

Stability of Periodic Oscillations of a Two-body Gravity Gradient System in the Plane of an Elliptic Orbit

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**Analytical Methods of Celestial Mechanics 2026, August 17-21, 2026
Euler International Mathematical Institute, St. Petersburg, RUSSIA**

1.0 Basic coordinate systems

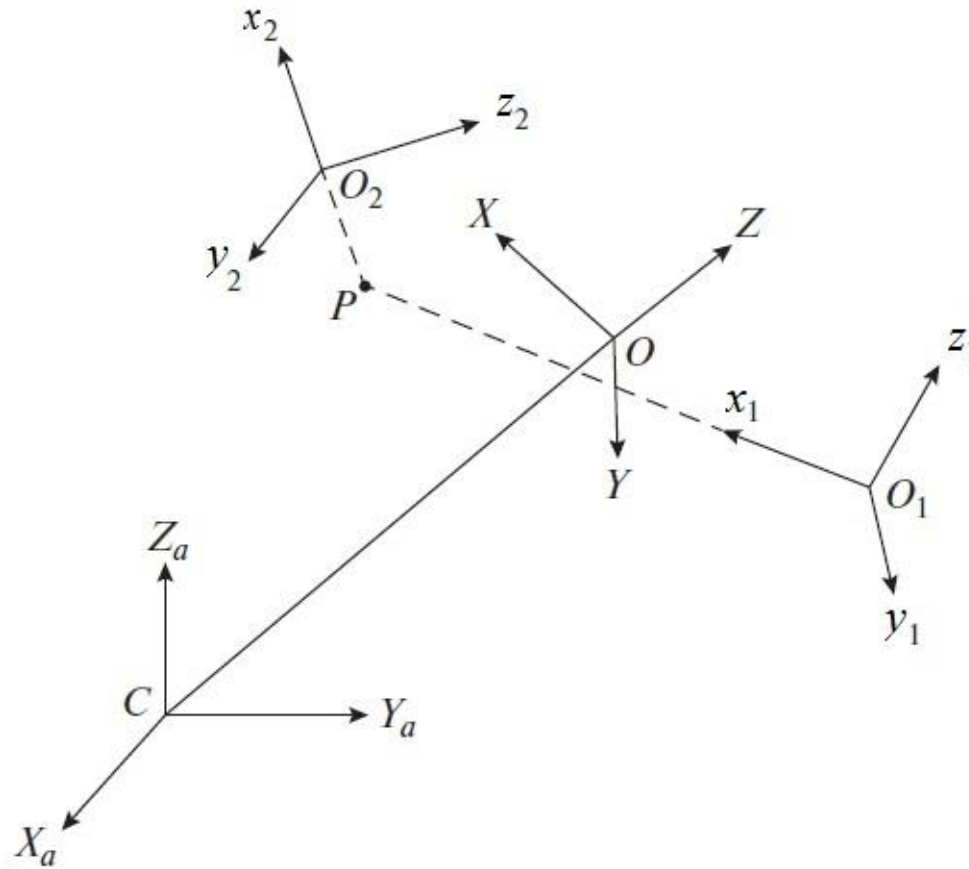


Fig. 1. Basic coordinate systems

$OXYZ$ - is the orbital coordinate system

(a_i, b_i, c_i) —are the coordinates of the spherical joint P in the body coordinate systems $O_i x_i y_i z_i$

1.1 Basic coordinate systems

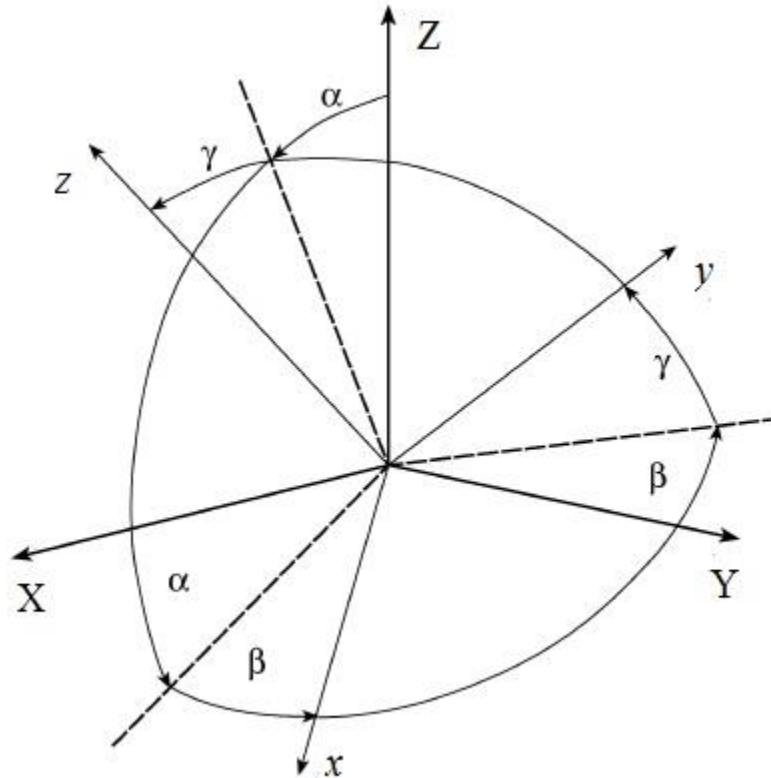


Fig. 2. Orientation of body-fixed axes with respect to the orbital coordinate system by the aircraft angles of α - pitch, β - yaw and γ - roll

