

Assessing Chirikov's Diffusion Theory on the Arnold Model

P. M. Cincotta & C. M. Giordano

Universidad Nacional de La Plata/Instituto de Astrofísica de La
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Extended presentation

Chaotic diffusion: Chirikov's approach

- An specific property of chaotic motion is the sensitivity to initial conditions: orbits starting in an *arbitrary small neighborhood* U of a given phase space point P , exhibit an exponential divergence, the maximum Lyapunov exponent at P , σ , is the exponential mean rate.
- The Lyapunov time, $T_L = 1/\sigma$, is a microscopical time-scale for the hyperbolic dynamics on the tangent space of the phase space at the point P . It is usually assumed to be a characteristic time of predictable dynamics for orbits starting in U . **A local property**
- *Chaotic diffusion* is a statistical description of a transport process in a *macroscopic volume element* V around P . Its origin is the interaction of resonances and provides the diffusion time, T_D , the time-scale for a macroscopic change in the orbital configuration of the system at V .
Physically Relevant!

- Chirikov's 1979 work, *A universal instability of many-dimensional oscillator systems*, established the physical framework for understanding the transition from integrability to instabilities driven by local or global chaos, through the concept of resonance interaction and diffusion. It has accumulated around 4000 citations and remains the best physical approach to the subject.
- Actually many of his results were earlier published as preprints of the Nuclear Physics Institute of the Siberian section of the USSR Academy of Science, available at:
www.quantware.ups-tlse.fr/chirikov/publbinp.html.
- It is often cited primarily for his well-known resonance overlap criterion, which identifies the threshold for global chaos, the so-called Chirikov regime in contrast to Nekhoroshev regime.
- The central objective of his theory is to characterize the diffusion along a single resonance in multidimensional systems within the weakly chaotic limit, when resonance overlap is nearly negligible. Such a diffusion is conjectured to share several features encountered in the Arnold mechanism.

- Chirikov's formulation of the diffusion rate shows a strong connection with the estimates from the theory developed by Nekhoroshev. Thus Chirikov's review is devoted to the Arnold diffusion.
- Here we first revisit Chirikov's approach to derive the diffusion rate $D \sim T_D^{-1}$ along a single resonance in the Arnold model.
- We discuss the validity of his main assumptions following Chirikov's own suggestion. Almost fifty years ago, he wrote:

Academician A.N. Kolmogorov has mentioned on an occasion that it is not so much important to be rigorous as to be right. A way to be convinced (and to convince others!) of the rightness of a solution without a rigorous theory is a tried method of science: the experiment. In the field of non-linear oscillations, the so-called numerical experiment, or numerical simulation i.e., numerical integration of the equations of motion of a dynamical system by computer, has recently been spreading more and more.

The Arnold model: $2\frac{1}{2}$ degrees of freedom

$$H(I_1, I_2, \vartheta_1, \vartheta_2, \tau; \varepsilon, \mu) = \textcolor{red}{H}_1(I_1, \vartheta_1; \varepsilon) + \textcolor{blue}{H}_2(I_2) + \varepsilon\mu U(\vartheta_1, \vartheta_2, \textcolor{violet}{\tau})$$

$$\textcolor{red}{H}_1(I_1, \vartheta_1; \varepsilon) = \frac{I_1^2}{2} + \varepsilon(\cos \vartheta_1 - 1)$$

$$\textcolor{blue}{H}_2(I_2) = \frac{I_2^2}{2}$$

$$U(\vartheta_1, \vartheta_2, \textcolor{violet}{\tau}) = (\cos \vartheta_1 - 1)(\sin \vartheta_2 + \cos \textcolor{violet}{\tau}).$$

$$I_1, I_2 \in \mathbb{R}, \vartheta_1, \vartheta_2, \textcolor{violet}{\tau} = \textcolor{violet}{\Omega}t + \textcolor{violet}{\tau}_0 \in \mathbb{S}^1, \quad 0 < \varepsilon\mu \ll \varepsilon \ll 1.$$

$$\text{Set (as Arnold/Chirikov)} \quad \textcolor{violet}{\Omega} = 1, \textcolor{violet}{\tau}_0 = 0 \rightarrow \textcolor{violet}{\tau} = t \in \mathbb{S}^1$$

$U = 0$ at $\vartheta_1 = 0$, relevant!.

$$\varepsilon = 0, \quad H(I_1, I_2; 0) = \textcolor{red}{H}_1(I_1; 0) + \textcolor{blue}{H}_2(I_2) \rightarrow \text{two free rotators.}$$

$$\varepsilon \neq 0, \mu = 0, \quad H(I_1, I_2, \vartheta_1, \varepsilon) = \textcolor{red}{H}_1(I_1, \vartheta_1; \varepsilon) + \textcolor{blue}{H}_2(I_2)$$

$$\textcolor{red}{H}_1: \text{NF for the resonant phase } \vartheta_1, \text{ small osc freq, } \omega_0(\varepsilon) = \sqrt{\varepsilon}$$

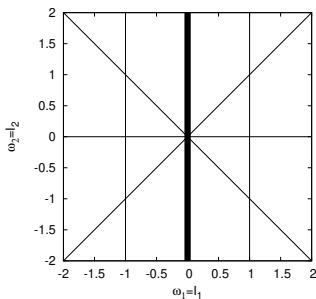
$$\textcolor{red}{H}_1 = h_1 < 0: \text{osc.}, \quad h_1 > 0: \text{circ.}, \quad h_1 = 0: \text{sx.}$$

