

Equilibrium configurations of rotating inhomogeneous liquid masses under gravity and shell elasticity

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1. Lamb H. Hydrodynamics. - M.-L.: 1947. - 928 p
2. Appel P. Equilibrium Figures of a Rotating Homogeneous Fluid. 1936. - 322 p.
3. Chandrasekhar S. Ellipsoidal Figures of Equilibrium. - M., 1973. - 288 p.
5. Chandrasekhar S. The Stability of a rotating liquid drop// Proc/ Roy. Soc. (London), A286, 1965. pp. 1-26.



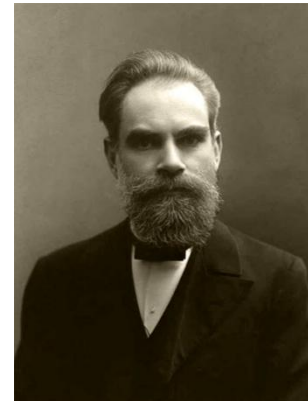
Colin Maclaurin
(1698–1746)
1742



Carl Gustav Jacobi
(1804-1851)
1834



Jules Henri
Poincaré
1854-1912
1885



Александр
Михайлович
Ляпунов
1857-1918
1884



Subrahmanyan
Chandrasekhar
(1910-1995)

1963

General Formulation of the Problem

A liquid mass rotates about an axis. The fluid is ideal. Then, the motion of such a system can be determined by the Lagrange equations with the Lagrangian function equal to the difference between the kinetic energy of the fluid and the potential energy. The angular variable is a cyclic coordinate. It can be eliminated by the Routh method, and the dynamics of the liquid mass will be determined by means of the Routh function, which is represented in the form

$$R = T_2 + T_1 - U, \quad U = \Pi + \frac{K^2}{2J}, \quad K = J_0 \Omega$$

Lyapunov-Poincaré Theorem on Stability Conditions.

If the equilibrium shape is the minimum point U , then the equilibrium shape is stable by the Lagrange-Dirichlet theorem and asymptotically stable when dissipative forces are added. If the equilibrium shape is not the minimum point of the function U and can take negative values, then when arbitrarily small resistance forces are added with complete dissipation, the equilibrium shape is unstable. The region of equilibrium forms corresponding to the minimum of the function U is called the region of secular stability.

Investigation of the U function on a family of triaxial ellipsoids

$$U = \Pi + \frac{K^2}{2J}, \quad a_1 a_2 a_3 = \text{const}, \quad K = J_0 \Omega$$

The energy of the self-gravitating mass

$$\Pi = -\frac{8\pi^2}{15} G \rho^2 \int_0^\infty \frac{(a_1 a_2 a_3)^2 du}{\sqrt{(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)}}$$

Ball Energy

$$\Pi_0 = -\frac{16\pi^2 G \rho^2 a^5}{15}$$

Moment of inertia

$$J = \frac{4\pi\rho}{15} a_1 a_2 a_3 (a_1^2 + a_2^2)$$

Maclaurin ellipsoids

$$a_1 = a_2, \quad x = a_3 / a_1$$

$$\delta \tilde{U} = \left(-\frac{1}{2} \tilde{\Omega}^2 \frac{d\tilde{J}}{dx} + \frac{d\tilde{\Pi}}{dx} \right) \delta x, \quad \tilde{J} = \frac{1}{2} x^{-2/3}, \quad \tilde{\Pi} = -\frac{x^{1/3} \arccos x}{\sqrt{1-x^2}},$$

$$\frac{\Omega^2}{\pi G \rho} = \boxed{\tilde{\Omega}^2 = 2 \frac{d\tilde{\Pi}}{dx} / \frac{d\tilde{J}}{dx}} = -\frac{6x^2}{1-x^2} + \frac{2x(1+2x^2) \arccos x}{(1-x^2)^{3/2}}$$

