

On the behavior of orbits of Vanhaecke system on integral surfaces

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Epigraph

Als ich studierte, galten die Abelschen Funktionen – in Nachwirkung der Jacobischen Tradition – als der unbestrittene Gipfel der Mathematik, und jeder von uns hatte den selbstverständlichen Ehrgeiz, hier selbst weiterzukommen. Und jetzt? Die junge Generation kennt die Abelschen Funktionen kaum mehr. – Felix Klein, 1926.

When I was a student, the Abelian functions were regarded – as a continuation of the Jacobi tradition – as the undisputed pinnacle of mathematics, and each of us had the self-evident ambition to make further progress here ourselves. And now? The younger generation scarcely knows the Abelian functions at all. – Felix Klein, 1926.

Outline of the talk

- How was the problem of **integration in Abelian functions** posed in the 19th century?
- Which Hamiltonian systems are integrable in terms of Abelian functions?
- How is this related to **Liouville's theorem and KAM theory**?
- What do the **trajectories** of such systems look like in numerical simulations, and what should they look like?

Inversion of the integral

Let R be a polynomial of degree n in x with simple zeros $e_1, e_2, \dots, e_n$ and

$$\int \frac{dx}{\sqrt{R(x)}} = u.$$

- ❶ If $n = 2$, then x is a **single-valued periodic** function of u (sine).
- ❷ If $n = 3$ or 4 , then x is a **doubly periodic** function of u (elliptic function).
- ❸ If $n > 4$, then x cannot be a single-valued function of u .

Obstacle: the numbers

$$\omega_i = 2 \int_{e_i}^{e_{i+1}} \frac{dx}{\sqrt{R(x)}}$$

are **periods**, and as n grows, there are more and more of them. . .

Abelian functions

Let R be a polynomial of degree n in x with simple zeros $e_1, e_2, \dots, e_n$ and

$$\int_{a_1}^{x_1} \frac{dx}{\sqrt{R(x)}} + \int_{a_2}^{x_2} \frac{dx}{\sqrt{R(x)}} = u_1,$$
$$\int_{a_1}^{x_1} \frac{x dx}{\sqrt{R(x)}} + \int_{a_2}^{x_2} \frac{x dx}{\sqrt{R(x)}} = u_2.$$

If $n = 5$ or 6 , then a **rational symmetric function** of x_1 and x_2 is a **single-valued function** of u_1 and u_2 :

$$s(x_1, x_2) = \text{Al}(u_1, u_2).$$

Periods of Abelian functions

A pair of complex numbers (ω_1, ω_2) is called a **period** if

$$\text{Al}(u_1 + \omega_1, u_2 + \omega_2) = \text{Al}(u_1, u_2).$$

For Abelian functions, such a pair is computed by the formula

$$\omega_1 = 2 \int_e^{e'} \frac{dx}{\sqrt{R(x)}}, \quad \omega_2 = 2 \int_e^{e'} \frac{x dx}{\sqrt{R(x)}},$$

where e, e' are **zeros** of R . There are four such pairs of periods, two of which are **real**. One can show that they are **linearly independent over $\mathbb{Z}$** .

Integration of dynamical systems

Problem (Weierstrass, 1880s)

Given a dynamical system

$$\frac{dx}{dt} = f(x, y, z), \quad \frac{dy}{dt} = g(x, y, z), \quad F(x, y, z) = 0,$$

it is required to choose its **parameters** so that its solution is expressed **algebraically** in terms of **Abelian functions** of the form

$$\text{Al} \left(\int \alpha dt, \int \beta dt \right),$$

where α, β are **algebraic functions** of t .

We have no algorithm for solving this problem, nor any understanding of why dynamical systems should be integrable in this form.

Examples

Known examples:

- 1888: the **Kovalevskaya top** — a dynamical system with **quadratic right-hand side**, integrable in Abelian functions.
- 1994: the **Vanhaecke system** — a **completely integrable Hamiltonian system** with two degrees of freedom and a polynomial Hamiltonian, integrable in Abelian functions.

Remarks:

- Both cases can be regarded as **particular cases** of the **Garnier system**.
- In both cases, the solutions are expressed as

$$Al(k_1t + c_1, k_2t + c_2).$$

Vanhaecke system: Hamiltonian

The system has **two degrees of freedom** (q_1, q_2 – coordinates, p_1, p_2 – momenta).

Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + (q_1^2 + q_2^2)^2 + aq_1^2 + bq_2^2, \quad a \neq b.$$

Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2.$$

Remark: when $a = b$, the system **degenerates** (we do not consider this case).