2.1 Equations of motion in the orbital plane

Consider the motion of the two bodies system connected by a spherical joint around its center of mass in the plane of an elliptic orbit when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, and $\beta_1 = \beta_2 = 0$, $\gamma_1 = \gamma_2 = 0$. The expressions for the force function, which determines the effect of the Earth gravitational field on the system of two bodies connected by a spherical joint in the case $b_1 = b_2 = c_1 = c_2 = 0$ have the form:

$$U = \frac{3M\mu}{2\rho^3} (a_1 \sin \alpha_1 - a_2 \sin \alpha_2)^2 - \frac{3\mu}{2\rho^3} ((A_1 - C_1) \sin^2 \alpha_1 - (A_2 - C_2) \sin^2 \alpha_2) + \frac{M\mu}{\rho^3} a_1 a_2 \cos(\alpha_1 - \alpha_2). \quad (1)$$

Here ρ is a radial distance between the center of mass of the Earth C and center of mass of the system O ; $\mu = fM_0$, where f is a gravitational constant, and M_0 is the mass of the Earth; $\omega = d\vartheta/dt = \omega_0(1 + e \cos \vartheta)^2$; $\dot{\alpha}_1 = d\alpha_1/dt$; $\dot{\alpha}_2 = d\alpha_2/dt$; $\mu/\rho^3 = \omega_0^2(1 + e \cos \vartheta)^3$; ϑ is the true anomaly and e is the orbital eccentricity. On the circular orbit $\omega = \omega_0$, $\mu/\rho^3 = \omega_0^2$, $\vartheta = \omega_0 t$. A_i, B_i, C_i are the principal central moments, $M = M_1 M_2 / (M_1 + M_2)$ and M_i is the mass of the i -th body; $\alpha_i, \beta_i, \gamma_i$ - are the aircraft angles of pitch, yaw and roll; $(a_i, 0, 0)$ - are the coordinates of the of the spherical joint of the i -th body in reference frame.

2.2 Equations of motion in the orbital plane

The expressions for the kinetic energy the system of two bodies connected by a joint in the case when $b_1=b_2=c_1=c_2=0$ have the form

$$T = \frac{1}{2}(B_1 + Ma_1^2)(\dot{\alpha}_1 + \omega)^2 + \frac{1}{2}(B_2 + Ma_2^2)(\dot{\alpha}_2 + \omega)^2 - Ma_1a_2 \cos(\alpha_1 - \alpha_2)(\dot{\alpha}_1 + \omega)(\dot{\alpha}_2 + \omega). \quad (2)$$

By using the kinetic energy expression (2) and the expression (1) for the force function, the equations of motion for this system can be written as Lagrange equations of the second kind by applying symbolic differentiation **D** in the *Wolfram Mathematica*

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{\alpha}_i} - \frac{\partial T}{\partial \alpha_i} - \frac{\partial U}{\partial \alpha_i} = 0, \quad i = 1, 2.$$

in the form of a system of second-order ordinary differential equations in two variables.

2.3 Equations of motion

The Lagrange equations have the form

$$(B_1 + Ma_1^2)\ddot{\alpha}_1 - Ma_1a_2\cos(\alpha_1 - \alpha_2)\ddot{\alpha}_2 - Ma_1a_2((\dot{\alpha}_2 + \omega)^2 - \frac{\mu}{\rho^3})\sin(\alpha_1 - \alpha_2) + 3\frac{\mu}{\rho^3}((A_1 - C_1 - Ma_1^2)\sin\alpha_1 + Ma_1a_2\sin\alpha_2)\cos\alpha_1 + \quad (3)$$

$$+((B_1 + Ma_1^2) - Ma_1a_2\cos(\alpha_1 - \alpha_2))\dot{\omega} = 0, \\ -Ma_1a_2\cos(\alpha_1 - \alpha_2)\ddot{\alpha}_1 + (B_2 + Ma_2^2)\ddot{\alpha}_2 + Ma_1a_2((\dot{\alpha}_1 + \omega)^2 - \frac{\mu}{\rho^3})\sin(\alpha_1 - \alpha_2) + 3\frac{\mu}{\rho^3}((A_2 - C_2 - Ma_2^2)\sin\alpha_2 + Ma_1a_2\sin\alpha_1)\cos\alpha_2 + \quad (4)$$

