s_{nm} \sin m\lambda'; \quad S_{nm} \equiv c_{nm} \sin m\lambda' - s_{nm} \cos m\lambda',$$

$$\mathbf{r}_{\text{ITRS}} \mathbf{S}_{\text{ITRS}} = r [\sin \varphi \sin \varphi' + \cos \varphi \cos \varphi' \cos(\lambda - \lambda')],$$

$$\mathbf{r}_{\text{ITRS}} \mathbf{u}_{\text{ITRS}} = r [\sin \varphi \cos \varphi' - \cos \varphi \sin \varphi' \cos(\lambda - \lambda')],$$

$$\mathbf{r}_{\text{ITRS}} \mathbf{v}_{\text{ITRS}} = r \cos \varphi \sin(\lambda - \lambda').$$

However, for practical applications of this equation we should obtain an explicit separation between the coordinates of the external bodies and those of an *arbitrary* point P , where the TGP will be calculated.

The TGP part due to the complete geopotential.

The final form

$$V^{\text{GP}}(r, \varphi, \lambda) = \frac{r}{R_{\text{eq}}} \sum_{n=2}^{\infty} \sum_{m=0}^n \{P_{10}(\sin \varphi) C_{10_{nm}} + P_{11}(\sin \varphi) [C_{11_{nm}} \cos \lambda + S_{11_{nm}} \sin \lambda]\},$$

where

$$C_{10_{nm}} = -\frac{\mu'}{R_{\text{eq}}} \left(\frac{R_{\text{eq}}}{r'}\right)^{n+2} (n-m+1) P_{n+1,m}(\sin \varphi') [\textcolor{red}{C}_{nm} \cos m\lambda' + \textcolor{red}{S}_{nm} \sin m\lambda'],$$

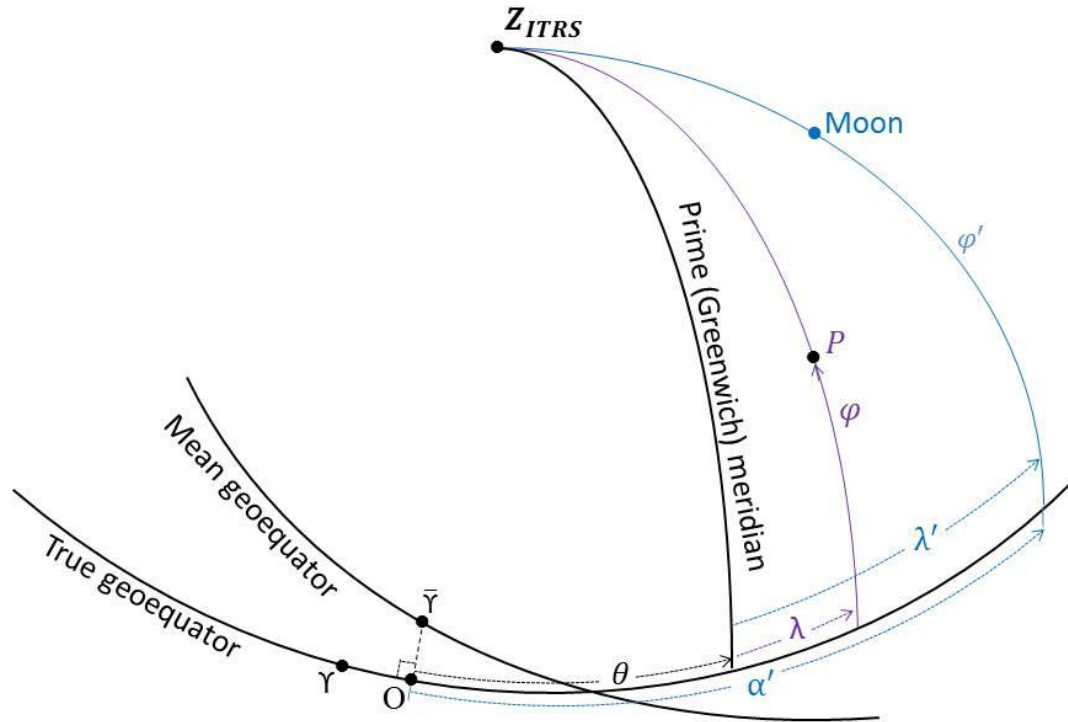
$$C_{11_{nm}} = -\frac{\mu'}{2R_{\text{eq}}} \left(\frac{R_{\text{eq}}}{r'}\right)^{n+2} \times \left\{ \begin{aligned} &P_{n+1,m+1}(\sin \varphi') [\textcolor{red}{C}_{nm} \cos(m+1)\lambda' + \textcolor{red}{S}_{nm} \sin(m+1)\lambda'] - \\ &(n-m+1)(n-m+2) P_{n+1,m-1}(\sin \varphi') [\textcolor{red}{C}_{nm} \cos(m-1)\lambda' + \textcolor{red}{S}_{nm} \sin(m-1)\lambda'] \end{aligned} \right\},$$