Set $\mu \neq 0$, averaging out free oscillations:

$$U = 0.5(\sin(\vartheta_1 + \vartheta_2) - \sin(\vartheta_1 - \vartheta_2) + \cos(\vartheta_1 + t) + \cos(\vartheta_1 - t) - 2 \sin \vartheta_2)$$

Res. $\mathcal{O}(\varepsilon, \varepsilon\mu)$: $m_1\omega_1 + m_2\omega_2 + m_3 = 0$, $\omega_1(I_1)$, $\omega_2 = I_2 \in \mathbb{R} \setminus \mathbb{Q}$

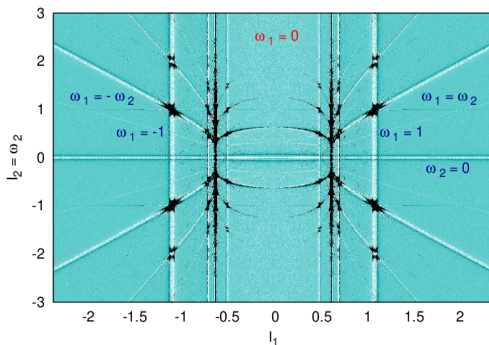
$$\mathbf{m} \in \mathcal{M} = \{(1, 0, 0), (1, 1, 0), (1, -1, 0), (1, 0, 1), (1, 0, -1), (0, 1, 0)\}$$



Theoretical resonance web of the Arnold model with harmonics in $\mathcal{M}$ are depicted. Amplitude of $\omega_1 = 0, \varepsilon$, while the rest $\mu\varepsilon \ll \varepsilon$.

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The actual resonance web



-Actual resonance web, $\varepsilon = 0.1, \mu = 0.01$. Dynamical indicator (MEGNO) contour plot, grid of 1000×1000 values of $I_1, I_2, \vartheta_1 = \pi, \vartheta_2 = 0$. White/light blue: periodic or quasiperiodic motion, dark blue/black: chaotic motion.

-The resonances with harmonics in $\mathcal{M}$: $\omega_1 = 0$, half-width, $\text{HW}_{\omega_1=0} = 2\sqrt{\varepsilon} \approx 0.63$, the remainder, $\text{HW} \sim \sqrt{\varepsilon\mu} \approx 0.03$.

-The motion near the separatrices is chaotic $\rightarrow$ the chaotic layer.

-Several other resonances are observed.

Resonances at $\mathcal{O}(\varepsilon^2\mu, \varepsilon^2\mu^2)$: $\omega_2 = 2\omega_1 + 1$, $\omega_2 = 2\omega_1$, $\omega_2 = 1$.

The resonance $\omega_2 = 1$ implies that H_1 is not valid near $(\omega_1, \omega_2) = (0, 1)$ (also near the origin).

At any order in $\varepsilon\mu$, the full resonance set is

$$\mathcal{R}_0 = \{k_1\omega_1(I_1) + k_2\omega_2(I_2) + k_3 = 0, \ k_1, k_2, k_3 \in \mathbb{Z}\}$$

$$\omega_1 \approx I_1, \omega_2 \approx I_2, \ |I_1| > 2r\sqrt{\varepsilon}, \ r \gtrsim 1 \text{ (away from } \omega_1 = 0)$$

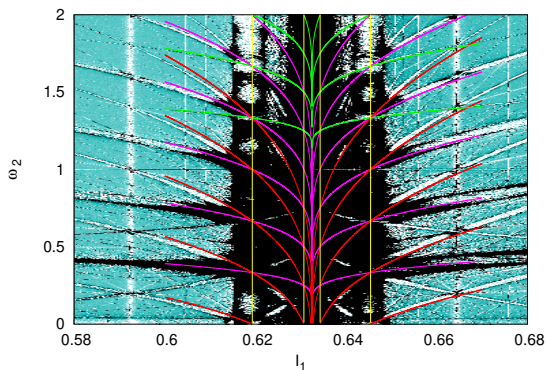
If $0 \leq |I_1| < 2r\sqrt{\varepsilon}$, $\omega_1 = \omega_p(h_1, \varepsilon)$, the pendulum frequency.

$$\mathcal{R} = \{k_1\omega_p(h_1, \varepsilon) + k_2\omega_2(I_2) + k_3 = 0, \ k_1, k_2, k_3 \in \mathbb{Z}\},$$

$\mathcal{R}$ accounts for the resonant frequency: second-level resonances,
 $\mathcal{R}_0$: first level ones.

$$\mathcal{R} \approx \mathcal{R}_0, \ |I_1| > 2r\sqrt{\varepsilon} \text{ since } \omega_p \approx I_1.$$

Resonances near the chaotic layer



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$$\omega_2(I_2) = k_1 \omega_p(h_1, \varepsilon) - 1, \quad k_1 = 3, \dots, 7$$

$$\omega_2(I_2) = k_1 \omega_p(h_1, \varepsilon) + 1, \quad k_1 = 1, \dots, 4$$

$$\omega_2(I_2) = k_1 \omega_p(h_1, \varepsilon), \quad k_1 = 1, \dots, 4$$

$$\omega_p(h_1, \varepsilon) = 1/k_1, \quad k_1 = 3, 4$$

$$\omega_p(h_1, \varepsilon) \approx 2\omega_{\text{sx}}(h_1, \varepsilon)$$

$$\omega_{\text{sx}}(h_1, \varepsilon) = \frac{\pi \omega_0(\varepsilon)}{\ln(32\varepsilon/|h_1|)}$$