Jacobi ellipsoids

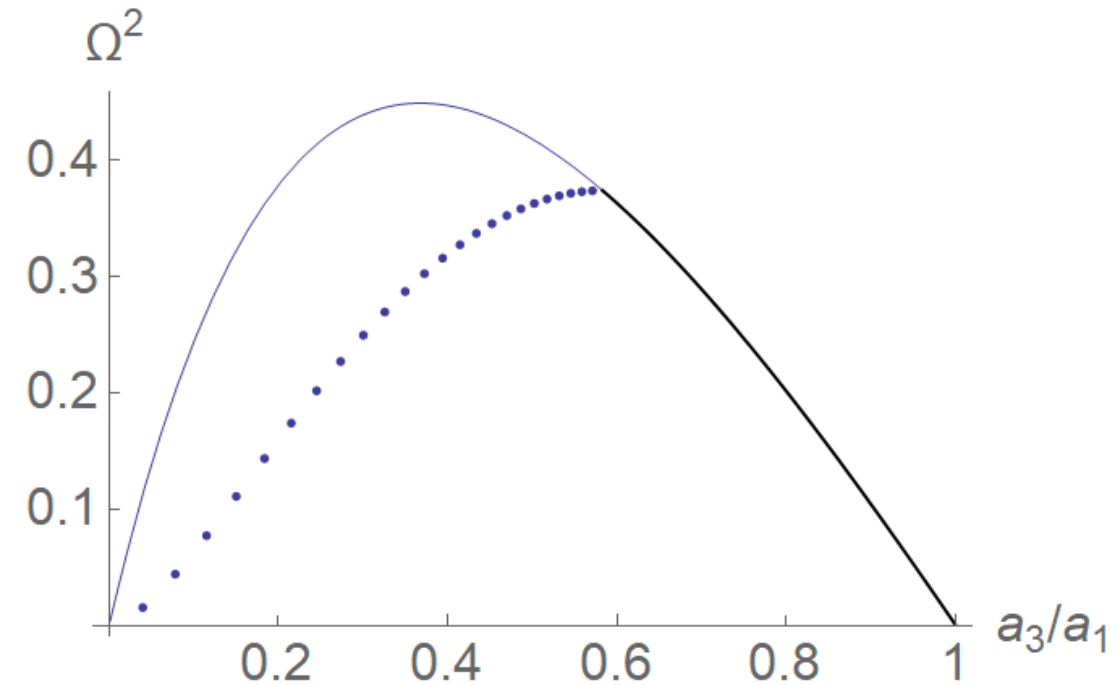
Equations of equilibrium

$$-\frac{1}{2}\tilde{\Omega}^2 \frac{\partial \tilde{J}}{\partial X} + \frac{\partial \tilde{\Pi}}{\partial X} = 0, \quad -\frac{1}{2}\tilde{\Omega}^2 \frac{\partial \tilde{J}}{\partial Y} + \frac{\partial \tilde{\Pi}}{\partial Y} = 0$$

We exclude it $\tilde{\Omega}^2$

$$\boxed{\frac{\partial \tilde{J}}{\partial Y} \frac{\partial \tilde{\Pi}}{\partial X} - \frac{\partial \tilde{J}}{\partial X} \frac{\partial \tilde{\Pi}}{\partial Y} = 0.}$$

$$X = \frac{a_3}{a_1}, \quad Y = \frac{a_3}{a_2}.$$



$$\tilde{\Omega}_*^2 = 0.37423, \quad x_* = a_3 / a_1 = 0.58272$$

Compression of the Earth with uniform density distribution

Ellipsoid with semi-axes $a_1 = a_2 = a(1 + \delta), \quad a_3 = a(1 + \delta)^{-2}$

$$\Pi_0 = -\frac{16\pi^2 G \rho^2 a^5}{15} \left(1 - \frac{4}{5} \delta^2\right), \quad J = \frac{8}{15} \pi \rho a^5 (1 + 2\delta)$$