$$S_{11_{nm}} = -\frac{\mu'}{2R_{\text{eq}}} \left(\frac{R_{\text{eq}}}{r'}\right)^{n+2} \times \left\{ \begin{aligned} &P_{n+1,m+1}(\sin \varphi') [\textcolor{red}{C}_{nm} \sin(m+1)\lambda' - \textcolor{red}{S}_{nm} \cos(m+1)\lambda'] - \\ &(n-m+1)(n-m+2) P_{n+1,m-1}(\sin \varphi') [\textcolor{red}{S}_{nm} \cos(m-1)\lambda' - \textcolor{red}{C}_{nm} \sin(m-1)\lambda'] \end{aligned} \right\}.$$

Functions $C_{10_{nm}}, C_{11_{nm}}, S_{11_{nm}}$ include all data related to external bodies. They do not depend on r, φ, λ and can be developed to harmonic series.

Using a new “slowly changing” variable

Let us introduce variable $\alpha' = \lambda' + \theta$, where $\theta \approx GMST$ (UT1, TDB).



Using “slow” variable α' instead of λ' in expressions for coefficients $C_{10nm}, C_{11nm}, S_{11nm}$ allows one to essentially decrease the sample step when tabulating the coefficients over a long-term interval (we shall develop them in harmonic series then). To calculate α' the TDB is only used (but not the much less stable UT1 which is necessary to obtain λ').

The TGP part due to geopotential if to use α' (1/2)

$$V^{\text{gp}}(r, \varphi, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^n V_{nm}^{\text{gp}}(r, \varphi, \lambda),$$

where

$$V_{nm}^{\text{gp}}(r, \varphi, \lambda) = \frac{r}{R_{\text{eq}}} \left\{ \begin{aligned} &\bar{P}_{10}(\sin \varphi) [\hat{C}_{10_{nm}} \cos m\theta + \hat{S}_{10_{nm}} \sin m\theta] \\ &+ \bar{P}_{11}(\sin \varphi) [\hat{C}_{11,1_{nm}} \cos(\lambda + m_+\theta) + \hat{S}_{11,1_{nm}} \sin(\lambda + m_+\theta)] \\ &+ \bar{P}_{11}(\sin \varphi) [\hat{C}_{11,2_{nm}} \cos(\lambda - m_-\theta) + \hat{S}_{11,2_{nm}} \sin(\lambda - m_-\theta)] \end{aligned} \right\},$$

$$m_+ \equiv m + 1, \quad m_- \equiv m - 1,$$

and *normalized* values of both associated Legendre functions $\bar{P}_{nm}$ and geopotential coefficients $\bar{c}_{nm}$, $\bar{s}_{nm}$ are used:

$$\bar{P}_{nm} = N_{nm} P_{nm}, \quad \bar{c}_{nm} = \frac{c_{nm}}{N_{nm}}, \quad \bar{s}_{nm} = \frac{s_{nm}}{N_{nm}},$$

$$N_{nm} = \sqrt{\frac{\delta_m (2n+1)(n-m)!}{(n+m)!}}, \quad \delta_m = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m \neq 0 \end{cases},$$