The chaotic layer of the main resonance

From $H = H_1(I_1, \vartheta_1; \varepsilon) + H_2(I_2) + \varepsilon\mu U(\vartheta_1, \vartheta_2, t)$; compute

$$\dot{H} = \varepsilon\mu\partial U/\partial t = \varepsilon\mu(\sin(\vartheta_1(t) - t) - \sin(\vartheta_1(t) + t))$$

$$\dot{H}_2 = -\varepsilon\mu\omega_2\partial U/\partial\vartheta_2 = -\varepsilon\mu\omega_2(\cos(\vartheta_1(t) + \vartheta_2(t)) + \cos(\vartheta_1(t) - \vartheta_2(t)))$$

$\int \dot{H}, \dot{H}_2$ over a half-period of osc near the sx of H_1 : $T = \pi/\omega_{\text{sx}}(h_1)$

$\vartheta_1(t) = 4 \arctan\left(e^{\omega_0(t-t_0)}\right)$, the unperturbed sx ($T \rightarrow \infty$), $I_2 = \omega_2$.

$$\Delta H = \frac{\sqrt{\varepsilon}\mu}{2} A_2\left(\frac{1}{\omega_0}\right) \sin t_0, \quad \Delta H_2 = \frac{\sqrt{\varepsilon}\mu\omega_2}{2} A_2\left(\frac{\omega_2}{\omega_0}\right) \cos \vartheta_{20}$$

t_0 is the value of t when the motion in H_1 crosses the equilibrium point,
 $\vartheta_{20} = \vartheta_2(t_0)$, $\omega_0 = \sqrt{\varepsilon}$ and $A_2(\lambda)$ is the MAI of index 2,

$$A_2(\lambda) = \int_{-\infty}^{\infty} \cos\left[(4 \arctan(e^t) - \pi) - \lambda t\right] dt \approx 4\pi\lambda \frac{e^{-\lambda\pi/2}}{\sinh(\pi\lambda)}$$

Then $\Delta H_1 = \Delta H - \Delta H_2$ describes the chaotic layer.

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Let

$$v(\omega_2, \varepsilon) = \frac{\omega_2 A_2 \left(\frac{\omega_2}{\omega_0} \right)}{A_2 \left(\frac{1}{\omega_0} \right)}, \quad \hat{W} = \frac{\sqrt{\varepsilon} \mu}{2} A_2 \left(\frac{1}{\omega_0} \right).$$

Being $\Delta H_1 = \Delta H - \Delta H_2$, writing $H'_1 = H_1(t+T)$, $H_1 = H_1(t)$

$$\Delta H_1 = H'_1 - H_1 = \hat{W}(\sin t_0 - v \cos \vartheta_{20}),$$

Introduce: $w = H_1/\varepsilon$, $W = \hat{W}/\varepsilon \approx 4\pi\mu\varepsilon^{-1}e^{-\pi/(2\sqrt{\varepsilon})}$, $s = w/W$

$$s' = s + \sin t_0 - v \cos \vartheta_{20}$$

Now, $\Delta t_0 \equiv t'_0 - t_0 = T(w)$, $\Delta \vartheta_{20} \equiv \vartheta'_{20} - \vartheta_{20} = \omega_2 T(w)$,

Set: $\lambda(\varepsilon) = 1/\sqrt{\varepsilon}$, $\eta = \lambda \ln(32/|W|)$, $T(s) = \lambda \ln(32/|sW|)$

$$t'_0 = t_0 - \lambda \ln |s'| + \eta$$

$$\vartheta'_{20} = \vartheta_{20} - \omega_2 \lambda \ln |s'| + \omega_2 \eta$$

Notice the similarity of this 3D map with the 2D sx map

Difference: the quasiperiodic term in ϑ_{20} of amplitude v .

Since $s' = s + \sin t_0 - v \cos \vartheta_{20}$ and $v(\omega_2, \varepsilon) \approx \omega_2^2 e^{\frac{(1-\omega_2)\pi}{2\sqrt{\varepsilon}}}$, $\varepsilon \ll 1$
 $v(\omega_2, \varepsilon) \gg 1$ if $0 < \omega_2 < 1$; $v(\omega_2, \varepsilon) \ll 1$ if $\omega_2 > 1$

the 3D map $\rightarrow$ two 2D sx maps depending on ω_2 ,

$$s' \approx s - v \cos \vartheta_{20}, \quad \vartheta'_{20} = \vartheta_{20} - \omega_2 \lambda \ln |s'| + \omega_2 \eta, \quad \omega_2 < 1$$

$$s' \approx s + \sin t_0, \quad t'_0 = t_0 - \lambda \ln |s'| + \eta, \quad \omega_2 > 1$$

The HW of the chaotic layer given by a sx map:

$$y' = y + \sin x, \quad x' = x - \lambda \ln |y'| + \eta \text{ lies in } \lambda < y_d < \lambda + 4$$

The finite HW implies strong correlations of the phase x

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If $\lambda \gg 1$, the HW of the chaotic layer is

$$s_d \approx \begin{cases} v(\omega_2, \varepsilon) \omega_2 \lambda(\varepsilon) & \omega_2 < 1 \\ \lambda(\varepsilon) & \omega_2 > 1 \end{cases}$$

that is independent of ω_2 for $\omega_2 > 1$.

