

On the integration of Hamiltonian systems using Liouville's theorem in CAS

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Ahmed Deina Mohamed EL Mokhtar,
Ksaverii Yu. Malyshev

Patrice Lumumba RUDN, SINP MSU

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The main theorem ([Whitakker, 1904])

Theorem (Liouville, 1855)

If a dynamical system, defined by the equations

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \quad (r = 1, 2),$$

the integral of energy is $H(q_1, q_2, p_1, p_2) = h$, and if

$V(q_1, q_2, p_1, p_2) = f$ denotes any other integral not involving the time, then the expression $p_1 dq_1 + p_2 dq_2$, where p_1 and p_2 have the values found from these integrals, is the exact differential of some function

$$\vartheta(q_1, q_2, h, f),$$

and the two remaining integrals of the system are

$$\frac{\partial \vartheta}{\partial f} = \text{const}, \quad \frac{\partial \vartheta}{\partial h} = t + \text{const}.$$

The Questions

- 1. Can we use Liouville's theorem in the given formulation as a computational tool, especially using CAS?
- 2. The theorem doesn't mention transitions to special coordinate systems or separation of variables. Does it work in the original coordinates?
- 3. To what extent are the calculations associated with this theorem feasible in symbolic form? Checking the condition of the total differential for $p_1 dq_1 + p_2 dq_2$, calculating the potential ϑ , finding g , f , $q_r(t)$, etc.
- 4. Is it possible to find substitutions of separating variables by analyzing the formulas associated with this theorem?

The class of Hamiltonian systems under consideration

- Condition 1. Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + V(q_1, q_2), \quad V \in \mathbb{Q}[q_1, q_2]$$

- Condition 2. There is an additional integral

$$F = ap_1^2 + bp_2^2 + cp_1p_2 + D, \quad a, b, c, D \in \mathbb{Q}[q_1, q_2],$$

$$\text{or } F = p_2q_1 - q_2p_1.$$

This class of Hamiltonian systems includes systems that satisfy the Bertrand-Darboux theorem, but may also include Hamiltonian systems that do not.

It includes almost all systems from the Hietarinta (1984) and many from Nakagava-Yoshida (2001) lists of integrable systems.

Calculating momenta for Liouville's theorem

Let be h, f are the values of integrals. We have ideal $(H - h, F - f)$ in polynomial rings $\mathbb{Q}[p_1, p_2, a, \dots, f]$, $\mathbb{Q}[p_2, p_1, a, \dots, f]$. Using the Groebner basis technique, it can be found

$$p_1 = \pm \sqrt{\frac{B_1 \pm \sqrt{\mathfrak{D}}}{A}}, \quad p_2 = \pm \sqrt{\frac{B_2 \pm \sqrt{\mathfrak{D}}}{A}}$$

$$A = 2(a - b)^2 + 2c^2,$$

$$B_1 = 2V(-2ab + 2b^2 + c^2) + 2D(a - b) + 2h(2ab - 2b^2 - c^2) + 2f(b - a),$$

$$B_2 = 2V(-2ab + 2a^2 + c^2) + 2D(b - a) + 2h(2ab - 2a^2 - c^2) + 2f(a - b),$$

$$\mathfrak{D} = (4c^4 - 16abc^2)V^2 + (8ac^2 + 8bc^2)VD - 4c^2D^2 +$$

$$(32abc^2h - 8c^4h - 8ac^2f - 8bc^2f)V - 8(ac^2h + bc^2h + c^2f)D -$$

$$16abc^2h^2 + 4c^4h^2 + 8ac^2hf + 8bc^2hf - 4c^2f^2.$$

Examples of systems for testing the theorem

Potentials: Ioshida systems (1984)

$$V_\alpha = \frac{q_1^4 + q_2^4}{4} + \alpha \frac{q_1^2 q_2^2}{2}, \quad \alpha = 0, 1, 3$$

