

On the behavior of orbits of Vanhaecke system on integral surfaces

Mikhail Malykh, Lubov Lapshenkova and Shiwei Wang

Abstract. In the 1990s, P. Vanhecke described a Hamiltonian system with two degrees of freedom and a polynomial Hamiltonian integrable in Abelian functions of two variables. We believe that the Vanhaecke system is a good test to hone methods for studying integral manifolds of conservative dynamical systems.

1. Introduction

The idea of the motion nature of conservative dynamical systems was formed in the 1970s [1]. A dynamical system is called transitive if almost any of its orbits fills the entire phase space densely. Transitivity of the system is prevented by conservation laws that hold the orbit on some integral manifold. Therefore, the study of a conservative dynamical system splits into two parts: 1) finding all conservation laws by analytical methods; 2) studying a dynamical system whose restriction to integral manifolds is transitive by statistical methods.

An issue of not less importance is the development of algorithms for finding manifolds on which the system is transitive. In fact, the problem is to find a complete set of conservation laws. Note that these manifolds can be seen by observing the trajectory over a long period of time. It is not difficult to roughly estimate the dimension of the manifold on which the integral curve “winds” and hence determine the number of independent conservation laws. However, the accumulation of errors in numerical methods makes conclusions based on these experiments not entirely convincing.

A natural first step in verifying such judgments is to apply numerical methods to dynamic systems that have been exactly integrated. Interesting and non-trivial from this point of view are nonlinear dynamic systems integrable in Abelian functions of two variables.

2. Vanhaecke system and its integral manifolds

In this talk, we consider the Vanhaecke system [2]. This is a completely integrable Hamiltonian system with Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + (q_1^2 + q_2^2)^2 + aq_1^2 + bq_2^2 \quad (1)$$

for $a \neq b$. This system has two polynomial integrals:

$$F = (p_2q_1 - p_1q_2)^2 - (2q_1^4 + 2q_1^2q_2^2 + 2aq_1^2 + p_1^2)(a - b)$$

and

$$G = (p_2q_1 - p_1q_2)^2 + (2q_1^2q_2^2 + 2q_2^4 + 2bq_2^2 + p_2^2)(a - b)$$

in involution.

Equations

$$F(p_1, p_2, q_1, q_2) = f, \quad G(p_1, p_2, q_1, q_2) = g \quad (2)$$

define in $\mathbb{R}^4$ a surface S , the integral surface of the Vanhaecke system. We calculated its projection into $q_1q_2p_1$ using the Gröbner basis in Sage [3], it is given by the formula

$$s_2p_1^4 + s_1p_1^2 + s_0 = 0, \quad (3)$$

where s_0, s_1, s_2 are polynomials in q_1, q_2 , whose coefficients depend on the parameters a, b, f, g .

One might think that Liouville's theorem allows finding two more integrals from two polynomial integrals in involution, and therefore the system has two more integrals depending on t . After eliminating t , we should obtain a one-dimensional integral manifold, not a surface. This kind of reasoning leads to the wrong result, since it suggests carrying over local results to the global case. In fact, Liouville's theorem leads to two additional integrals of motion, which are very complicated Abelian quadratures, written explicitly in [4].

Vanhaecke's passage to the Jacobi problem is based on a guess, which we will describe as follows: we pass from the variables q_1, q_2 to two new variables x_1, x_2 , which are roots of the equation

$$x^2 + (q_1^2 + q_2^2 + a + b)x + aq_2^2 + bq_1^2 + ab = 0.$$

The problem of integrating the Vanhaecke system on the integral surface can be written as a Jacobi problem

$$\begin{cases} \int_{x_1^0}^{x_1} \frac{d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{d\xi}{\sqrt{R(\xi)}} = 0, \\ \int_{x_1^0}^{x_1} \frac{\xi d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{\xi d\xi}{\sqrt{R(\xi)}} = t, \end{cases}$$

where t is the time passed between the initial position of the system (x_1^0, x_2^0) and the final position (x_1, x_2) . Here R is a polynomial of degree 5 in ξ . All necessary calculations can be performed in the Sage system.

The fundamental theorem of the theory of Abelian functions states that every rational symmetric function s of quantities x_1, x_2 satisfying the Jacobi problem

$$\begin{cases} \int_{x_1^0}^{x_1} \frac{d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{d\xi}{\sqrt{R(\xi)}} = u, \\ \int_{x_1^0}^{x_1} \frac{\xi d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{\xi d\xi}{\sqrt{R(\xi)}} = v, \end{cases}$$

is a meromorphic function of u, v :

$$s = \text{Al}(u, v),$$

referred to as Abelian function. The zeros of R define the periods of the Abelian function: if e_1, e_2 are zeros of R and there are no other zeros in the interval $[e_1, e_2]$, then the numbers

$$\omega_1 = 2 \int_{e_1}^{e_2} \frac{d\xi}{\sqrt{R(\xi)}}, \quad \omega_2 = 2 \int_{e_1}^{e_2} \frac{\xi d\xi}{\sqrt{R(\xi)}}$$

form a system of common periods of the Abelian function:

$$\text{Al}(u + \omega_1, v + \omega_2) = \text{Al}(u, v).$$

Returning to our dynamical system, we see that

$$x_1 + x_2 = \text{Al}_1(0, t), \quad x_1 x_2 = \text{Al}_2(0, t).$$

Thus the squares of q_1 and q_2 are expressed linearly through these functions, which allows expressing the general solution of the Vanhaecke system in terms of two Abelian functions of two arguments.

3. Results of computer experiments and discussion

For arbitrary initial data, the orbit is not closed and fills at $t \in \mathbb{R}$ some piece of the algebraic surface of the surface everywhere densely. In other words, for almost any initial data, the restriction of the Vanhaecke system to an integral surface is transitive. This does not contradict either the complete integrability of the Vanhaecke system in the sense of Liouville or its solvability in classical transcendental functions. The very intricate course of the orbit on the surface is described analytically by means of Abelian functions of two variables and is determined by four real numbers, a set of periods of Abelian functions. The key point here is the analytical description of functions of one variable t using periodic functions of two variables.

The behavior of orbits in numerical experiments is determined not so much by the rationality or irrationality of the ratio of two periods ω_1 and ω'_1 , but by the possibility of approximating this number with a rational fraction with high accuracy. Therefore, for arbitrary initial data, we see a seemingly periodic solution with a very large period and a closed orbit that runs along the integral surface densely enough for this surface to become visible, but not densely enough to allow saying that it covers the surface everywhere densely.

Details of the experiments can be found in [3].

References

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Mikhail Malykh

Peoples' Friendship University of Russia named after Patrice Lumumba

Moscow, Russia

e-mail: `malykh-md@rudn.ru`

Lubov Lapshenkova

Peoples' Friendship University of Russia named after Patrice Lumumba

Moscow, Russia

e-mail: `lapshenkova-la@rudn.ru`

Shiwei Wang

Peoples' Friendship University of Russia named after Patrice Lumumba

Moscow, Russia

e-mail: `1995wsw@gmail.com`