The TGP part due to geopotential if to use α' (2/2)

$$\begin{pmatrix} \hat{C}_{10nm} \\ \hat{S}_{10nm} \end{pmatrix} = -\frac{\mu'}{R_{eq}} \left(\frac{R_{eq}}{r'} \right)^{n+2} \sqrt{\frac{2n+1}{3(2n+3)} (n-m_-)(n+m_+)} \bar{P}_{n+1,m}(\sin \varphi') \times \\ \times \begin{pmatrix} \bar{c}_{nm} \cos m\alpha' + \bar{s}_{nm} \sin m\alpha' \\ \bar{c}_{nm} \sin m\alpha' - \bar{s}_{nm} \cos m\alpha' \end{pmatrix},$$

$$\begin{pmatrix} \hat{C}_{11,1nm} \\ \hat{S}_{11,1nm} \end{pmatrix} = -\frac{\mu'}{2R_{eq}} \left(\frac{R_{eq}}{r'} \right)^{n+2} \sqrt{\frac{\delta_m(2n+1)}{6(2n+3)} (n+m_+)(n+m_++1)} \bar{P}_{n+1,m_+}(\sin \varphi') \times \\ \times \begin{pmatrix} \bar{c}_{nm} \cos m_+\alpha' + \bar{s}_{nm} \sin m_+\alpha' \\ \bar{c}_{nm} \sin m_+\alpha' - \bar{s}_{nm} \cos m_+\alpha' \end{pmatrix},$$

$$\begin{pmatrix} \hat{C}_{11,2nm} \\ \hat{S}_{11,2nm} \end{pmatrix} = \frac{\mu'}{2R_{eq}} \left(\frac{R_{eq}}{r'} \right)^{n+2} \sqrt{\frac{\delta_m(2n+1)}{3\delta_{m_-}(2n+3)} (n-m_-)(n-m_-+1)} \bar{P}_{n+1,m_-}(\sin \varphi') \times \\ \times \begin{pmatrix} \bar{c}_{nm} \cos m_-\alpha' + \bar{s}_{nm} \sin m_-\alpha' \\ -\bar{c}_{nm} \sin m_-\alpha' + \bar{s}_{nm} \cos m_-\alpha' \end{pmatrix}.$$

Development of the TGP in harmonic series

The “TGP coefficients” $\hat{C}_{10_{nm}}, \hat{S}_{10_{nm}}, \hat{C}_{11,1_{nm}}, \hat{S}_{11,1_{nm}}, \hat{C}_{11,2_{nm}}, \hat{S}_{11,2_{nm}}$ include all data related to external bodies. Their numerical values are calculated by using planetary ephemerides DE441 (Park et al. 2021) over 30,000+ years (-13200 – 17191) with a sampling step of one day. Next, by employing a modification of the spectral analysis method (Kudryavtsev 2004, 2007b) the thus tabulated values of the coefficients are approximated by harmonic series, where both amplitudes and arguments of the series’ terms are high-degree polynomials of time:

$$\hat{C}(t) = \sum_i [(A_{0i}^{cc} + A_{1i}^{cc}t + \dots + A_{5i}^{cc}t^5) \cos \omega_i^c(t) + (A_{0i}^{cs} + A_{1i}^{cs}t + \dots + A_{5i}^{cs}t^5) \sin \omega_i^c(t)],$$

$$\hat{S}(t) = \sum_i [(A_{0i}^{sc} + A_{1i}^{sc}t + \dots + A_{5i}^{sc}t^5) \cos \omega_i^s(t) + (A_{0i}^{ss} + A_{1i}^{ss}t + \dots + A_{5i}^{ss}t^5) \sin \omega_i^s(t)],$$

$$\omega_i^c(t) = v_{1i}^c t + v_{2i}^c t^2 + \dots + v_{6i}^c t^6, \quad \omega_i^s(t) = v_{1i}^s t + v_{2i}^s t^2 + \dots + v_{6i}^s t^6.$$