Vanhaecke system (Harnier, Anharmon. oscillator, (1912-1994))

$$V_{\mu,\nu} = (q_1^2 + q_2^2)^2 + \mu q_1^2 + \nu q_2^2$$

Additional Integrals (Ioshida, Gorielli):

$$F_0 = \frac{p_1^2 - p_2^2}{4} + \frac{q_1^4 - q_2^4}{4}, \quad F_1 = p_2 q_1 - p_1 q_2, \quad F_3 = p_1 p_2 + q_1 q_2 (q_1^2 + q_2^2);$$

$$F_{\mu,\nu} = (q_1 p_2 - q_2 p_1)^2 + (\mu - \nu)(p_1^2 + 2q_1^4 + 2q_1^2 q_2^2 + 2\mu q_1^2).$$

Liouville Integration for Yoshida-0

It is the separable case.

$$\vartheta = \int \sqrt{-1/2q_1^4 + f + h} dq_1 + \int \sqrt{-1/2q_2^4 - f + h} dq_2$$

First remaining integral:

$$c1 = - \int_{q_{10}}^{q_1(t)} \frac{dq_1}{\sqrt{-1/2q_1^4 + f + h}} + \int_{q_{20}}^{q_2(t)} \frac{dq_2}{\sqrt{-1/2q_2^4 - f + h}}$$

Second remaining integral:

$$c2 + t = - \int_{q_{10}}^{q_1(t)} \frac{dq_1}{\sqrt{-1/2q_1^4 + f + h}} - \int_{q_{20}}^{q_2(t)} \frac{dq_2}{\sqrt{-1/2q_2^4 - f + h}}$$

For initial conditions $q_1(0) = 3, q_2(0) = 4, p_1(0) = 1, p_2(0) = 2$ the formulas with sn are:

$$q_1(t) = C_1 \operatorname{sn}(\sqrt{2} \cdot 264^{1/4}(t - 1.155360369)/4, I)$$

$$q_2(t) = C_2 \operatorname{sn}(\sqrt{2} \cdot 264^{1/4}(t - 0.8583887884)/4, I)$$

Liouville integration for Yoshida-1

$$\begin{aligned} \vartheta = & -f(\arctan(q_1/q_{20}) - \arctan(q_{10}/q_{20}) - \\ & \arctan(q_2/q_1) - \arctan(q_{20}/q_1) + \\ & \int_{q_{10}}^{q_1} \left(\frac{q_1 \sqrt{-2(q_1^2 + q_2^2)^3 + 8h(q_1^2 + q_2^2) - 4f^2}}{(2(q_1^2 + q_2^2))} \right)_{q_2=q_{20}} dq_1 + \\ & + \int_{q_{20}}^{q_2} \left(\frac{q_2 \sqrt{-2(q_1^2 + q_2^2)^3 + 8h(q_1^2 + q_2^2) - 4f^2}}{(2(q_1^2 + q_2^2))} \right)_{q_1=const} dq_2 \end{aligned}$$

In this case, the variables are not separable. But this integral is reduced to an elliptic integral depending on $(q_1^2 + q_2^2)$.

It is true also for two «remaining integrals».

This case integrable in Cartesian coordinates, but they are not separable.

Liouville integration for losida-3

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^4 + q_2^4}{4} + 3\frac{q_1^2 q_2^2}{2}$$

$$F_3 = p_1 p_2 + q_1 q_2 (q_1^2 + q_2^2)$$

This is the first case with double radicals.

$$\vartheta = \int \left(\sqrt{B_1 + \sqrt{\mathfrak{D}}} dq_1 + \sqrt{B_2 + \sqrt{\mathfrak{D}}} dq_2 \right),$$

$$\mathfrak{D} = (q_1 + q_2)^4 - 4f - 4h)((q_1 - q_2)^4 + 4f - 4h),$$

factorisation in Sage;

$$B_1 = -q_1^4 - 6q_1^2 q_2^2 - q_2^4 + 4h = B_2.$$

We do not see some ideas for integration. But we will give the answer after several frames.

How to work with nested quadratures

- We have the parameters a, b, c, D, V . Using the initial conditions find f, h .
- Select one of the 4 branches for each 4-valued function. We wrote a function in the Sage for such a choice, over the field $\mathbb{Q}$. The choice boils down to checking the consistency of the initial conditions.
- The area of possible movement is determined using the polynomial $\mathfrak{D}$. For now, we have limited ourselves to visual inspection of this area.
- It can be proven that near the initial point there exists a subregion of the region of possible motion that is stellar relative to it. In this region, the potential values for the φ and its derivatives with respect to the e can be found numerically.
- This way, we can try to obtain the values of local integrals. We are not currently discussing global restoration of motion.