Equations of equilibrium $-\frac{1}{2} \tilde{\Omega}^2 \frac{d\tilde{J}}{d\delta} + \frac{d\tilde{\Pi}}{d\delta} = 0 \Rightarrow \delta = \frac{5\Omega^2}{16\pi G \rho}$

For the Earth, $\Omega = 7.29 \times 10^{-5} \text{ rad s}^{-1}$, $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$

$$\rho = 5514 \text{ kg m}^{-3}$$

$$\delta = \frac{5\Omega^2}{16\pi G \rho} = \delta = 0.00143, \quad a_1 / a_3 = (1 + \delta)^3 \approx 1 + 3\delta = 1 + 1/232.$$

Forms of equilibrium of a rotating drop

The Virial
method

Potential energy $\tilde{U} = \tilde{S} + \frac{\tilde{K}^2}{2\tilde{J}}, \quad \tilde{J} = \frac{8}{15} \left(\frac{X^{2/3}}{Y^{4/3}} + \frac{Y^{2/3}}{X^{4/3}} \right),$

$$\tilde{S} = \frac{(X^p + Y^p + 1)^{\frac{1}{p}}}{3^{1/p} (XY)^{1/3}}, \quad \tilde{K} = \tilde{J}_0 \tilde{\Omega}, \quad \tilde{\Omega}^2 = \frac{\Omega^2 \rho a_1 a_2 a_3}{8\sigma}, \quad X = \frac{a_3}{a_1}, \quad Y = \frac{a_3}{a_2}$$



Subrahmanyan
Chandrasekhar
(1910-1995)

