

On the integration of Hamiltonian systems using Liouville's theorem in CAS

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Abstract. Liouville's theorem allows one to reduce Hamiltonian systems to quadratures if there is a required number of independent integrals in involution. However, in practice, integration is usually performed using other methods. The question arises: is it possible to attempt to obtain the same results directly from Liouville's theorem? We consider several examples of systems with two degrees of freedom and polynomial integrals. For these systems, we obtain additional integrals using Liouville's theorem and analyze them using the CAS `SageMath`.

Introduction

Let us consider a Hamiltonian system with two degrees of freedom. Following Whittaker [2, p. 275], we formulate Liouville's theorem [1].

Theorem 1 (Liouville, 1855). *If a dynamical system, defined by the equations*

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \quad (r = 1, 2),$$

the integral of energy is $H(q_1, q_2, p_1, p_2) = h$, and if $V(q_1, q_2, p_1, p_2) = c$ denotes any other integral not involving the time, then the expression $p_1 dq_1 + p_2 dq_2$, where p_1 and p_2 have the values found from these integrals, is the exact differential of some function

$$\vartheta(q_1, q_2, h, c),$$

and the two remaining integrals of the system are

$$\frac{\partial \vartheta}{\partial c} = \text{const}, \quad \frac{\partial \vartheta}{\partial h} = t + \text{const}.$$

Theoretically, this statement can be regarded as a method for integrating canonical equations when an additional integral of motion is known. This becomes

particularly promising when the known integrals of motion are polynomials quadratic in the momenta [3]. On the other hand, in the polynomial case, it becomes possible to obtain additional (remaining) integrals of motion in the form of integrals of algebraic functions.

Typically, such systems are integrated using other methods, which yield results in Abelian functions [4, 5]. Integration is based on the transition to special coordinate systems [6, 7], cf. [8]. Is it possible to obtain the known results by implementing Liouville's theorem as a method for integrating a canonical system in CAS?

Examples

We considered Yoshida Hamiltonians (1984, [9, 10], [11, p. 264]) as the first examples

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^4 + q_2^4}{4} + \alpha \frac{q_1^2 q_2^2}{2},$$

which yields integrable systems only for three values of the parameter $\alpha = 0, 1, 3$.

The second example we considered is the Vanhaeke system [4, 5], with Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + (q_1^2 + q_2^2)^2 + a q_1^2 + b q_2^2,$$

which is integrable for any values of a, b and for $a = b = 0$ it is reduced to the Yoshida system with $\alpha = 1$.

The additional integrals of the indicated systems are known and have degree no higher than second in momenta. So the equations for the integrals are explicitly solved for the momenta, which allows one to write quadratures from Liouville's theorem.

In the case of $\alpha = 3$ for Yoshida system and for the Vanhaeke system with arbitrary values a, b , it is easy to obtain the hyperelliptic quadrature

$$\vartheta(q_1, q_2, h, c) = \int_{(q_1, q_2)}^{(q_1, q_2)} \sqrt{\frac{B_1 + \sqrt{D}}{A}} dq_1 + \int_{(q_1, q_2)}^{(q_1, q_2)} \sqrt{\frac{B_2 + \sqrt{D}}{A}} dq_2.$$

Here A, B_1, B_2, D are polynomials of q_1, q_2, h, c, a, b . It is necessary to monitor the choice of branches of 4-valued functions and carry out the integration path within the region of possible motion. These quadratures consist of several hundred symbols. Further, it is not difficult to construct quadrature expressions for additional integrals obtained in Liouville's theorem.

These quadratures can be simplified to elliptic integrals, for example, in the case of a particular solution $q_1 = p_1 = 0$, when q_2 is expressed in elliptic functions for Vanhaecke and Yoshida systems. Note that in the case of $\alpha = 0$, the nested quadrature disappears.

Conclusion

We have obtained expressions for additional (remained) integrals of motion, and in some special cases, the systems can indeed be integrated using known functions. Moreover, sometimes for integration there is no need to switch to special coordinate systems. The calculation results are compared with better-known results related to these systems. The interplay between the two faces of the same solution – Abelian quadratures in separated variables and hyperelliptic quadratures in Cartesian space – may well prove to be a subject of independent interest.

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