General Vanhaecke system

$$H = \frac{p_1^2 + p_2^2}{2} + (q_1^2 + q_2^2)^2 + \mu q_1^2 + \nu q_2^2$$

$$F_{\mu,\nu} = (q_1 p_2 - q_2 p_1)^2 + (\mu - \nu)(p_1^2 + 2q_1^4 + 2q_1^2 q_2^2 + 2\mu q_1^2).$$

For p_1, p_2 :

$$A = (q_2^2 - q_1^2 - \mu + \nu)^2 + 4q_1^2 q_2^2$$

We cannot give here the expressions for $\mathfrak{D}$ and B_1, B_2 because they are too large.

Branch Selection

```
branch_selection_1(q10=1,q20=-3,p10=-8,p20=2,a=1,b=2)
```

```
-20*sqrt(1/13)
```

```
20*sqrt(1/13)
```

```
-8
```

```
8
```

```
vp12
```

```
-sqrt(2)*sqrt(-(q1^8 + q1^2*q2^6 - q1^6 + (3*q1^4 + 2*q1^2)*q2^4  
+ sqrt(-528*q1^6 - 1140*q2^6 - 4*(701*q1^2 + 855)*q2^4 - 4*(547  
0)*q1*q2 + 439*q1^2 - 285)/(q1^4 + q2^4 + 2*(q1^2 + 1)*q2^2 - 2
```

- The branche selection is possible, when $q_1 q_2 \neq 0$.
- Else: we have two variants for branche.

Remaining Integrals

After the Branch selection, we can reconstruct $\frac{\partial \vartheta}{\partial c}, \frac{\partial \vartheta}{\partial h}, \vartheta$. This is possible near the starting point.

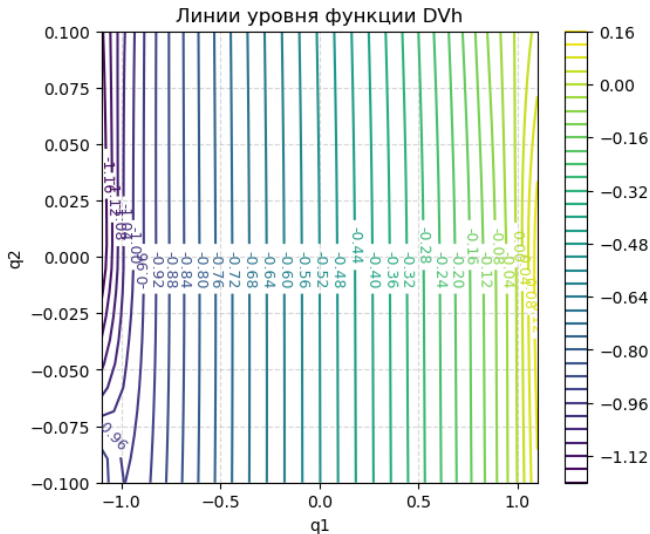
To do this, we integrate the corresponding differential forms along a straight (not broken) path from the starting point (q_{10}, q_{20}) to the target point (q_1, q_2) .

This can be done using the built-in numerical integrator, and then we can obtain level curves for the remaining integrals.

Let us now present the level lines for a difficult case of initial conditions:

