

Stability of Periodic Oscillations of a Two-body Gravity Gradient System in the Plane of an Elliptic Orbit

Sergey A. Gutnik

Abstract. The dynamics of a two-body system connected by a spherical hinge moving within a gravitational field in an elliptical orbit is investigated. The dynamics of the system is described by a second-order system of differential equations with periodic coefficients. By applying perturbation techniques and symbolic computation, a periodic solution to the equations of motion is constructed as a power series in a small parameter. The conditions under which such periodic motion occurs were determined and its stability in the linear approximation was investigated. The stability analysis involves the computation of the Hamiltonian of the system, which determines the linearized differential equations of the perturbed motion with periodic coefficients, and the evaluation of the monodromy matrix eigenvalues.

Introduction

In this work, we study the oscillations of a system of two bodies connected by a spherical hinge that moves within a gravitational field in the plane of an elliptical orbit. The investigation of the motion of a system of two connected bodies in an elliptical orbit is of practical interest for the design of composite schemes for a passive gravitational attitude control systems. While the dynamics of various gravitational orientation schemes for satellite-stabilizer configurations has been widely discussed in the literature, some reviews can be found in [1, 2, 3].

The oscillations of a satellite (one rigid body) in an elliptical orbit were investigated in detail in [4]. Research on the planar oscillations of two connected bodies has largely been restricted to cases where their centers of mass coincide [5, 6, 7]. In [8], the eigen-oscillations of a system of two bodies were studied and optimal parameters for transition to equilibrium were found.

The present work extends this research by considering the planar oscillations of a two-body system in an elliptical orbit, specifically where the spherical hinge is located at the intersection of the principal central axes of inertia of the first and second bodies. Expanding upon our previous study [9], which was limited to a first-order linear approximation, this paper calculates the cubic approximation and provides a solution accurate to the third order. By employing perturbation techniques and symbolic computation with the help of Wolfram Mathematica [10], we construct the periodic solution in the form of a power series in a small parameter and investigate its stability in the linear approximation.

Equations of Motion

The equations of motion of a system of two bodies connected by a spherical hinge in the plane of an elliptical orbit relative to the center of mass in the orbital coordinate system: $OXYZ$ (where the OZ axis is directed along the radius vector connecting the center of mass of the Earth and the center of mass of the two-body system O ; the axis OX is directed along the vector of the linear velocity of the center of mass, and the axis OY coincides with the normal to the plane of the orbit) have the following form:

$$\begin{aligned}
& - (1 + e \cos \vartheta) \alpha_2'' + 2e \alpha_2' \sin \vartheta + \frac{B_1 + Ma_1^2}{Ma_1 a_2} ((1 + e \cos \vartheta) \alpha_1'' - 2e \alpha_1' \sin \vartheta) \\
& - e(1 + e \cos \vartheta) (\alpha_2' + 1)^2 + 2e \sin \vartheta \left(1 - \frac{B_1 + Ma_1^2}{Ma_1 a_2}\right) + 4e \\
& + \frac{3e(A_1 - C_1 - Ma_1^2)}{Ma_1 a_2} = 0, \\
& - (1 + e \cos \vartheta) \alpha_1'' + 2e \alpha_1' \sin \vartheta + \frac{B_2 + Ma_2^2}{Ma_1 a_2} ((1 + e \cos \vartheta) \alpha_2'' - 2e \alpha_2' \sin \vartheta) \\
& + e(1 + e \cos \vartheta) (\alpha_1' + 1)^2 + 2e \sin \vartheta \left(1 - \frac{B_2 + Ma_2^2}{Ma_1 a_2}\right) + 2e \\
& + \frac{3e(A_2 - C_2 - Ma_2^2)}{Ma_1 a_2} = 0,
\end{aligned} \tag{1}$$

The axes of the coordinate systems $O_1 x_1 y_1 z_1$ and $O_2 x_2 y_2 z_2$, are directed along the principal central axes of inertia of the first and the second body, respectively. The orientation of the coordinate system $O_i x_i y_i z_i$ with respect to the orbital coordinate system is determined by the aircraft angles α_i (pitch), β_i (yaw), and γ_i (roll) (see [3]). The index $i = 1$ ($i = 2$) refers to body 1 (body 2).

In equations (1) α_1, α_2 are the aircraft pitch angles of orientation of the axis of inertia of the first and second bodies relative to the orbital coordinate system, ϑ is the true anomaly and e is the orbital eccentricity. Here A_i, B_i, C_i are principal central moments of inertia and $M = M_1 M_2 / (M_1 + M_2)$ is the reduced mass, where M_i is the mass of the i th body. The case where the hinge is located at the

intersection of the principal central axes of inertia of the satellite and stabilizer is considered. Then the coordinates of the spherical hinge P in the coordinate system associated with each body are $(a_1, 0, 0)$, $(a_2, 0, 0)$. The prime in (1) denotes differentiation with respect to ϑ . System (1) admits the zero stationary solution

$$\alpha_1 = \alpha_2 = 0. \quad (2)$$

In an elliptical orbit, a zero equilibrium orientation (2) of the system (1) does not exist due to the non-uniform rotation of the radius vector connecting the centers of mass of the Earth and the two-body system. The equilibrium orientation that exists in a circular orbit transforms into periodic eccentric oscillations in the orbital plane. These eccentric oscillations represent a particular periodic solution of the system of equations (1).

The objective of this work is to construct a periodic solution of the system of equations of motion (1) in the form of power series in a small eccentricity parameter e ($e \ll 1$) in the neighborhood of the zero stationary solution (2) (which we denote by the dynamic equilibrium position) using procedures for symbolic calculations of power series, and to analyze the stability of the zero dynamic equilibrium position in the linear approximation.