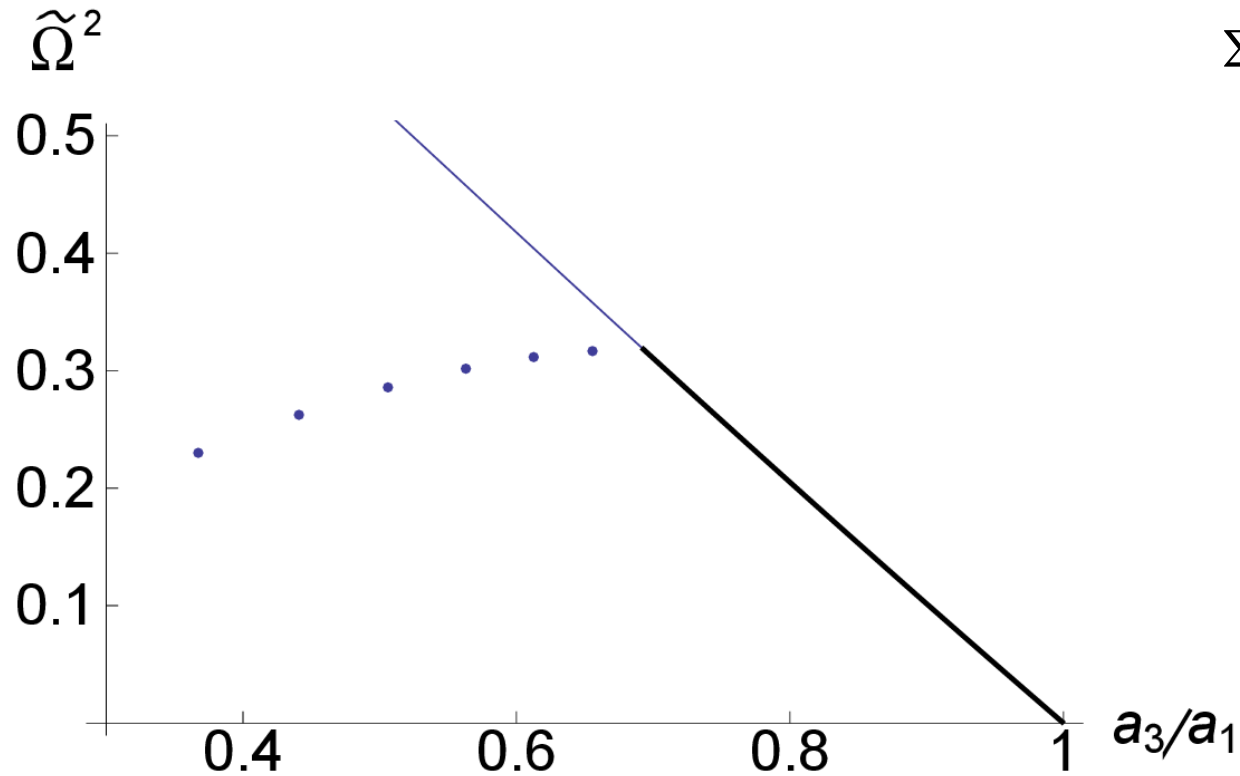
Axisymmetric series

$$\tilde{\Omega}^2 = 2\tilde{S}'(X) / \tilde{J}'(X) = \frac{15}{8 \times 3^{1/p}} (1 - X^p)(1 + 2X^p)^{1/p-1}.$$

Stability $\delta^2 U = a_{11}(\delta X^2 + \delta Y^2) + a_{12} \delta X \delta Y > 0,$

$$a_{11} - a_{12} = \frac{(1 + 2X^p)^{1/p-1}}{2 \times 3^{1/p} X^{8/3}} ((2 + p)X^p - 2) > 0, \quad 1 > X > X_* = \left(\frac{2}{2 + p} \right)^{1/p} = 0.693$$

Axisymmetric and non-axisymmetric series of equilibrium figures



$$\Sigma = \frac{\Omega^2 \rho a_1^3}{8\sigma} < 0.4587, \quad \Sigma = \tilde{\Omega}^2 / X < 0.4592$$

$$a_1 / l = a_1 / (a_1^2 a_3)^{1/3} = X^{1/3}$$

Σ	a_1 / l	a_{ch} / l
0.1611	1.049	1.050
0.2104	1.063	1.064
0.2756	1.081	1.082
0.3667	1.105	1.106

Axisymmetric series $1 > X > X_* = 0.693, \quad 0 < \Omega^2 < 0.318$

Non-axisymmetric series

$$\frac{\partial \tilde{J}}{\partial Y} \frac{\partial \tilde{S}}{\partial X} - \frac{\partial \tilde{J}}{\partial X} \frac{\partial \tilde{S}}{\partial Y} = 0 \Rightarrow Y^2(1 - Y^p) = X^2(1 - X^p), \quad X = \frac{a_3}{a_1}, \quad Y = \frac{a_3}{a_2}$$

How to find a non – axisymmetric series (X,Y) ?

Non-axisymmetric series

$$\frac{\partial \tilde{J}}{\partial Y} \frac{\partial \tilde{S}}{\partial X} - \frac{\partial \tilde{J}}{\partial X} \frac{\partial \tilde{S}}{\partial Y} = 0 \Rightarrow Y^2(1 - Y^p) = X^2(1 - X^p), \quad X = \frac{a_3}{a_1}, \quad Y = \frac{a_2}{a_1}$$

Substitution $X = ZY \Rightarrow (1 - Y^p) = Z^2(1 - Z^p Y^p) \Rightarrow 1 - Z^2 = Y^p(1 - Z^{p+2})$

$$\frac{a_2}{a_1} = Z, \quad \frac{a_3}{a_1} = X = Z \left(\frac{1 - Z^2}{1 - Z^{p+2}} \right)^{1/p}, \quad \frac{a_3}{a_2} = Y = \left(\frac{1 - Z^2}{1 - Z^{p+2}} \right)^{1/p}$$

a_2 / a_1	a_3 / a_1	Ω^2	D
0.9	0.655592	0.316691	0.002432
0.8	0.612582	0.311575	0.011182
0.7	0.562994	0.301721	0.030005
0.6	0.506012	0.28582	0.066902
0.5	0.440935	0.262418	0.140778
0.4	0.367326	0.230056	0.303513