$$q_{10} = 1, q_{20} = 1/100, p_{10} = 1/100, p_{20} = 1$$

Remaining Integrals. $\frac{\partial \vartheta}{\partial h}$



$$\frac{\partial v}{\partial h}$$

Последняя контрольная точка: 5 часов наз

8 -0.7526315789473685 -0.015789473684210534 -0.87708001983277271
8 4 -0.6368421052631579 -0.015789473684210534 -0.8157426287705242
8 5 -0.5210526315789474 -0.015789473684210534 -0.758577897579582
8 6 -0.4052631578947369 -0.015789473684210534 -0.703746061662153
8 7 -0.2894736842105263 -0.015789473684210534 -0.6506563652685219
8 8 -0.17368421052631577 -0.015789473684210534 -0.5983116547324723
8 9 -0.05789473684210522 -0.015789473684210534 -0.546431470914227
8 10 -0.05789473684210522 -0.015789473684210534 -0.4947035199992565
8 11 -0.17368421052631589 -0.015789473684210534 -0.44284688275179696
8 12 -0.2894736842105263 -0.015789473684210534 -0.3905491393167026
8 13 -0.405263157894737 -0.015789473684210534 -0.33741163633104265
8 14 -0.5210526315789474 -0.015789473684210534 -0.2828404464274714
8 15 -0.6368421052631581 -0.015789473684210534 -0.225878436923392
8 16 -0.7526315789473685 -0.015789473684210534 -0.16487455248109845
8 17 -0.868421052631579 -0.015789473684210534 -0.09647932308840415
8 18 -0.9842105263157896 -0.015789473684210534 -0.012537161282845287
8 19 1.1 -0.015789473684210534 0.14016087517791806
9 0 -1.1 -0.005263157894736845 -1.1881680109453314
9 1 -0.9842105263157895 -0.005263157894736845 -1.0335441887706476
9 2 -0.868421052631579 -0.005263157894736845 -0.9463067485497361
9 3 -0.7526315789473685 -0.005263157894736845 -0.877059164954052
9 4 -0.6368421052631579 -0.005263157894736845 -0.8156459232494697
9 5 -0.5210526315789474 -0.005263157894736845 -0.7584435676135614
9 6 -0.4052631578947369 -0.005263157894736845 -0.7037146394574857
9 7 -0.2894736842105263 -0.005263157894736845 -0.6504764025618033
9 8 -0.17368421052631577 -0.005263157894736845 -0.5981123261389002
9 9 -0.05789473684210522 -0.005263157894736845 -0.5462120830817703

Remaining Integrals. Calculation

The very possibility of such calculations, on a 20×20 grid, without parallelism, in about 15 minutes, speaks to the possible use of Liouville quadratures for research.

What About Vanhaecke coordinates transformation

- Main thesis. We noticed that as a result of the Vaneke transformation, the zero-level line of the discriminant $\mathfrak{D}$ goes into the boundary of the rectangle with sides parallel to the coordinate axes.
- Our hypothesis: this allows us to hopefully obtain a coordinate transformation by solving a purely algebraic problem about the zeros of the polynomial $\mathfrak{D}$.
- This helped us integrate Yoshida-3. In this case we should only rotate Cartesian coordinate system.

What About Vanhaecke coordinates transformation

Vanhaecke transformation:

$$t_1 + t_2 = -q_1^2 - q_2^2 - \mu - \nu$$

$$t_1 t_2 = \nu q_1^2 + \mu q_2^2 + \mu \nu$$

Theorem

$$\begin{aligned} \mathfrak{D} = & 16(2\mu\nu t_1 + 2\mu t_1^2 + 2\nu t_1^2 + 2t_1^3 - 2\nu h - 2ht_1 - g) \times \\ & (2\mu\nu t_2 + 2\mu\nu t_2^2 + 2\nu t_2^2 + 2t_2^3 - 2\nu h - 2ht_2 - g) \times \\ & (\mu + t_1)(\mu + t_2)(\nu + t_1)(\nu + t_2)/(\mu - \nu)^2 \end{aligned}$$

Yosida-3 Integration using our hypothesis

Hamiltonian equations:

$$\frac{d^2 q_1}{dt^2} = q_1(-q_1^2 - 3q_2^2)$$

$$\frac{d^2 q_2}{dt^2} = q_2(-q_2^2 - 3q_1^2)$$

Замена:

$(q_1 * (-q_1^2 - 3 * q_2^2)).subs(q_1 = (U + V)/2).subs(q_2 = (U - V)/2).simplify\_full()$

After transformation :

$$\frac{d^2(U + V)}{dt^2} = -U^3 - V^3$$

$$\frac{d^2(U - V)}{dt^2} = -U^3 + V^3$$

We have integration in elliptic Jacobi functions.

Simple periodic solutions for Vanhaecke

We cannot find general solution for Vanhaecke system. But, analyze our computations, we find the solution

$$q_2 = p_2 = 0;$$
$$t = \int_{q_1(0)}^{q_1(t)} \frac{d\xi}{\sqrt{-2\mu\xi^4 - 2\mu\xi^2 + 2h}}$$

In terms of Vanhaecke coordinates:

$$t_1 = -\nu, \quad t_2 = -q_1^2\mu - \mu\nu$$

Using Liouville we can find it, but using transformations – it is too difficult.