Results: catalog of new TGP terms, format *K25*

The catalog of obtained TGP terms in the *K25* format includes amplitudes $A_0^{cc}, A_1^{cc}, \dots, A_6^{cs}$ ($A_0^{sc}, A_1^{sc}, \dots, A_6^{ss}$) and frequencies $\nu_1^c, \nu_2^c, \dots, \nu_6^c$ ($\nu_1^s, \nu_2^s, \dots, \nu_6^s$) of every term of the final series.

The minimum threshold for amplitudes is set equal to $1 \times 10^{-9} \text{ m}^2 \text{ s}^{-2}$. (In previous *HW95*, *KSM03* expansions of the TGP it is $1 \times 10^{-8} \text{ m}^2 \text{ c}^{-2}$).

The number of the TGP terms due to harmonics $\bar{c}_{nm}, \bar{s}_{nm}$ (format *K25*)

<i>n</i>	<i>m</i>			
	0	1	2	3
2	834 (321 ^a)	-	156 (40 ^b)	
3	5	12	6	8

^a The number of terms in the *KSM03* development (Kudryavtsev 2004);

^b The number of terms in the *KSM03ext* development (Kudryavtsev 2025a).

When $n \geq 4$, no term of amplitude above the chosen threshold is found. In total we obtained 1021 terms of the Earth TGP due to the geopotential.

The catalog: [http://lnfm1.sai.msu.ru/neb/ksm/tgp/k25_\(format-k25\).zip](http://lnfm1.sai.msu.ru/neb/ksm/tgp/k25_(format-k25).zip).

Results: catalog of new TGP terms, format *HW95* (1/3)

Format *HW95* (Hartmann & Wenzel 1995, 1996) represents the TGP in the *rotating* ITRS. It is used in some popular software packages for analyzing the tides measurements, e.g., ETERNA (Wenzel 1996).

In a generalized version of the *HW95* format the TGP is represented as:

$$V^{\text{gp}}(r, \varphi, \lambda, T_{UT1}, T) = \frac{r}{R_{\text{eq}}} \sum_{k=0}^1 \bar{P}_{1k}(\sin \phi) \sum_{i=1}^{N_k} \left[\begin{aligned} &(C_{0ki} + C_{1ki}T + C_{2ki}T^2) \cos B_{ki}(T_{UT1}, T) + \\ &+ (S_{0ki} + S_{1ki}T + S_{2ki}T^2) \sin B_{ki}(T_{UT1}, T) \end{aligned} \right],$$

where

T is TDB from epoch J2000.0, T_{UT1} is UT1 from epoch J2000.0, $C_{0ki}, C_{1ki}, \dots, S_{2ki}$ are numerical coefficients obtained in the solution, B_{ki} are arguments of the series' terms, which are functions of the east longitude of the observation point P and 14 fundamental variables:

Results: catalog of new TGP terms, format *HW95* (2/3)

$$B_{ki}(T_{UT1}, T) = k\lambda + k_{1_{ki}}\tau(T_{UT1}, T) + k_{2_{ki}}s(T) + k_{3_{ki}}h(T) + k_{4_{ki}}p(T) + \\ + k_{5_{ki}}N'(T) + k_{6_{ki}}p_s(T) + k_{7_{ki}}\lambda_1(T) + \dots + k_{13_{ki}}\lambda_8(T) + k_{14_{ki}}p_A(T)$$

where

$k_{1_{ki}}, k_{2_{ki}}, \dots, k_{14_{ki}}$ are some integer multipliers;

τ is the mean lunar time at P ,

s is the mean lunar longitude,

h is the mean solar longitude,

p is the mean longitude of lunar perigee,

N' is the negative mean longitude of ascending node of the Moon,

p_s is the mean longitude of solar perigee,

p_A is the general precession in longitude,

$\lambda_1, \dots, \lambda_8$ are the mean longitudes of seven major planets (all but Earth).