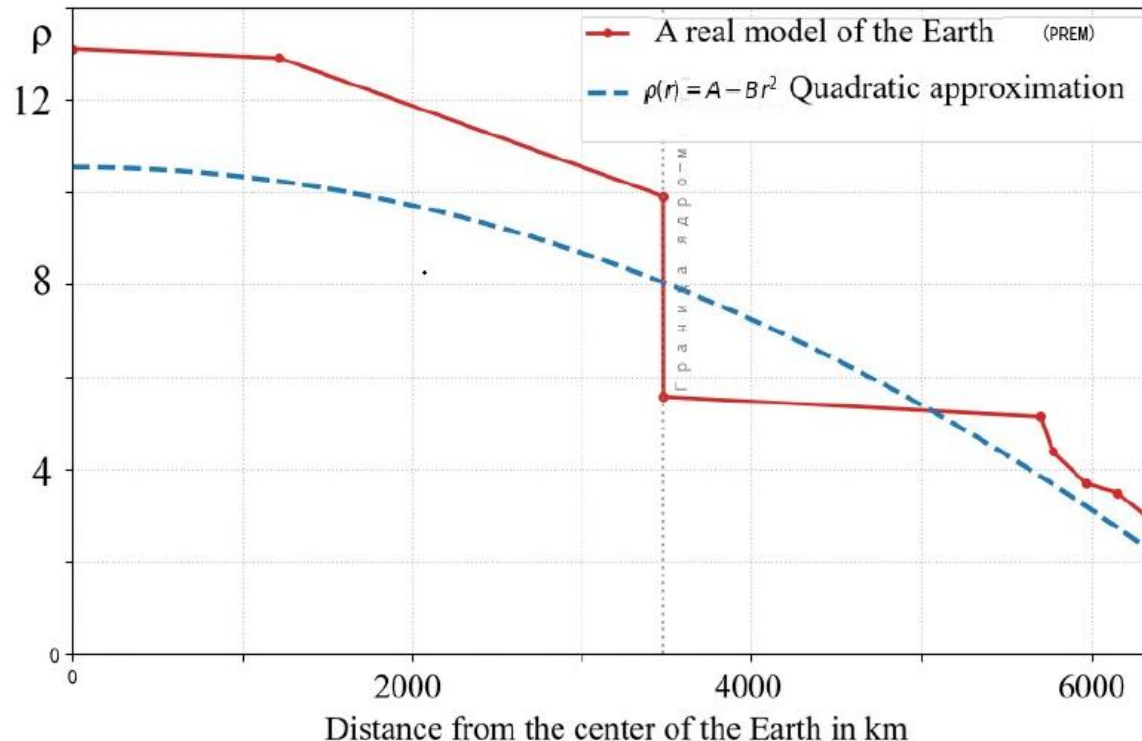
Earth's flattening for an inhomogeneous density distribution

Model in which the density depends quadratically on the distance r from the center:

$$\rho(r) = A - Br^2, \quad 0 \leq r \leq a$$

The constants A and B are determined from two global integral characteristics:
the total mass M and the moment of inertia I . For the Earth, this gives

$$A \approx 10530 \text{ kg m}^{-3}, \quad B \approx 2.06 \times 10^{-10} \text{ kg m}^{-5}, \quad a = 6.37 \times 10^6 \text{ m}$$



Roshe E. *Mémoire sur la loi de la densité à l'intérieur de la Terre*//Mémoires de l'Académie des Sciences de Montpellier. Vol 1. 1848

Marchenko A.N. "Earth's radial density profiles based on Gauss' and Roche's distributions" // *Boll di Geod e Scien Affini*, 2000, Vol. 59, No. 3