This solution describes a periodic motion accross the double line of branches.

Conclusions

- Our study indicates the possibility of integrating Hamiltonian systems using Liouville's theorem without knowledge of specific changes of variable.
- We believe there are no particular obstacles to implementing this theorem using numerical methods.
- We hope to understand the relationship between a special change of variables and discriminant zeros. We were able to find a solution in elliptic functions for the Yoshida-3 system using a change that rotates the coordinates.
- We give an expression for factorization $\mathfrak{D}$, give periodic simple solutions for Vanhaecke system, give solution for Yoshida-3 system.

Separation of variables. Tsyganov, Grigoriev package

Typically, canonical systems are integrated by switching to a special coordinate system in which the variables are separated. Integration is then possible without resorting to Liouville's theorem.

Substitutions are usually represented as the result of a lucky guess. However, finding the necessary substitution is possible, for example, using the Livi-Civita criterion (1903). The Levi-Civita criterion has been translated into the language of modern differential geometry. In 2005, Grigoriev and Tsyganov published the *seppwar.mw* package for Maple for working with L -systems. In it, they calculated the substitution separating the variables for the Vanhaecke system.

We were able to obtain all additional integrals for the systems under consideration. But were unable to find the separating variables for the Yosida V_0 , V_3 systems. Therefore, an attempt to directly integrate the system using Liouville's theorem seems justified.

Attempt to find a replacement for the Yoshida system V_3

As an example we consider the Hénon-Héles integrable model with the Hamiltonian $H = T + V$, where

```
> Tn:=add(p||i^2,i=1..n)/2;
> Vn:=1/4*(q1^4+q2^4)+3*q1^2*q2^2/2;
```

$$T_n := \frac{p_1^2}{2} + \frac{p_2^2}{2}$$

$$V_n := \frac{1}{4} q_1^4 + \frac{1}{4} q_2^4 + \frac{3}{2} q_1^2 q_2^2$$

Substituting these parts of the Hamilton function into the prepared above list of equations `ListEq` one gets a set of polynomial equations of t be identically satisfied for all admissible values of variables p . It means that coefficients for the second, first and zeroth power of p_i vanish algebraic and partial differential equations

```
> System:= NULL;
for i from 1 to NumEq do
    u:=expand(dvalue(subs({T(var)=Tn, V(q)=Vn},ListEq[i])));
    System:= System,coeffs(u,{p});
end do;
nops({System});
```

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At $n=2,3,4$ the resulting system of equation consists of 13, 51 and 136 equations. Even though some of these equations may be dependent, in course, for a generic potential this system is heavily overdetermined and has only the trivial solution, which means that the Hamilton function

In the case of polynomial potentials we can easily solve this system by means of the standard Maple procedure

```
> Ans:=pdsolve({System},{seq(seq(L[i,j](q), i=1..n),j=1..n)});
Ans := {L_1,1(q1,q2) = -C1, L_1,2(q1,q2) = -C2, L_2,1(q1,q2) = -C2, L_2,2(q1,q2) = -C1}
```

Substituting this answer into the L-tensor one gets

```
> unassign(Ln): Ln:=array(1..n,1..n):
Trans:={seq(seq(L[i,j](q)=Ln[i,j], i=1..n),j=1..n)}:
Ans2:=simplify(map2(subs,Trans,Ans)):
Ln:=map2(subs,Ans2,evalm(Ln));
```

$$L_n := \begin{bmatrix} -C1 & -C2 \\ -C2 & -C1 \end{bmatrix}$$

The eigenvalues of this L-tensor are the separation coordinates

```
> Q:=1/2*Eigenvalues(Ln);
```

$$Q := -C1 + -C2, -C1 - -C2$$

Thus one gets translated parabolic coordinates, which are the separation coordinates for the model.