In terms of $w = W\varepsilon$, the HW is

$$w_d \approx \begin{cases} \frac{4\pi\mu\omega_2^3}{\varepsilon^{3/2}} e^{-\pi\omega_2/(2\sqrt{\varepsilon})}, & 0 < |\omega_2| < 1, \\ \frac{4\pi\mu}{\varepsilon^{3/2}} e^{-\pi/(2\sqrt{\varepsilon})}, & |\omega_2| > 1. \end{cases}$$

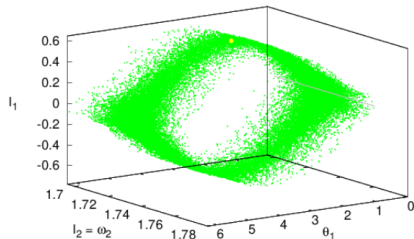
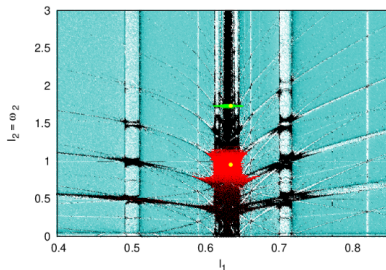
The 2D sx map, $(y, x) \rightarrow (y', x')$, describes the chaotic layer of a pendulum $H_p(p, \psi)$, when a perturbation of phase x is introduced: $H_p(p, \psi) + \mu V(\psi, x)$.

For the resonance $\omega_1 = 0$,

- If $\omega_2 < 1$, only ϑ_{20} appears in the map $\rightarrow$ harmonic $\vartheta_1 - \vartheta_2$ is considered $\rightarrow$ the “resonance” $\omega_1 = \omega_2$ produces the chaotic layer.
- If $\omega_2 > 1$, only t_0 appears in the map $\rightarrow$ harmonic $\vartheta_1 - t$ is considered $\rightarrow$ the “resonance” $\omega_1 = 1$ leads to the chaotic layer.

In Chirikov's language, $\omega_1 = 0$ is the **guiding resonance**, while the **layer resonance** is: $\omega_1 = \omega_2$ if $\omega_2 < 1$, $\omega_1 = 1$ if $\omega_2 > 1$

Diffusion on the chaotic layer



Left: Diffusion along and across the chaotic layer of the GR.

- Parameters: $\varepsilon = 0.10$, $\mu = 0.01$, ensemble of $n_p = 400$ random initial values around $I_1 = 2\sqrt{\varepsilon} \approx 0.63$, $\vartheta_1 = \pi$, $\vartheta_2 = 0$.
- $\omega_2 = 0.55\sqrt{3}$, $\omega_2 = \sqrt{3}$, restricted to $|\vartheta_1 - \pi| + |\vartheta_2| < 0.02$.
- Integration time 10^7 .

Right: Similar to the left but for $\omega_2 = \sqrt{3}$ and restricted to $|\vartheta_2| < 10^{-5}$.

For $0 < \omega_2 < 1$ the diffusion spreads away from the GR.

Diffusion is much faster in $0 < \omega_2 < 1$ than when $\omega_2 > 1$.

[return](#)

The diffusion *across* the GR is understood through the 3D sx map.

The diffusion *along* the GR can be explained in two different ways.

- Heuristically, according to Chirikov:

Arguing that since $U(\vartheta_1, \vartheta_2, t)$, it causes not only variations of I_1 but also of I_2 . Due to the nature of the motion inside the chaotic layer, (I_1, ϑ_1) , the variation of I_2 should be also chaotic since both degrees of freedom are coupled through the perturbation U .

- Formally, through the so-called **Arnold mechanism**.

Arnold mechanism (briefly)

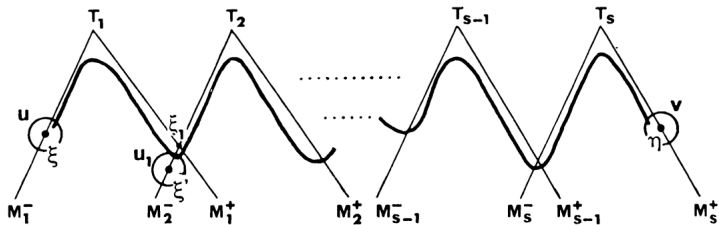
Since $U = 0$ at $\vartheta_1 = 0$, all points $(I_1, I_2, \vartheta_1, \vartheta_2) = (0, \omega_2, 0, \vartheta_2)$ are whiskered tori. A *transition chain* exists, such that if ω_2 is irrational and μ is exponentially small with respect to ε , then all tori $I_1 = 0, I_2 = \omega_2 > 0$ are transition tori. ●

Transition torus is a whiskered torus satisfying that, the images of a neighbourhood of a given point on its arriving whisker M^- , intersects any manifold which is transverse to the departing whisker M^+ at given point on it.

Transition chain is a set of s transition tori satisfying that M_l^+ of the l -transition torus intersects transversally M_{l+1}^- of the $(l+1)$ -transition torus with $l = 0, \dots, s-1$.

For large t , a neighborhood of $\xi = (0, \omega_2, 0, \vartheta_2)$ on M_1^- is mapped by the flow to a neighborhood of $\eta = (0, \omega'_2, 0, \vartheta_2)$ on M_s^+ , provided that $|\omega'_2 - \omega_2| \sim \mu e^{-1/\sqrt{\varepsilon}}$ and $\mu \sim e^{-1/\sqrt{\varepsilon}}$.

It requires quite small values of μ so that the distance between the tori, $|\omega'_2 - \omega_2|$, remains small enough for $M_{\omega_2}^+$ to intersect $M_{\omega'_2}^-$.



Sketch of the transition chain that leads to the Arnold mechanism, $T_i, i = 1, \dots, s$ are the whiskered tori labeled by the value of ω_2 . The bold curve represent an orbit. Figure taken from Arnold and Avez, *Ergodic Problems of Classical Mechanics*.