First integrals

The system possesses two **polynomial integrals**:

$$F = (q_1 p_2 - q_2 p_1)^2 - (a - b)(2q_1^4 + 2q_1^2 q_2^2 + 2a q_1^2 + p_1^2),$$

$$G = (q_1 p_2 - q_2 p_1)^2 + (a - b)(2q_1^2 q_2^2 + 2q_2^4 + 2b q_2^2 + p_2^2).$$

The energy is expressed linearly through them:

$$F - G = 2(b - a)H.$$

Since the Poisson bracket $\{F, G\}$ is zero, the Vanhaecke system is **completely Liouville integrable**.

Invariant surface

Fix the values of the integrals:

$$F = f, \quad G = g.$$

In $\mathbb{R}^4$, these equations define a two-dimensional surface S .

For visualisation, we construct the projection onto (q_1, q_2, p_1) .

Using **Gröbner basis**, we obtain:

$$s_2(q_1, q_2) p_1^4 + s_1(q_1, q_2) p_1^2 + s_0(q_1, q_2) = 0,$$

where s_0, s_1, s_2 are polynomials in q_1, q_2 with parameters a, b, f, g .

The surface is symmetric (even powers) and has **double lines**.

Numerical simulation

We use the `fdm` for Sage package.

- **4th-order Runge–Kutta method.**
- Parameters: $a = 1$, $b = 2$.
- Initial conditions: $q_1(0) = 0$, $q_2(0) = 1$, $p_1(0) = 1$, $p_2(0) = 0$.
- Integration time: $0 < t < 200$ with a small step.

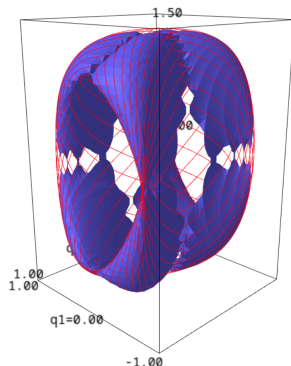
Result: the trajectory winds onto the surface, passing through the **double lines**.

In 4D the trajectory has no self-intersections (by the **Cauchy theorem**).

However, when projected to 3D, double lines appear – these are sections by coordinate planes.

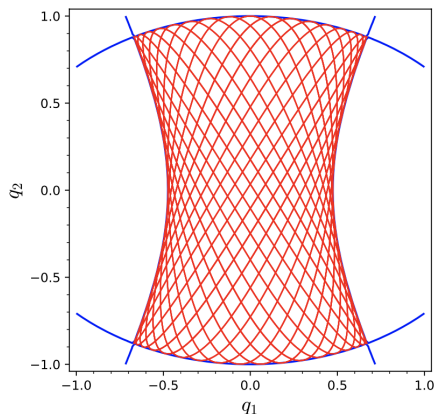
The surface consists of 8 “vareniki” glued into a cylinder; the trajectory passes from the outer side to the inner side through the double lines.

Visualization: projection and trajectory



Projection of the integral manifold into the space $q_1q_2p_1$ (blue surface) and one of the integral curves (red line) that winds onto this manifold. The trajectory appears to be closed, but this may be an illusion.

Projection onto the (q_1, q_2) -plane



Projection of the integral manifold onto the plane q_1q_2 and one of the integral curves (red line) that winds onto this manifold

Old version of Liouville's theorem

Suppose that two polynomial integrals F, G of a Hamiltonian system with 2 degrees of freedom are known, with $\{F, G\} = 0$. To recover the remaining two integrals, one should:

- 1 express from the system of equations

$$F(p_1, p_2, q_1, q_2) = f, \quad G(p_1, p_2, q_1, q_2) = g$$

the momenta p_1, p_2 as functions of q_1, q_2, f, g , and substitute them into the 1-form

$$p_1 dq_1 + p_2 dq_2 - H dt,$$

- 2 find the potential S for this exact 1-form

$$S = \int (p_1 dq_1 + p_2 dq_2 - H dt);$$

- 3 write down the two integrals

$$\frac{\partial S}{\partial f} = f', \quad \frac{\partial S}{\partial g} = g'.$$

Application to the Vanhaecke system

The system

$$\begin{aligned} f &= (q_1 p_2 - q_2 p_1)^2 - (a - b)(2q_1^4 + 2q_1^2 q_2^2 + 2a q_1^2 + p_1^2), \\ g &= (q_1 p_2 - q_2 p_1)^2 + (a - b)(2q_1^2 q_2^2 + 2q_2^4 + 2b q_2^2 + p_2^2). \end{aligned}$$

allows one to express p_1, p_2 **in radicals** in terms of q_1, q_2 , and these expressions can be substituted into the **quadrature**

$$S = \frac{f - g}{2(b - a)} t + \int (p_1 dq_1 + p_2 dq_2).$$

Liouville's theorem gives a systematic way, but the explicit formulas are too large to be useful.

Ref.: Ahemed Deina Mohamed El Mokhtar. Integration of the Vanhaecke system in Sage. RUDN, 2026.