Finding a replacement for the Vanhaecke system $V_{\mu,\nu}$

```

> Tn:=add(p||i^2,i=1..n)/2;
Vn:=add(a||i*q||i^2,i=1..n)+(add(q||i^2,i=1..n))^2;

```

$$T_n := \frac{p_1^2}{2} + \frac{p_2^2}{2}$$

$$V_n := a_1 q_1^2 + a_2 q_2^2 + (q_1^2 + q_2^2)^2$$

```

> System:= NULL:
  for i from 1 to NumEq do
    u:=expand(dvalue(subs({T(var)=Tn, V(q)=Vn},ListEq[i])));
    System:= System,coeffs(u,{p});
  end do;

```

It is easy to calculate solution of the complete system of equations

```

> Ans:=pdsolve({System},{seq(seq(L[i,j](q), i=1..n),j=1..n)});

```

$$Ans := \left\{ L_{1,1}(q_1, q_2) = (q_1^2 + a_1 - a_2) \cdot C_2 + C_1, L_{1,2}(q_1, q_2) = -C_2 q_2 q_1, L_{2,1}(q_1, q_2) = -C_2 q_2 q_1, L_{2,2}(q_1, q_2) = -C_2 q_2^2 + C_1 \right\}$$

The corresponding L tensor has the form

```

> unassign(Ln): Ln:=array(1..n,1..n):
Trans:={seq(seq(L[i,j](q)=Ln[i,j], i=1..n),j=1..n)}:
Ans2:=simplify(map2(subs,Trans,Ans)):
Ln:=map2(subs,Ans2,evalm(Ln));

```

$$Ln := \begin{bmatrix} (q_1^2 + a_1 - a_2) \cdot C_2 + C_1 & -C_2 q_2 q_1 \\ -C_2 q_2 q_1 & -C_2 q_2^2 + C_1 \end{bmatrix}$$

At $C_1 = a_n$ and $C_2 = 1$ the eigenvalues Q_i of the tensor L are defined as zeroes of the equation

```

> eq:=linalg[det](Ln-lambda*ed)/mul(lambda-a||i,i=1..n):
convert(subs({_C1=a||n,_C2=1},eq=0),fullparfrac,lambda);

```

$$1 - \frac{q_1^2}{\lambda - a_1} - \frac{q_2^2}{\lambda - a_2} = 0$$

Here $\lambda_{1,2}$ are the Vanhaecke coordinates.

Thank you for your time and
attention!

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Problem formulation

Vanhaecke hamiltonian:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + aq_1^2 + bq_2^2 + (q_1^2 + q_2^2)^2, \quad a \neq b.$$

Our problems:

- 1. Verify the results of Liouville's theorem for the Vanhaecke system in symbolic form.
- 2. Implement Liouville's theorem as a method for integrating a canonical system with known integrals by quadratures in the CAS Sage.

Solution to problem 1

Integrals, integral surface $M_{f,g}$:

$$F = (q_1 p_2 - q_2 p_1)^2 + (b - a)(p_1^2 + 2q_1^4 + 2q_1^2 q_2^2 + 2a q_1^2),$$

$$G = (q_1 p_2 - q_2 p_1)^2 + (a - b)(p_2^2 + 2q_2^4 + 2q_2^2 q_1^2 + 2b q_2^2),$$

$$F - G = 2H(b - a).$$

Two independent integrals. Perfect differential condition (result of the Liouville's Theorem), on the integral surface:

$$\frac{\partial p_1}{\partial q_2} = \frac{\partial p_2}{\partial q_1}.$$

The computations of partial differentiation are too complicated.
This equation is verified using the Groebner basis technique.

Solution to problem 1. Groebner basis.

```
k=PolynomialRing(QQ, [p2,p1,q2,q1,a,b,f,g], order='lex') |
```

```
J=k*[F==f,G==g] |
```

```
J.groebner_basis() |
```

$$\Rightarrow F_1(q_1, q_2, p_1) = 0.$$

F_1 is a biquadratic equation for p_1 . It consists of 183 monomials.
 Differentiation:

$$\frac{\partial p_1}{\partial q_2} = -\frac{\frac{\partial F_1}{\partial q_2}}{\frac{\partial F_1}{\partial p_1}} \in \text{FractionField}(k).$$

Solution to problem 1. Groebner basis.

