

Equilibrium configurations of rotating inhomogeneous liquid masses under gravity and shell elasticity

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Abstract. Equilibrium configurations of a rotating liquid mass subject to gravitational or elastic forces are considered. Routh reduction is used to eliminate the cyclic angular coordinate. By the Lyapunov–Poincaré theorem and within the Ritz approximation, the equilibrium and stability problem is reduced to minimizing a function of two dimensionless semiaxes of a triaxial ellipsoid. For a uniform density distribution, the proposed approach reproduces the known exact solutions: the families of Maclaurin spheroids and Jacobi ellipsoids. For a drop held together by surface tension, an approximate analytical solution of the equilibrium and stability problem is obtained in elementary functions and agrees with the results tabulated by Chandrasekhar. For an inhomogeneous mass distribution, a quadratic Roche model is used to obtain an analytical estimate of the Earth's flattening that is close to the observed value.

Introduction

This study concerns the shape of a rotating self-gravitating liquid mass. The motion of such a system can be described by the Lagrange equations, with the Lagrangian equal to the difference between the kinetic and potential energies of the liquid. The angular variable is a cyclic coordinate. It can be eliminated by Routh reduction, after which the dynamics of the liquid mass is governed by the Routhian

$$R = T_2 + T_1 - U, \quad U = \Pi + \frac{K^2}{2J}, \quad K = J_0\Omega,$$

where T_2 is a positive-definite quadratic form and T_1 is a linear form in the generalized velocities. The function U is independent of the velocities. It is the effective potential energy and includes the rotational term determined by the moment of inertia.

The quantity K is the constant angular momentum of the ellipsoid. An equilibrium configuration is determined by the stationarity condition for U . If the equilibrium configuration is a local minimum of U , it is stable by the Lagrange–Dirichlet theorem and becomes asymptotically stable when dissipative forces are added. If the equilibrium configuration is not a minimum of U and the second variation can take negative values, then arbitrarily small resistance forces with complete dissipation make it unstable. The set of equilibrium configurations corresponding to minima of U is called the region of secular stability. This result is given in Appell’s monograph [1]. Appell referred to the formulation of the secular stability conditions for a rotating liquid mass in the presence of completely dissipative resistance forces as the Poincaré theorem.

We seek the minimum of the functional within a family of triaxial ellipsoids, which is the Ritz approximation. The problem is thereby reduced to minimizing a function of the ellipsoid semiaxes a_1 , a_2 , and a_3 .

For axisymmetric configurations, the problem reduces to minimizing a function of one variable, x , equal to the ratio of the ellipsoid semiaxes:

$$\begin{aligned}\tilde{U} &= \tilde{\Pi} + \frac{\tilde{K}^2}{2\tilde{J}}, & \tilde{K} &= \tilde{J}_0\tilde{\Omega}, & \tilde{\Omega} &= \frac{\Omega}{\sqrt{\pi G\rho}}, \\ \tilde{J} &= \frac{1}{2}x^{-2/3}, & \tilde{\Pi} &= -\frac{x^{1/3}\arccos x}{\sqrt{1-x^2}}, & x &= \frac{a_3}{a_1}.\end{aligned}$$

Here J , Π , Ω , and K denote the moment of inertia, potential energy, angular velocity, and angular momentum, respectively [2, 3]. Tildes denote dimensionless quantities; in particular, $\tilde{\Pi}$ is normalized by the magnitude of the gravitational energy of a sphere of radius $(a_1 a_2 a_3)^{1/3}$.

The necessary stationarity condition for $\tilde{U}$ yields the family of Maclaurin equilibrium configurations:

$$\tilde{\Omega}^2 = -\frac{6x^2}{1-x^2} + \frac{2x(1+2x^2)\arccos x}{(1-x^2)^{3/2}}.$$

The value $x_* = 0.582724$ corresponds to the eccentricity $e_* = 0.81267$. The Maclaurin formula then gives $\tilde{\Omega}_*^2 = 0.37423$. The point x_* is a bifurcation point from which the family of asymmetric Jacobi ellipsoids branches. Thus, the Maclaurin family is secularly stable for $x_* < x < 1$.