KAM–Liouville theorem

Assume that a Hamiltonian system with two degrees of freedom admits two polynomial first integrals F and G such that

- 1 $\{F, G\} = 0$,
- 2 the common level set

$$F(p_1, p_2, q_1, q_2) = f, \quad G(p_1, p_2, q_1, q_2) = g$$

is compact and connected in $\mathbb{R}^4$.

Then this level set is diffeomorphic to a two-dimensional torus. In suitable coordinates, the equations of motion become

$$\dot{\varphi}_1 = \omega_1, \quad \dot{\varphi}_2 = \omega_2,$$

where the frequencies ω_1, ω_2 depend only on f and g .

Crucially, the diffeomorphism is not constructed explicitly, but its existence is guaranteed.

Application to the Vanhaecke system (1/2)

The common level set

$$\begin{aligned}f &= (q_1 p_2 - q_2 p_1)^2 - (a - b)(2q_1^4 + 2q_1^2 q_2^2 + 2a q_1^2 + p_1^2), \\g &= (q_1 p_2 - q_2 p_1)^2 + (a - b)(2q_1^2 q_2^2 + 2q_2^4 + 2b q_2^2 + p_2^2)\end{aligned}$$

is compact and connected in $\mathbb{R}^4$; hence the Liouville–Arnold theorem applies.

Application to the Vanhaecke system (2/2)

- 1 For generic values of f, g , the frequency ratio $\tau = \omega_1/\omega_2$ is irrational, so every trajectory is everywhere dense on the invariant torus.
- 2 For exceptional values of f, g , the ratio τ becomes rational; the motion is then periodic, and the trajectories are no longer dense on the torus.

The set of exceptional values has measure zero in the parameter space. Why, then, do numerical experiments predominantly exhibit the periodic case (case 2)?

Ref.: Kolmogorov, A. N. General theory of dynamical systems and classical mechanics, 1954, §2.

Vanhaecke change of variables

Introduce new variables x_1, x_2 as the roots of the equation

$$x^2 + (q_1^2 + q_2^2 + a + b)x + (aq_2^2 + bq_1^2 + ab) = 0.$$

Then

$$x_1 + x_2 = -q_1^2 - q_2^2 - a - b, \quad x_1 x_2 = bq_1^2 + aq_2^2 + ab.$$

From this, using Gröbner basis techniques, one can reduce the original problem to the Jacobi problem:

$$\frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}} = 0, \quad \frac{x_1 dx_1}{\sqrt{R(x_1)}} + \frac{x_2 dx_2}{\sqrt{R(x_2)}} = dt.$$

Ref.: Shiwei, W. et al. On the behavior of orbits of Vanhaecke system on integral surfaces. arXiv, 2025. DOI: 10.48550/arXiv.2504.15065.

Relation to Liouville's theorem

Since the system possesses four independent integrals of motion, the integrals

$$\int \frac{dx_1}{\sqrt{R(x_1)}} + \frac{dx_2}{\sqrt{R(x_2)}} = c_1, \quad \int \frac{x_1 dx_1}{\sqrt{R(x_1)}} + \frac{x_2 dx_2}{\sqrt{R(x_2)}} = t + c_2$$

are linearly related to the integrals obtained from Liouville's theorem, namely

$$k_1 t + \frac{\partial}{\partial f} \int (p_1 dq_1 + p_2 dq_2) = c'_1, \quad k_2 t + \frac{\partial}{\partial g} \int (p_1 dq_1 + p_2 dq_2) = c'_2.$$

The Vanhaecke change of variables achieves separation of variables in the Liouville quadratures: the first term depends only on x_1 , the second only on x_2 .

Explicit form of $R(x)$

The polynomial $R(x)$ is of degree 5:

$$R(x) = \frac{4}{a-b}(x+b)(x+a) \times \\ \times \left(2(a-b)x^3 + 2(a^2-b^2)x^2 + 2(a^2b-2ab^2+f-g)x + bf-ag \right).$$

Degree 5 means we are in the hyperelliptic case. Inversion requires Abelian functions (not elliptic ones). Elliptic integrals (degree 3 or 4) invert to elliptic functions; degree 5 already gives Abelian functions.

Solution in terms of Abelian functions

The fundamental theorem in the theory of Abelian functions states that any rational symmetric function of x_1, x_2 satisfying the Jacobi problem is an Abelian function of two variables.

For our system,

$$x_1 + x_2 = \text{Al}_1(0, t), \quad x_1 x_2 = \text{Al}_2(0, t).$$

From this, q_1^2 and q_2^2 are readily obtained.

Periods of Abelian functions

Take two roots e_1, e_2 of the polynomial R . Define a pair of real periods:

$$\omega_1 = 2 \int_{e_1}^{e_2} \frac{d\xi}{\sqrt{R(\xi)}}, \quad \omega_2 = 2 \int_{e_1}^{e_2} \frac{\xi d\xi}{\sqrt{R(\xi)}}.$$

The polynomial R and the limits of integration depend on f and g ; hence the periods are functions of f and g .