The estimates for μ and $|\omega'_2 - \omega_2|$ can be found as follows:

Recalling ΔH_2 : ●

$$\Delta H_2 = \frac{\sqrt{\varepsilon}\mu\omega_2}{2} A_2 \left(\frac{\omega_2}{\omega_0} \right) \cos \vartheta_{20},$$

it was estimated using $\vartheta_1 = \vartheta_1^{\text{sx}}(t; \varepsilon)$ and $I_2 = \omega_2$.

Perturbed solution: $\vartheta_1 = \vartheta_1^{\text{sx}}(t; \varepsilon) + \mathcal{O}(\mu)$, $I_2 = \omega_2 + \mathcal{O}(\mu)$,

$$\Delta H_2 = \frac{\sqrt{\varepsilon}\mu\omega_2}{2} A_2 \left(\frac{\omega_2}{\omega_0} \right) \cos \vartheta_{20} + \mathcal{O}(\mu^2),$$

being $A_2 \left(\frac{\omega_2}{\omega_0} \right) \approx \frac{8\pi\omega_2}{\sqrt{\varepsilon}} e^{-\pi\omega_2/(2\sqrt{\varepsilon})}$ then $\mu \sim \mathcal{O} \left(e^{-C/\sqrt{\varepsilon}} \right)$.

Assuming $\langle \Delta H_2 \rangle \approx 0$, setting

$$H_2(\omega_2) + \langle \Delta H_2^2(\omega_2) \rangle^{1/2} = H_2(\omega'_2) - \langle \Delta H_2^2(\omega'_2) \rangle^{1/2} \quad \omega'_2 \gtrsim \omega_2$$

follows $|\omega'^2_2 - \omega^2_2| \sim \pi\mu\omega^2_2 e^{-\pi\omega_2/(2\sqrt{\varepsilon})} \rightarrow |\omega'_2 - \omega_2| \sim \mu e^{-1/\sqrt{\varepsilon}}$.

Chirikov's diffusion estimates

The changes in the energies were computed after a motion time

$$T(h_1, \varepsilon) = \frac{1}{\sqrt{\varepsilon}} \ln \left(\frac{32\varepsilon}{|h_1|} \right) = \frac{1}{\sqrt{\varepsilon}} \ln \left(\frac{32}{|\xi|w_d} \right), \quad |h_1| = \varepsilon \xi w_d, \quad 0 < \xi < 1$$

The average period of motion in the chaotic layer:

$$T_a(\varepsilon, \mu) \approx \int_0^1 T(\xi, \varepsilon, \mu) d\xi = \frac{1}{\sqrt{\varepsilon}} \ln \left(\frac{32e}{w_d(\varepsilon, \mu)} \right).$$

$$T_a(\varepsilon, \mu) \sim \begin{cases} \frac{\pi\omega_2}{2\varepsilon}, & \omega_2 > 1 \\ \frac{\pi}{2\varepsilon}, & \omega_2 > 1. \end{cases}$$

return

Such changes are:

$$\Delta H_2 = \hat{W}(\varepsilon, \mu) v(\omega_2, \varepsilon) \cos \vartheta_{20}, \quad \Delta H = \hat{W}(\varepsilon, \mu) \sin t_0,$$

$$\hat{W}(\varepsilon, \mu) = \frac{\sqrt{\varepsilon}\mu}{2} A_2 \left(\frac{1}{\sqrt{\varepsilon}} \right), \quad v(\omega_2, \varepsilon) = \omega_2 \frac{A_2 \left(\frac{\omega_2}{\sqrt{\varepsilon}} \right)}{A_2 \left(\frac{1}{\sqrt{\varepsilon}} \right)},$$

After N periods T_a , assuming (wrongly) nearly random phases,

$$\underbrace{\langle (H_2(t) - H_2(0))^2 \rangle}_{\Delta H_2^2(t)} = \hat{W}^2 v^2 \sum_{k=1}^N \langle \cos^2 \vartheta_{20}(k) \rangle \equiv \hat{W}^2 v^2 Z_{\vartheta_2}(N),$$

$$\underbrace{\langle (H(t) - H(0))^2 \rangle}_{\Delta H^2(t)} = \hat{W}^2 \sum_{k=1}^N \langle \sin^2 t_0(k) \rangle \equiv \hat{W}^2 Z_t(N).$$

Motion confined to the layer: $\langle \Delta H_1^2(t) \rangle \sim \varepsilon^2 w_d^2 \rightarrow 3\text{D map} \rightarrow 2\text{D sx maps}$ ●

if $\omega_2 < 1$, ϑ_{20} is correlated (the largest term),

if $\omega_2 > 1$, t_0 correlated (the largest term).

The phase of the neglected term (the smallest) can be taken random:

$$t_0 \text{ if } \omega_2 < 1; \quad \vartheta_{20} \text{ if } \omega_2 > 1.$$

In the *nearly normal diffusion* (limiting stochasticity approximation)

$$Z_t(N) \approx N/2, \quad \omega_2 < 1; \quad Z_{\vartheta_2}(N) \approx N/2, \quad \omega_2 > 1.$$

Defining

$$D_1 = \frac{\langle \Delta H_1^2 \rangle}{NT_a}, \quad D_2 = \frac{\langle \Delta H_2^2 \rangle}{NT_a}, \quad D = \frac{\langle \Delta H^2 \rangle}{NT_a}$$

For large N (large times)

$$D_1 \rightarrow 0, \quad D_2 = \frac{v^2 \hat{W}^2}{T_a} \frac{Z_{\vartheta_2}(N)}{N}, \quad D = \frac{\hat{W}^2}{T_a} \frac{Z_t(N)}{N},$$