Moment of inertia $J(\delta) \approx J_0(1 + 2\delta), \quad J_0 = \frac{8\pi a^5}{15} \left(A - \frac{5}{7} B a^2 \right),$

The energy of the self-gravitating $\Pi(\delta) \approx \Pi_0 \left(1 - \frac{4}{5} \delta^2 \right), \quad \Pi_0 = -\frac{16\pi^2 G}{15} a^5 \left(A^2 - \frac{10}{7} A B a^2 + \frac{25}{63} B^2 a^4 \right).$

$\delta = -5J_0\Omega^2 / (8\Pi_0).$ For the Earth $J_0 = 8.0 \times 10^{37}, \quad \Pi_0 = -2.47 \times 10^{32}$

$\delta = 0.00108, \quad (a_1 - a_3) / a_1 = 3\delta = 0.00323$

Newton

The corresponding flattening $(a_1 - a_3) / a_1 = 0.00335, \quad (1 / 232 = 0.00431)$

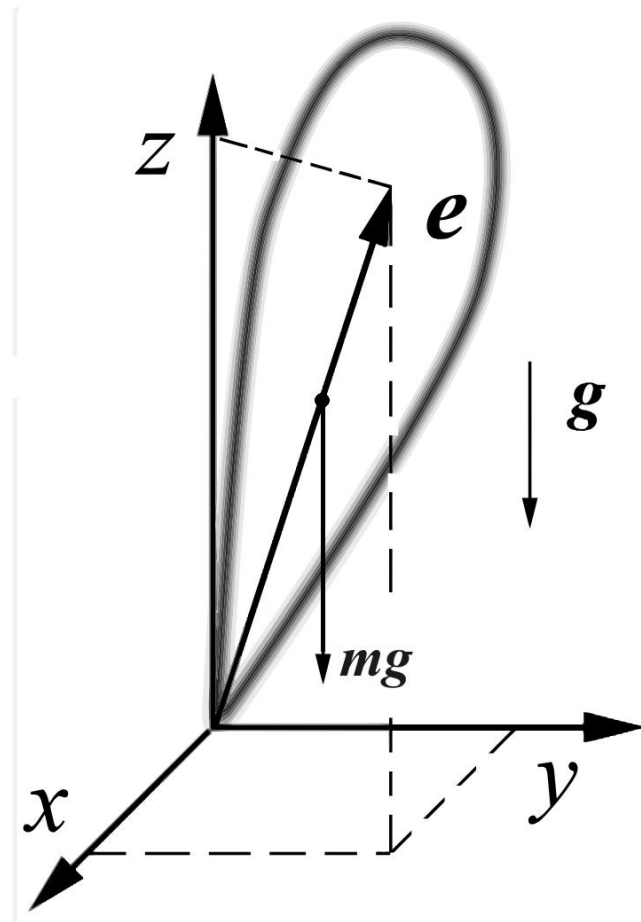
Example: Lagrange's top

$$\frac{d\mathbf{K}}{dt} = A\mathbf{e} \times \frac{d^2\mathbf{e}}{dt^2} + Cr \frac{d\mathbf{e}}{dt} = \mathbf{Mom} = mgl\mathbf{e} \times \mathbf{k},$$

$$\varepsilon = \frac{|\mathbf{Mom}|}{Cr\omega} = \frac{mgl \sin \theta}{Cr\omega}, \quad \varepsilon_1 = \frac{A\omega}{Cr},$$

$$\frac{d\alpha}{dt} = - \frac{2mgl}{Cr \left(1 + \sqrt{1 + \frac{2Amgl \cos \theta}{C^2 r^2}} \right)}$$

$$= - \frac{2mgl}{Cr \left(1 + \sqrt{1 + \varepsilon \varepsilon_1 \operatorname{ctg} \theta} \right)} \approx - \frac{mgl}{Cr}$$