Similarly, for the roots e_3, e_4 we introduce the second pair of real periods (ω'_1, ω'_2) .

The resulting values are key to determining the periodicity of solutions of the Vanhaecke system.

When is a trajectory closed?

Theorem

The solution is periodic if and only if the frequency ratio $\tau = \omega_1/\omega'_1$ is rational. More precisely:

① If

$$n\omega_1 + m\omega'_1 = 0,$$

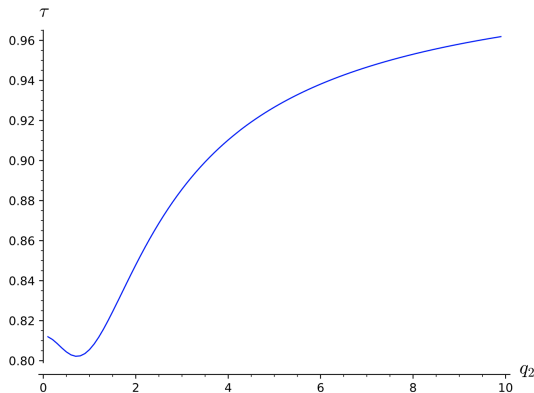
then the trajectory is closed, and its period in time is

$$T = n\omega_2 + m\omega'_2.$$

② If τ is irrational, then the trajectory is everywhere dense on the invariant torus.

Observation: the larger the integers n, m , the denser the trajectory covers the integral surface.

Dependence of τ on parameters



Dependence of τ on the initial value $q_2(0)$ with $q_1 = 0$, $p_1 = 1$, $p_2 = 0$. Since τ varies continuously, arbitrarily small changes in the parameters can result in either rational or irrational values.

Why do we observe "nearly closed" trajectories?

Even if τ is irrational, by Jacobi's theorem one can find integers n, m such that

$$\Delta u = n\omega_1 + m\omega'_1$$

is arbitrarily small. The corresponding change in time

$$\Delta v = n\omega_2 + m\omega'_2$$

need not be small. Then

$$\text{Al}(0, t + \Delta v) = \text{Al}(-\Delta u, t) \simeq \text{Al}(0, t),$$

that is, by continuity of **Abelian functions**, the trajectory almost returns to itself.

Example: illusion of periodicity (1/2)

Over a finite time interval, e.g., $0 < t < 200$, the trajectory appears to be closed, giving an illusion of periodicity.

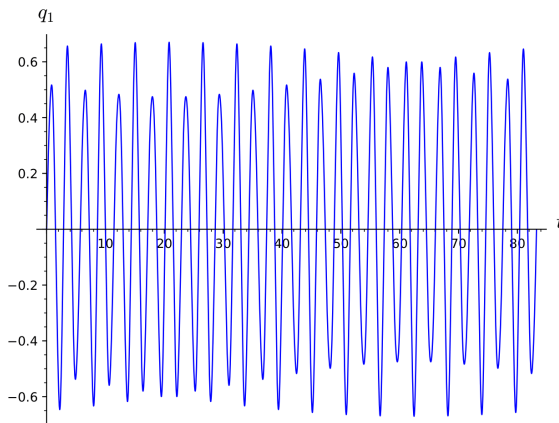
For the chosen parameters, the frequency ratio is

$$\tau \approx \frac{29}{72} \quad \text{with an accuracy of } 10^{-4},$$

so the trajectory exhibits an almost periodic behaviour.

The next slide shows the corresponding time evolution of $q_1(t)$ and discusses the periods of the coordinates and their squares.

Example: periodicity in practice (2/2)



Time evolution of $q_1(t)$ for the Vanhaecke system with $a = 1$, $b = 2$, initial conditions $(0, 1, 1, 0)$. The period of $q_1(t)$ is $T \approx 83.25$. Owing to the symmetry $q_1 \rightarrow -q_1$, the squared coordinates have period $T/2 \approx 41.625$, which agrees with the Abelian function solution.

Conclusion

- 1 The idea, going back to Jacobi and Weierstrass, of finding solutions of dynamical systems in the form of a function of two variables (say, $Al(k_1t, k_2t)$) is natural in the case where trajectories are everywhere dense on the invariant surfaces.
- 2 The Vanhaecke system is trivial from the standpoint of modern mechanics, yet quite challenging from the viewpoint of analytic computations and any attempt to obtain concrete numerical results.
- 3 The Vanhaecke change of variables can be interpreted as separation of variables in the Liouville quadratures. Is it possible to find an algorithm for obtaining such a change of variables?
- 4 Interpretation of numerical experiments without recourse to solutions in terms of Abelian functions turns out to be quite questionable.