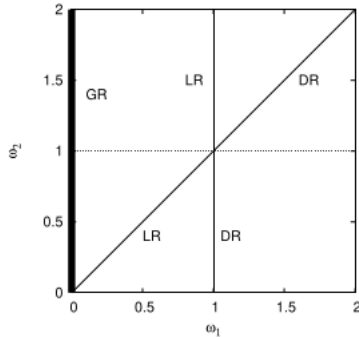
so $D_2 \approx D \rightarrow v^2 Z_{\vartheta_2}(N) \approx Z_t(N)$. Then

$$Z_{\vartheta_2}(N) \approx \begin{cases} \frac{N}{2v^2}, & 0 < |\omega_2| < 1 \ (v \gg 1), \\ \frac{N}{2}, & |\omega_2| > 1, \end{cases}$$

Being $v^2 \approx \omega_2^4 e^{\frac{(1-\omega_2)\pi}{\sqrt{\varepsilon}}}$, $\hat{W}^2 \approx 16\pi^2 \mu^2 e^{-\frac{\pi}{\sqrt{\varepsilon}}}$,

$$D_2(\omega_2, \varepsilon, \mu) \approx \begin{cases} \frac{8\pi^2 \mu^2}{T_a} e^{-\pi/\sqrt{\varepsilon}}, & 0 < |\omega_2| < 1, \\ \frac{8\pi^2 \mu^2 \omega_2^4}{T_a} e^{-\pi\omega_2/\sqrt{\varepsilon}}, & |\omega_2| > 1. \end{cases}$$

where $T_a(\varepsilon, \mu)$.



Role of the resonances in the Arnold model depending on the value of $\omega_2 > 0$:

GR: guiding resonance, $\omega_1 = 0$,

LR: layer resonance, $\omega_1 = \omega_2$ for $\omega_2 < 1$ and $\omega_1 = 1$ for $\omega_2 > 1$,

DR: driving resonance, $\omega_1 = 1$ if $0 < \omega_2 < 1$ and $\omega_1 = \omega_2$ if $\omega_2 > 1$.

Chirikov's main assumptions:

- 1) The closest harmonic to the actual resonance (GR), produces the chaotic layer, named LR. The 3D map $\rightarrow$ two 2D sx maps
- 2) $H_1 \leq \varepsilon w_d$ finite: the diffusion *across* the layer is bounded
- 3) The phase of the LR is strongly correlated
- 4) The phase of the distant harmonic to the GR is almost random: the diffusion *along* the layer is driven by the DR
- 5) Normal diffusion: $\langle \Delta H_2^2 \rangle \sim D_2 t$

Main result:

$$D_2(\omega_2, \varepsilon, \mu) \approx \begin{cases} \frac{8\pi^2 \mu^2}{T_a} e^{-\pi/\sqrt{\varepsilon}}, & 0 < |\omega_2| < 1, \\ \frac{8\pi^2 \mu^2 \omega_2^4}{T_a} e^{-\pi\omega_2/\sqrt{\varepsilon}}, & |\omega_2| > 1. \end{cases}$$

Numerical experiments

We investigate the dependence of D_2 on ω_2 , the computation: Ensembles of $n_p = 400$ i.c. in the center of the GR chaotic layer

Integration time: 5×10^6

Parameters: $\varepsilon = 0.10, 0.05$, $\mu = 0.01$, $0.2 \leq \omega_2 < 1.5$

Arguments of $A_2(\lambda)$: $1/\omega_0 = 1/\sqrt{\varepsilon} \approx 3.16$; $0.63 < \omega_2/\omega_0 < 4.75$

$1/\omega_0 \approx 4.47$; $0.89 < \omega_2/\omega_0 < 6.71$

Two different ways:

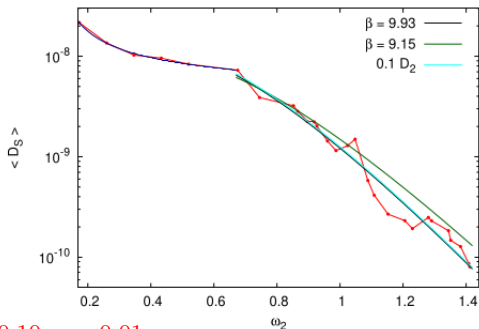
- A Shannon entropy approach to get diffusion coefficient $\langle D_S \rangle$. It rests on a *local* normal diffusion approximation.

- A power law for the variance evolution of I_2 : $\text{var}[I_2(t)] = 2D_a t^b$.

$b=1$: normal diffusion, $b \neq 1$: anomalous. D_a diffusion-like coefficient.

For comparison of both estimates, compute

$$T_D^S = \frac{(\Delta I_2^2)_0}{2\langle D_S \rangle}, \quad T_D^a = \left[\frac{(\Delta I_2^2)_0}{2D_a} \right]^{1/b}, \quad (\Delta I_2)_0 = 0.01.$$



$\langle D_S \rangle$ for $\varepsilon = 0.10, \mu = 0.01$.