1. Earth’s flattening for a uniform density distribution

Consider an ellipsoid with semiaxes $a_1 = a_2 = a(1+\delta)$ and $a_3 = a(1+\delta)^{-2}$, where δ is a small deformation parameter. In terms of this parameter, the potential energy and moment of inertia, to leading order, are

$$\Pi = -\frac{16\pi^2 G\rho^2 a^5}{15} \left(1 - \frac{4}{5}\delta^2\right), \quad J = \frac{8}{15}\pi\rho a^5(1+2\delta).$$

The equilibrium equation for an ellipsoidal liquid mass rotating with angular velocity Ω is

$$\frac{d}{d\delta} \left(\frac{1}{2} J(\delta) \Omega^2 - \Pi(\delta) \right) = 0, \quad \delta = \frac{5\Omega^2}{16\pi G\rho}.$$

For the Earth, $\Omega = 7.29 \times 10^{-5} \text{ rad s}^{-1}$, $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, and $\rho = 5514 \text{ kg m}^{-3}$. Substitution gives $\delta = 0.00143$ and hence $a_1/a_3 = (1 + \delta)^3 \approx 1 + 3\delta = 1 + 1/232$. This result was already known to Newton and also follows from the Maclaurin formula. The observed ratio is approximately $a_1/a_3 = 1 + 1/298$. The discrepancy is explained by the nonuniform density distribution inside the Earth.

2. Earth's flattening for an inhomogeneous density distribution

Consider a model of the density distribution inside the planet in which the density depends quadratically on the distance r from the center: $\rho(r) = A - Br^2$, $0 \leq r \leq a$, where a is the mean radius of the planet. The constants A and B are determined from two global integral characteristics: the total mass M and the moment of inertia I . For the Earth, this gives

$$A \approx 10\,530 \text{ kg m}^{-3}, \quad B \approx 2.06 \times 10^{-10} \text{ kg m}^{-5}, \quad a = 6.37 \times 10^6 \text{ m}.$$

Figure 1 compares the density distribution in the Preliminary Reference Earth Model (PREM) with the quadratic approximation $\rho(r) = A - Br^2$.

According to PREM [7], the density at the Earth's center is about 13 g cm^{-3} , whereas the density of the crust near the surface is about 2.7 g cm^{-3} . The quadratic model slightly underestimates the density at both the center and the surface. To preserve the correct total mass and moment of inertia, it averages these values and produces a smoother profile. Nevertheless, this classical approximation is useful for analytical calculations in celestial mechanics and accurately reproduces integral quantities.

To leading order in δ , the moment of inertia $J(\delta)$ and the gravitational potential energy $\Pi(\delta)$ are

$$J(\delta) \approx J_0(1 + 2\delta), \quad J_0 = \frac{8\pi a^5}{15} \left(A - \frac{5}{7} B a^2 \right),$$

$$\Pi(\delta) \approx \Pi_0 \left(1 - \frac{4}{5} \delta^2 \right), \quad \Pi_0 = -\frac{16\pi^2 G}{15} a^5 \left(A^2 - \frac{10}{7} A B a^2 + \frac{25}{63} B^2 a^4 \right).$$

The equilibrium equation gives $\delta = -5J_0\Omega^2/(8\Pi_0)$. For the Earth, $J_0 = 8.0 \times 10^{37}$, $\Pi_0 = -2.47 \times 10^{32}$, and $\delta = 0.00108$. The corresponding flattening, $f = (a_1 - a_3)/a_1 \approx 3\delta = 0.00323$, is substantially closer to the observed value $1/298 \approx 0.00335$.

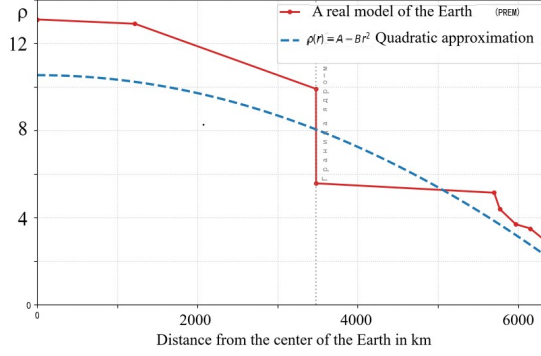


FIGURE 1. Comparison of the PREM density profile with the quadratic approximation for the Earth.

3. Effect of shell elasticity and other factors

Chandrasekhar [4] investigated the shape of a rotating liquid drop using the tensor virial method. The same approach can be extended to account for the elasticity of the Earth's shell by adding to Π the potential energy of the shell, which is proportional to its surface area [5]. V. A. Solonnikov and I. V. Denisova [7] studied the existence of axisymmetric equilibrium configurations of two-layer incompressible and compressible viscous liquids, taking into account surface tension and self-gravity.

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