Periodic solutions

It can be verified that a general solution of the nonlinear system (1) cannot be found in analytic form. To find solutions to system (1), we use methods of perturbation theory and algorithms for symbolic calculations of power series proposed in [11, 12, 13]. The periodic solution determining the eccentric oscillations of the system of two connected bodies was obtained as an expansion in a series in powers of a small parameter e up to a cubic term:

$$\alpha_i(\vartheta) = e\alpha_i^{(1)}(\vartheta) + e^2\alpha_i^{(2)}(\vartheta) + e^3\alpha_i^{(3)}(\vartheta). \quad (3)$$

The computation of the unknown functions $\alpha_i^{(k)}(\vartheta)$ in (3) is done in accordance with the techniques proposed in [13] and involves complex symbolic computations.

In this study, all symbolic computations were performed using the Wolfram Mathematica environment, specifically using functions such as: **Expand**, **TrigExpand**, **Series**, **Map**, **Simplify**, **Replace**, **DSolve**.

In the present work, we have derived the expressions for the periodic motion of the system in linear, quadratic and cubic approximations.

Stability Analysis

The existence of a periodic solution to the equations of motion (1) means that for a given value of the parameter e one can choose initial conditions such that the system remains in a state of dynamical equilibrium, with the bodies oscillating near the zero stationary position (2).

The stability analysis of the dynamic equilibrium position was conducted according to the following scheme. Using expressions for the kinetic energy and force function, and performing the Legendre transformation, the Hamiltonian H was determined. To study the stability of periodic solutions, new canonical variables q_1, q_2, p_1, p_2 were defined.

Rewriting the Hamiltonian H in the new canonical variables and expanding it into a power series in terms of q_i, p_i up to the second order inclusive yields $H = H_0 + H_1 + H_2$, where H_k is the k th order homogeneous polynomial with respect to the canonical variables q_i, p_i which are considered as small perturbations of the periodic solution. The quadratic part H_2 of the Hamiltonian determines the linearized differential equations of the perturbed motion with periodic coefficients. The perturbed motion of the system is described by a linear system of four differential equations with periodic coefficients.

The behavior of the solutions to this system of linearized equations is governed by their characteristic multipliers, which are the eigenvalues of the monodromy matrix. By calculating the monodromy matrix and analyzing its eigenvalues, we obtain, according to Lyapunov's theory, the stability conditions for periodic solutions in the linear approximation. Symbolic-numerical simulation indicate that this periodic motion is stable in the linear approximation.

References

- [1] Sarychev, V.A.: Problems of orientation of satellites, *Itogi Nauki i Tekhniki. Ser. Space Research*, Vol. 11. VINITI, Moscow (1978) (in Russian)
- [2] Rauschenbakh, B.V., Ovchinnikov, M.Yu., McKenna-Lawlor, S.: *Essential Spaceflight Dynamics and Magnetospherics*. Kluwer Academic Publishers (2003)
- [3] Gutnik, S.A., Sarychev, V.A.: Symbolic computations of the equilibrium orientations of a system of two connected bodies moving on a circular orbit around the Earth. *Math. Comput. Sci.* **15**(2), 407–417 (2021)
- [4] Zlatoustov, V.A., Markeev, A.P.: Stability of planar oscillations of a satellite in an elliptic orbit. *Celest. Mech.* **7**(1), 31–45 (1973)
- [5] Sarychev, V.A.: Relative equilibrium orientations of two bodies connected by a spherical hinge on a circular orbit. *Cosmic Research* **5**, 360–364 (1967)
- [6] Penkov, V.I.: Compensation for eccentric oscillations of a satellite with a gravitational stabilization system. *Cosmic Research* **15**(3), 376–383 (1977)
- [7] Clark, J.P.C.: Response of a two-body gravity gradient system in a slightly eccentric orbit. *J. Spacecraft and Rockets.* **7**(3), 294–298 (1970)
- [8] Sarychev, V.A., Mirer, S.A., Sazonov, V.V.: Plane oscillations of a gravitational system satellite-stabilizer with maximal speed of response. *Acta Astronaut.* **3**(9-10), 651–669 (1976)
- [9] Gutnik, S.A.: Investigation of the Periodic Planar Oscillations of a Two-Body System in an Elliptic Orbit Using the Polynomial Algebra Methods. In: 17th International Conference Polynomial Computer Algebra 2024, PCA 2024, St. Petersburg, April

- 15-20 2024. Proceedings, Vassiliev, N.N. (eds.), pp. 61–66. VVM Publishing, St. Petersburg (2024)
- [10] Wolfram, S.: An elementary introduction to the Wolfram Language. 2nd ed. Champaign, IL, USA, Wolfram media (2017)
 - [11] Nayfeh, A.H.: Introduction to Perturbation Techniques. John Wiley and Sons, New York (1993)
 - [12] Prokopenya, A.N.: Symbolic computation in studying stability of solutions of linear differential equations with periodic coefficients. *Programming and Computer Software* **33**(2), 60–66 (2007)
 - [13] Prokopenya, A.N.: Construction of a periodic solution to the equations of motions of generalized Atwood’s machine using computer algebra. *Programming and Computer Software* **46**(2), 120–125 (2020)

Sergey A. Gutnik
76, Prospekt Vernadskogo, 119454
Department of Mathematics, Econometrics and Informatics
MGIMO University
Moscow, Russia
e-mail: s.gutnik@inno.mgimo.ru