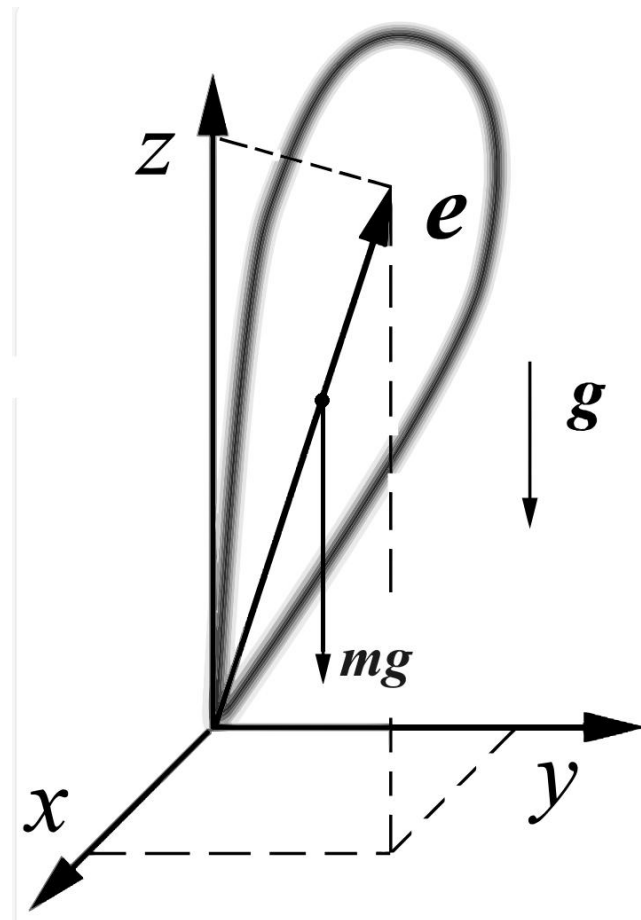


Example: Lagrange's top

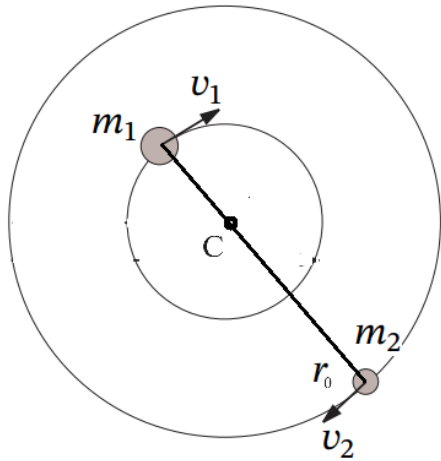
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Circular two-body problem



earth

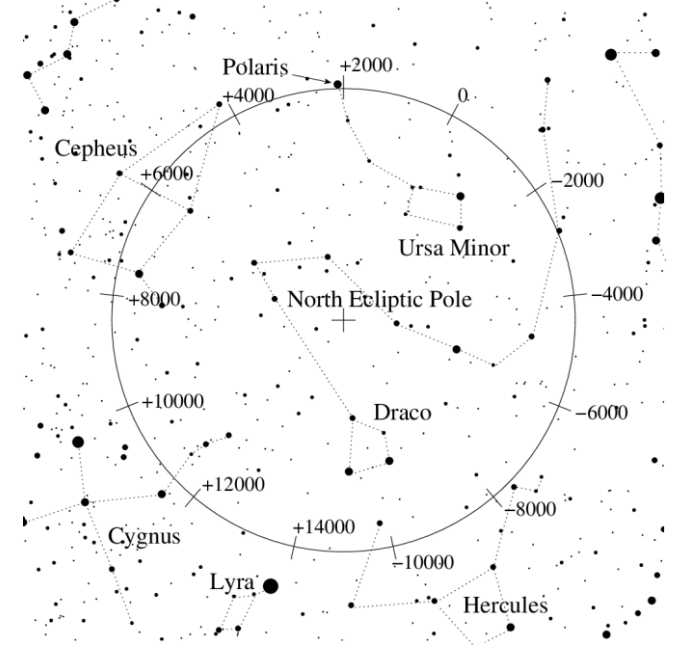
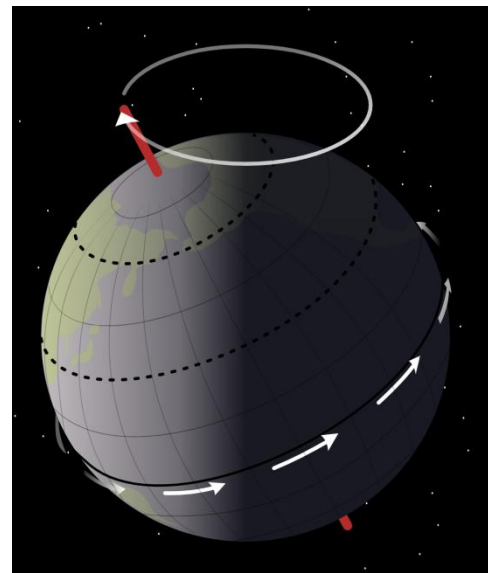
$$\mathbf{F} = -\gamma m_1 m_2 \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