Numerical law for $\omega_2 > 0.67$:

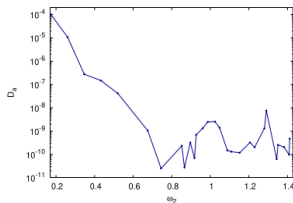
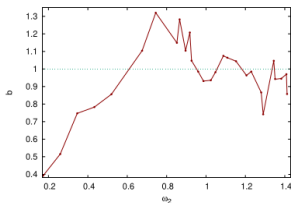
$$\langle D_S \rangle(\omega_2) = a \omega_2^4 e^{-\beta \omega_2} \quad D_2 = \frac{8\pi^2 \mu^2 \omega_2^4}{T_a(\varepsilon \mu)} e^{-\pi \omega_2 / \sqrt{\varepsilon}}, \quad \omega_2 > 1$$

Black: fit with $\beta = \pi/\sqrt{\varepsilon} \approx 9.93$, $a = 2.44 \times 10^{-5} \pm 9.15 \times 10^{-6}$.

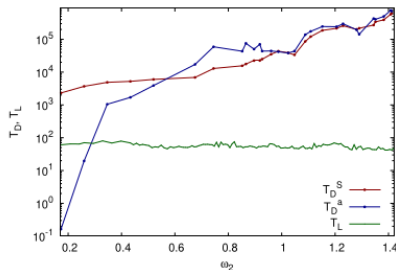
Green: fitting both, $\beta = 9.15 \pm 0.31$, $a = 1.26 \times 10^{-5} \pm 3.43 \times 10^{-6}$.

Cyan: theoretical D_2 for $\omega_2 > 1$ multiplied by a reduction factor $R = 0.1$.

For $\omega_2 < 0.67$ the fitting law does not match the expected one.



Exponent b and D_a for the law $\text{var}(I_2) = 2D_a t^b$

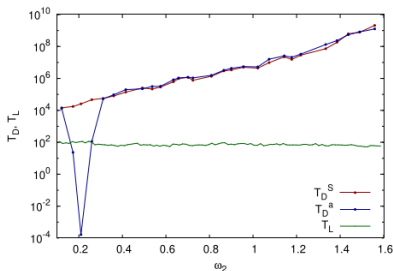
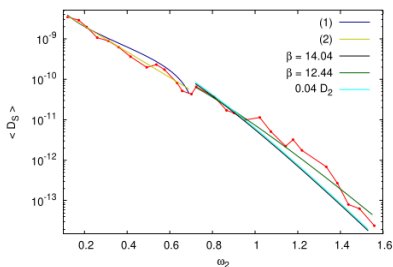


Diffusion times for $(\Delta I_2)_0 = 0.01$

$$T_D^S = \frac{(\Delta I_2)_0^2}{2\langle D_S \rangle}$$

$$T_D^a = \left[\frac{(\Delta I_2)_0^2}{2D_a} \right]^{1/b}$$

Lyapunov time: $T_L = mLE^{-1}$



Left: $\langle D_S \rangle$ for $\varepsilon=0.005, \mu=0.01$. Right: Diffusion and Lyapunov times.

Numerical law for $\omega_2 > 0.67$:

$$\langle D_S \rangle(\omega_2) = a\omega_2^4 e^{-\beta\omega_2}.$$

Black: fit with $\beta = \pi/\sqrt{\varepsilon} \approx 14.04$, $a = 7.19 \times 10^{-6} \pm 3.92 \times 10^{-7}$.

Green: fitting both, $\beta = 12.44 \pm 0.93$, $a = 1.85 \times 10^{-6} \pm 1.46 \times 10^{-6}$.

Cyan: theoretical D_2 for $\omega_2 > 1$ multiplied by a reduction factor $R=0.04$.

For $\omega_2 < 0.67$ the fitting laws do not match the expected one.

In both experiments, numerical results agree quite well with theoretical estimates for $\omega_2 > 0.67$

In the case $\omega_2 < 0.67$, no agreement was found: Resonance interaction is too strong ●

Further studies:

i) HW, w_d , of the chaotic layer via the 3D sx map for $s = w/W$:

$$s' = s + \sin t - v \cos \vartheta_2$$

$$t' = t - \lambda \ln |s'| + \eta \quad \lambda = 1/\sqrt{\varepsilon}$$

$$\vartheta_2' = \vartheta_2 - \omega_2 \lambda \ln |s'| + \omega_2 \eta$$

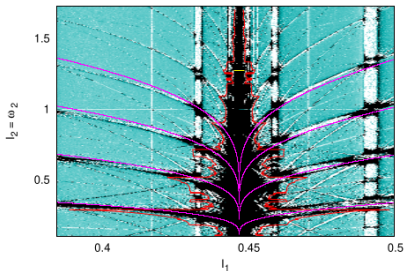
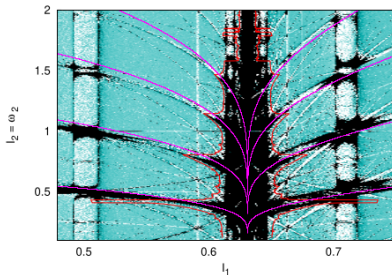
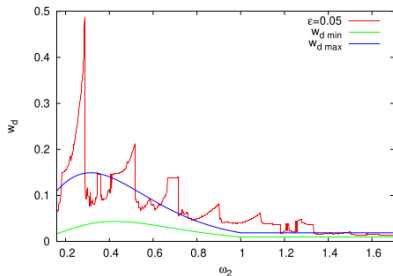
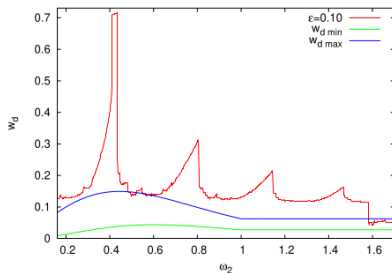
$$\text{- For } \omega_2 < 1, v \gg 1 \rightarrow \omega_2 v \lambda < s_d < \omega_2 v (\lambda + 4)$$

$$\text{- For } \omega_2 > 1, v \ll 1 \rightarrow \lambda < s_d < (\lambda + 4)$$

Ensembles of $n_p = 100$ i.c around $s \approx 0, t = \pi/2, \vartheta_2 = 0$ and 10^6 iterates.