Results: catalog of new TGP terms, format *HW95* (3/3)

The catalog includes both amplitudes $C_0, S_0, C_1, S_1, C_2, S_2$ and indexes $k, k_1, k_2, \dots, k_{14}$ for every term of the series.

The minimum threshold for amplitudes is equal to $1 \times 10^{-9} \text{ m}^2 \text{ s}^{-2}$.

The number of the TGP terms due to harmonics $\bar{c}_{nm}, \bar{s}_{nm}$ (format *HW95*)

n	m			
	0	1	2	3
2	722 (365 ^a / 352 ^b)	-	90 (19 ^c)	
3	4	6	3	4

^a The number of terms in the *KSM03* development (Kudryavtsev 2007a);

^b The number of terms in the *HW95* development (Hartmann and Wenzel 1995);

^c The number of terms in the *KSM03ext* development (Kudryavtsev 2025a).

In total we obtained 829 TGP terms due to the geopotential represented in the generalized *HW95* format.

The catalog is available on:

[http://lnfm1.sai.msu.ru/neb/ksm/tgp/k25_\(format-hw95gen\).dat](http://lnfm1.sai.msu.ru/neb/ksm/tgp/k25_(format-hw95gen).dat).

Special case 1: geopotential harmonics of order $m = 0$

$$V_{n0}^{\text{gp}}(r, \varphi, \lambda) = \frac{r}{R_{\text{eq}}} \left\{ \bar{P}_{10}(\sin \varphi) \hat{C}_{10_{n0}} + \bar{P}_{11}(\sin \varphi) [\hat{C}_{11_{n0}} \cos(\lambda + \theta) + \hat{S}_{11_{n0}} \sin(\lambda + \theta)] \right\},$$

where

$$\hat{C}_{10_{n0}} = \frac{\mu'}{R_{\text{eq}}} \bar{J}_n \left(\frac{R_{\text{eq}}}{r'} \right)^{n+2} (n+1) \sqrt{\frac{2n+1}{3(2n+3)}} \bar{P}_{n+1,0}(\sin \varphi'), \quad (\bar{J}_n \equiv -\bar{c}_{n0})$$

$$\begin{pmatrix} \hat{C}_{11_{n0}} \\ \hat{S}_{11_{n0}} \end{pmatrix} = \frac{\mu'}{R_{\text{eq}}} \bar{J}_n \left(\frac{R_{\text{eq}}}{r'} \right)^{n+2} \sqrt{\frac{(n+1)(n+2)(2n+1)}{6(2n+3)}} \bar{P}_{n+1,1}(\sin \varphi') \begin{pmatrix} \cos \alpha' \\ \sin \alpha' \end{pmatrix}.$$

If a constant term appears in the harmonic development of $\hat{C}_{10_{n0}}$, then there is a part of the TGP which does *not* depend on time nor the Earth's rotation angle θ . It leads to a “*permanent tide*” on the Earth's surface. The amplitude of this tide is a function of an observation point latitude φ . The classic TGP leads to some permanent tides depending on φ too, and all of them appear due to TGP terms of an *even* degree l (2, 4, ...). Here we get a permanent tide due to TGP terms of an *odd* degree ($l = 1$).