Another order of variables:

```
k=PolynomialRing(QQ, [p1,p2,q2,q1,a,b,f,g], order='lex') |
```

```
J=k*[F==f,G==g] |
```

```
J.groebner_basis() |
```

$$\Rightarrow F_2(q_1, q_2, p_2) = 0.$$

F_2 is a biquadratic equation for p_2 . It consists of 183 monomials.

Differentiation:

$$\frac{\partial p_2}{\partial q_1} = -\frac{\frac{\partial F_2}{\partial q_1}}{\frac{\partial F_2}{\partial p_2}} \in \text{FractionField}(k).$$

Solution to problem 1. Polynomial ideals.

It is necessary to check the equality of fractions

$$\frac{\frac{\partial F_1}{\partial q_2}}{\frac{\partial F_1}{\partial p_1}} = \frac{\frac{\partial F_2}{\partial q_1}}{\frac{\partial F_2}{\partial p_2}} \in \text{FractionField}(k).$$

This equality must be satisfied on the integral surface. This equality must be satisfied on the integral surface. For arbitrary parameter values, it is false.

We can consider the difference of two fractions. If the common numerator of this difference belongs to the ideal generated by the equations of the integral surface, then the equality can be considered true on the integral surface. In this form, we have a classic computer algebra problem.

Solution to problem 1. Polynomial ideals.

By reformulating the problem of testing Liouville equality as a problem of whether a polynomial belongs to some ideal, we can complete the calculations. Note that the common numerator for the difference of two fractions is a polynomial in 8 variables containing 1856 terms.

```
Ввод [14]: len(QQ[a, b, f, g, p1, p2, q1, q2](w).monomials())
```

```
Out[14]: 1856
```

```
Ввод [15]: k=PolynomialRing(QQ,[p1,p2,q2,q1,a,b,f,g])  
          J=k*[F==f,G==g]
```

```
Ввод [16]: w in J
```

```
Out[16]: True
```

The first problem is solved.

To Problem 2. Finding potential.

So, the form $p_1 dq_1 + p_2 dq_2$ is a perfect differential on the integral surface $M_{f,g}$. For integration by quadratures using Liouville's theorem we need the V — potential, $p_1 dq_1 + p_2 dq_2 = dV$. The quadrature has the form:

$$\int_{L \subset M_{f,g}} p_1 dq_1 = \int_{L \subset M_{f,g}} \sqrt{\frac{P_8(q_1) + \sqrt{P_6(q_1)}}{P_4(q_1)}} dq_1,$$

$$P_j \in \mathbb{Q}[q_2][q_1], \quad \text{degree}(P_j) = j.$$

The total number of symbols required to write the integrand is from 573 to 8470 elements. The integrand has 4 branches, we wrote a function to select the correct branch in the computer algebra system.

To Problem 2. Search for simplification of symbolic expressions.

Even using specific parameter values results in extremely large, unwieldy expressions. The question arises: could the calculations be simplified by using specialized coordinate systems well known from the literature? We examined several replacements recommended specifically for the Vanhaeke system. We hoped to simplify the differential form as a result of the canonical transformation.

Conclusions (SPRING-2026)

- For the Hamiltonian Vanhaeke system, the results of Liouville's theorem were verified to be true; this turned out to be possible only with the help of the polynomial ideal technique
- The quadratures needed for integration using Liouville's theorem are written out. Symbolic expressions for all their components are given.
- Classical coordinate substitutions associated with the Vanhaeke system are converted into canonical transformations using the Sage computer algebra system. This has not resulted in any simplification of the symbolic computations associated with this system.

Liouville's Theorem [Whittaker, 1917]

$$H = H(q_1, \dots, q_n, p_1, \dots, p_n)$$

Dynamical system:

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r}, \quad r = \overline{1, n}.$$

Integrals (in involution):

$$\begin{aligned} \varphi_r(q_1, \dots, q_n, p_1, \dots, p_n, t) &= a_r, \quad r = \overline{1, n} \Rightarrow \\ \Rightarrow p_r &= f_r(q_1, \dots, q_n, a_1, \dots, a_n, t), \quad r = \overline{1, n}. \end{aligned}$$

THEN the perfect differential:

$$p_1 dq_1 + \dots + p_n dq_n - H dt = dV$$

The remaining integrals of the system are:

$$\frac{\partial V}{\partial a_r} = b_r.$$

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Thank you for your time and
attention!

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