Symmetry of the map: $(t, \vartheta_2, s) \rightarrow (t + \pi, \vartheta_2 + \pi, -s)$, then

$$s_d = \max(s, -s) \rightarrow w_d = W s_d.$$



Top: HW chaotic layer GR, 10^6 iterations, $n_p = 100$, $w_d = s_d W$.

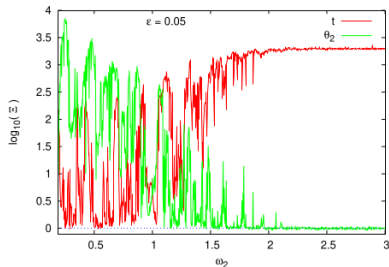
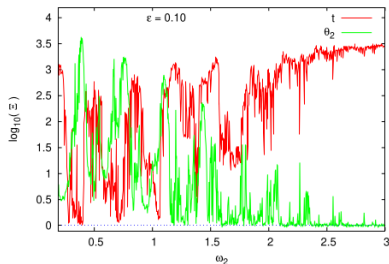
Bottom: DI contour plot, the HW and low order resonances

$\omega_2 = k_1 \omega_p$, $k_1 = 1, \dots, 4$.

ii) Phase correlations in the 3D sx map via an entropy approach:

$\Xi_x \sim$ entropy of phase x relative to entropy of a random phase:

$\Xi_x \approx 1 \rightarrow x \sim$ nearly random $\Xi_x \gg 1 \rightarrow x$ strongly correlated



Correlations of ϑ_2 and t .

For $\omega_2 < 1$ correlations persist in both phases.

Chirikov's assumption:

$\omega_2 < 1$, ϑ_2 is correlated, $t \sim$ random;

$\omega_2 > 1$, $\vartheta_2 \sim$ random, t is correlated.

Discussion

Chirikov's formulation works in high-frequency regime, $\omega_2 > 1$.

Weak high-order resonance interaction, LR and DR play their role.

For $\omega_2 > 1$ the phases behave as assumed, so D_2 is well estimated.

The reduction factor $R < 1$, introduced in Chirikov's theory, takes into account correlations that are always present in the external part of the chaotic layer of the GR.

In the low-frequency regime, $\omega_2 < 1$, the interaction of high-order resonances is strong, driving the diffusion away from the GR.

Resonances among the 3 frequencies $\omega_2(I_2) = k_1\omega_p(h_1, \varepsilon) + k_3$ are relevant.

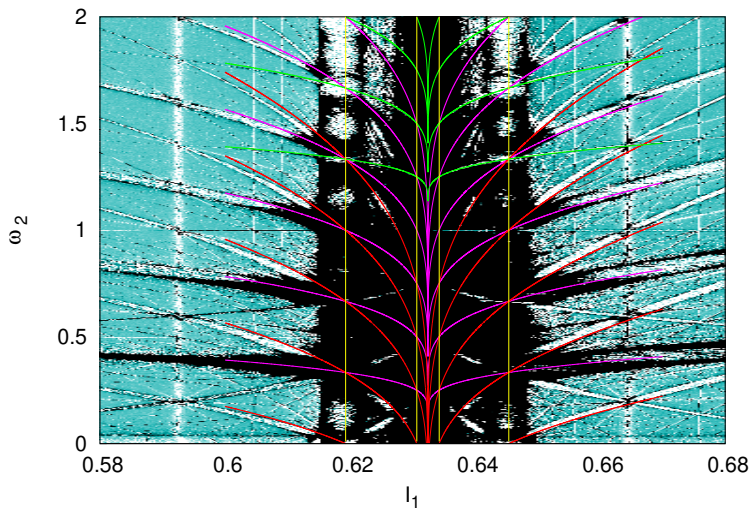
$\langle \Delta H_1^2 \rangle$ is not bounded by the expected threshold.

The chaotic layer is due to the overlap of the resonances

$\omega_2(I_2) = k_1\omega_p(h_1, \varepsilon) + k_3$ and $\omega_p(h_1, \varepsilon) = -k_3/k_1$.

Both phases exhibit correlation and uncorrelation (the derivation of D_2 rests on t nearly random).

Then, in such a domain, the concepts of LR and DR are not well defined.



Chirikov's formulation includes as perturbing resonances $\omega_2 = \omega_1$ and $\omega_1 = 1$. Actually, $\omega_1 = 1/2$ is closest to the GR and, for the adopted values of ε , it does not overlap with the GR.

Though the resonance $\omega_1 = 1/2$ is of smaller amplitude, the changes in H and H_2 depend on the frequency ratio through the MAI.

Resonances $\omega_2(I_2) = k_1\omega_p(h_1, \varepsilon) + k_3, k_3 = 0, -1$ help to explain why diffusion is faster when $\omega_2 < 1$:

These resonances approach the sx at $I_1 = 2\sqrt{\varepsilon}, \omega_2 \rightarrow 0, -1$, so their stable and unstable manifolds become aligned in the I_2 direction.

Chirikov applies his formulation, with the same assumptions, to a generic Hamiltonian. In such a case, only a single perturbation parameter exists and Arnold's mechanism would not work. However, some numerical examples show that they apply when considering LR and DR of very high order and far from a resonance crossing.

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