Special case 2: geopotential harmonics of order $m = 1$

$$V_{n1}^{\text{gp}}(r, \varphi, \lambda) = \frac{r}{R_{\text{eq}}} \left\{ \begin{aligned} &\bar{P}_{10}(\sin \varphi) [\hat{C}_{10n1} \cos \theta + \hat{S}_{10n1} \sin \theta] \\ &+ \bar{P}_{11}(\sin \varphi) [\hat{C}_{11,1n1} \cos(\lambda + 2\theta) + \hat{S}_{11,1n1} \sin(\lambda + 2\theta)] \\ &+ \bar{P}_{11}(\sin \varphi) [\hat{C}_{11,2n1} \cos \lambda + \hat{S}_{11,2n1} \sin \lambda] \end{aligned} \right\},$$

where

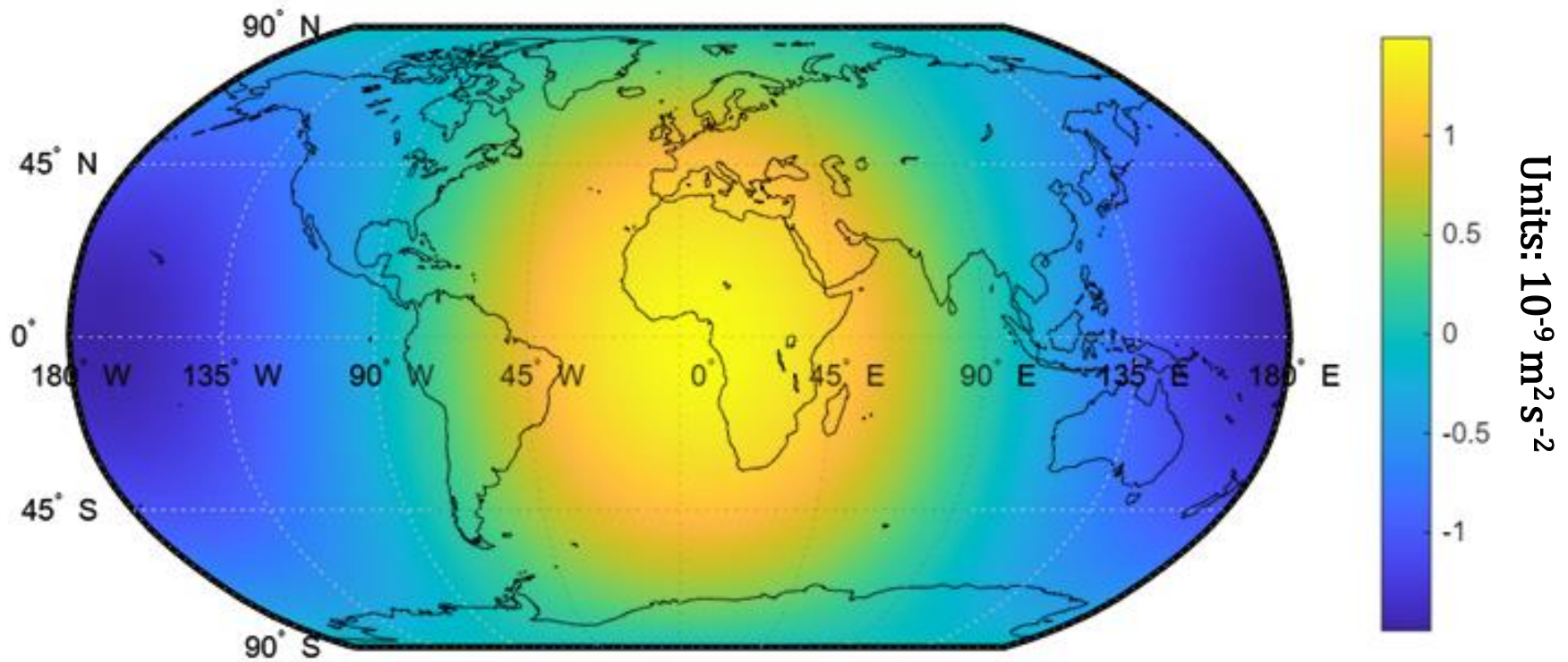
$$\begin{pmatrix} \hat{C}_{10n1} \\ \hat{S}_{10n1} \end{pmatrix} = -\frac{\mu'}{R_{\text{eq}}} \left(\frac{R_{\text{eq}}}{r'} \right)^{n+2} \sqrt{\frac{n(n+2)(2n+1)}{3(2n+3)}} \bar{P}_{n+1,1}(\sin \varphi') \begin{pmatrix} \bar{c}_{n1} \cos \alpha' + \bar{s}_{n1} \sin \alpha' \\ \bar{c}_{n1} \sin \alpha' - \bar{s}_{n1} \cos \alpha' \end{pmatrix},$$