$$\mathbf{Mom}(\omega t) = \frac{3\gamma m_1 m_2}{R^3} (A - C) ((\mathbf{r}_0 / R) \cdot \mathbf{e}) ((\mathbf{r}_0 / R) \times \mathbf{e})$$

$$\frac{d\mathbf{K}}{dt} = \mathbf{Mom}, \quad \mathbf{K} = A\mathbf{e} \times \frac{d\mathbf{e}}{dt} + C\mathbf{r}\mathbf{e}$$

$$\dot{\alpha} \sin \theta = M_2$$

$$\frac{d\alpha}{r dt} = \frac{3}{2} \left(\frac{\omega}{r} \right)^2 \delta \cos \theta \left(1 + \frac{m_1}{m_2} \right)^{-1}, \quad \delta = \frac{A - C}{C}$$



The period of lunisolar precession

$$\theta = 23.5^\circ, \quad \omega_1 = (360 / N_1)^\circ / \text{day}, \quad \omega_2 = (360 / N_2)^\circ / \text{day},$$

$$r = 360^\circ / \text{day}, \quad N_1 = 365 \text{day}, \quad N_2 = 28 \text{day}, \quad \delta = 0.0033$$

$$\frac{d\alpha}{dt} = \frac{3}{2} \delta \cos \theta \left(\frac{\omega_1^2}{r} + \frac{\omega_2^2}{r} (1 + m_e / m_l)^{-1} \right) \frac{^\circ}{\text{day}},$$

$$P = 26000 \text{year}, \quad m_e / m_l = 81$$

Таутохронные движения

(Петров А.Г. Доклады РАН 2024)

«Единственной зависящей только от x силой, вызывающей таутохронное движение, является притяжение, пропорциональное расстоянию» Аппель. ТМ, Том 1, стр 299.

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$$\ddot{x} + kx = 0$$

$$\ddot{x} + kx = nx^{-3} \quad x^2 = \frac{1}{2} \sqrt{n/k} \left(\xi + 1/\xi + (\xi - 1/\xi) \cos 2\sqrt{k}t \right),$$

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Гамильтонова форма

$$H_1 = \frac{1}{2} (p^2 + (1+q)^2 + (1+q)^{-2}),$$

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

$$R(Q, P) = P^2 + 4Q^2 + 2 + 2Q\sqrt{P^2 + 4Q^2 + 4},$$

$$q(Q, P) = \sqrt{R(Q, P)/2} - 1, \quad p(Q, P) = P \sqrt{\frac{P^2 + 4Q^2 + 4}{2R(Q, P)}}$$

$$\tilde{H} = \frac{1}{2} (P^2 + 4Q^2)$$