$$\begin{pmatrix} \hat{C}_{11,1n1} \\ \hat{S}_{11,1n1} \end{pmatrix} = -\frac{\mu'}{2R_{\text{eq}}} \left(\frac{R_{\text{eq}}}{r'} \right)^{n+2} \sqrt{\frac{(n+2)(n+3)(2n+1)}{3(2n+3)}} \bar{P}_{n+1,2}(\sin \varphi') \begin{pmatrix} \bar{c}_{n1} \cos 2\alpha' + \bar{s}_{n1} \sin 2\alpha' \\ \bar{c}_{n1} \sin 2\alpha' - \bar{s}_{n1} \cos 2\alpha' \end{pmatrix},$$

$$\begin{pmatrix} \hat{C}_{11,2n1} \\ \hat{S}_{11,2n1} \end{pmatrix} = \frac{\mu'}{R_{\text{eq}}} \left(\frac{R_{\text{eq}}}{r'} \right)^{n+2} \sqrt{\frac{n(n+1)(2n+1)}{6(2n+3)}} \bar{P}_{n+1,0}(\sin \varphi') \begin{pmatrix} \bar{c}_{n1} \\ \bar{s}_{n1} \end{pmatrix}.$$

Geopotential coefficients of order 1 (mostly $\bar{c}_{31}$, $\bar{s}_{31}$) cause a permanent tide, which depends on not only the latitude φ but the longitude λ of the observation point P too. It is a new type of the permanent tides on Earth.

A time-independent part of the Earth TGP («permanent tide»), caused by $\bar{c}_{31} / \bar{s}_{31}$



The maximum value of the Earth TGP part due to $\bar{c}_{31}$, s_{31} geopotential coefficients ($\approx 1.5 \times 10^{-9} \text{ m}^2 \text{ s}^{-2}$) is reached at a point $\varphi = 0^\circ$, $\lambda \approx 6.97^\circ \text{E}$ near São Tomé island off the west equatorial coast of Africa.

Conclusions (1/2)

- Explicit expressions are derived for calculating the Earth TGP jointly caused by the external bodies (the Moon, the Sun, major planets) and an *arbitrary* spherical harmonic coefficient of the geopotential. The formulae completely separate the “TGP coefficients”, which depend on masses and positions of the external bodies, from the coordinates of the observation point, where the TGP is estimated;
- The TGP coefficients related to the external bodies are developed into harmonic series over a time interval of more than 30,000 years. Modern values for masses and ephemerides of the Moon and planets DE441 are used as the source;
- The minimum threshold for amplitudes of the expansion series' terms is decreased by one order of magnitude, up to $1 \times 10^{-9} \text{ m}^2 \text{ s}^{-2}$. Using it, taking into account the effect of the complete geopotential, and employing modern ephemerides allowed us to add new terms in the Earth TGP harmonic development and update the existing ones;

Conclusions (2/2)

- The new development is available on the web-server of SAI MSU and given in two formats: *K25* (Kudryavtsev 2004, 2025a,b) and generalized *HW95* (Hartmann &Wenzel 1995). The former format allows one to represent the TGP coefficients over long-term intervals (thousands and tens of thousands of years), and the latter format is used in popular software packages for analyzing tides measurements;
- A new type of “permanent tides” on the Earth’s surface (i.e., where the local TGP value does not depend on time) is found. It depends on not only the latitude of the observation point (like any other permanent tide known before), but its geographical longitude as well.

More details: Kudryavtsev S., *J. Geodesy*, 100:74 (2026).

Thank you for your attention!