$$+((B_2 + Ma_2^2) - Ma_1a_2\cos(\alpha_1 - \alpha_2))\dot{\omega} = 0.$$

which determine the oscillations of the system in the plane of the elliptic orbit in the orbital coordinate system. In (3-4), the dot denotes differentiation with respect to time t . This system has the zero stationary solution

$$\alpha_1 = \alpha_2 = 0. \quad (5)$$

3.1 Periodic solutions

The objective of this work is to calculate the periodic solution of the equations of motion (3-4) in the form of a power series in a small parameter e ($e \ll 1$) in the neighborhood of the zero stationary solution (5) and to analyze its stability.

We assume that the oscillations are small and replace the sine and cosine in (3-4) by their expansions in power series.

Doing in (3-4) the substitution

$$dt = \frac{d\mathcal{G}}{\omega_0(1 + \cos \mathcal{G})^2},$$

we change the independent variable from t to $\mathcal{G}$ and reduce the system to the form of two second order of differential equations.

(The planar oscillations of a rigid body on an elliptic orbit are described by

$$(1 + e \cos \mathcal{G})\alpha'' + 2e\alpha' \sin \mathcal{G} + 3(A - C)/B \sin \alpha \cos \alpha - 2e \sin \mathcal{G} = 0. \quad (6)$$

Beletsky equation. In the non-resonant case ($m = \frac{3(A - C)}{B} \neq 1$) the solutions were found in the form of a power series in e . In the resonant case in the form of a power series in $e^{1/3}$). It was shown that after the substitution $\alpha = z(1 + e \cos \mathcal{G})^{-1}$ (6) reduces to the inhomogeneous *Hill* equation

$$z'' + (m + e \cos \mathcal{G})(1 + e \cos \mathcal{G})^{-1}z - 2e \sin \mathcal{G} = 0. \quad (7)$$

3.2 Periodic solutions

System (3-4) reduces to the system on independent variable ϑ

$$\begin{aligned} & -(1 + e \cos \vartheta) \alpha_2'' + 2e \alpha_2' \sin \vartheta + \frac{3(A_1 - C_1 - Ma_1^2)}{Ma_1 a_2} + 4 + \\ & + \frac{B_1 + Ma_1^2}{Ma_1 a_2} ((1 + e \cos \vartheta) \alpha_1'' - 2e \alpha_1' \sin \vartheta) - \\ & - e(1 + e \cos \vartheta) (\alpha_2' + 1)^2 + e(2 \sin \vartheta (1 - \frac{B_1 + Ma_1^2}{Ma_1 a_2})) = 0, \\ & -(1 + e \cos \vartheta) \alpha_1'' + 2e \alpha_1' \sin \vartheta + \frac{3(A_2 - C_2 - Ma_2^2)}{Ma_1 a_2} + 2 + \\ & + \frac{B_2 + Ma_2^2}{Ma_1 a_2} ((1 + e \cos \vartheta) \alpha_2'' - 2e \alpha_2' \sin \vartheta) + \\ & + e(1 + e \cos \vartheta) (\alpha_1' + 1)^2 + e(2 \sin \vartheta (1 - \frac{B_2 + Ma_2^2}{Ma_1 a_2})) = 0. \end{aligned} \tag{8}$$

It is possible to check that a general solution of nonlinear system (8) cannot be found in analytic form. To find a solution of (8) it is convenient to apply the Krylov-Bogolyubov perturbation techniques and symbolic computations.

3.3 Periodic solutions

The periodic solution determining the eccentric oscillations of the system of two connected bodies was obtained as an expansion in a series in powers of a small parameter e

$$\alpha_1(\mathcal{G}) = e\alpha_1^{(1)}(\mathcal{G}) + e^2\alpha_1^{(2)}(\mathcal{G}) + \dots, \quad (9)$$

$$\alpha_2(\mathcal{G}) = e\alpha_2^{(1)}(\mathcal{G}) + e^2\alpha_2^{(2)}(\mathcal{G}) + \dots$$

Computation of unknown functions $\alpha_i^{(k)}(\mathcal{G})$ in (9) was done in accordance with the techniques proposed by Prokopenya, A.N. [8], which requires quite tedious symbolic computations.

In our work the symbolic computations are performed using *Wolfram Mathematica* functions:

Expand; TrigExpand, Series; Normal, Replace; DSolve; NDSolve

Substituting (9) into (8) and collecting coefficients of equal powers of e , we obtain the set of systems of linear differential equations which can be solved in succession.

3.4 Periodic solutions

Substitute (9) into (8) we use *Mathematica* **pure** and **MAP** functions

$$rul1 = \alpha_1 \rightarrow (e\alpha_1^{(1)}[\#] + e^2\alpha_1^{(2)}[\#]\&),$$

$$rul2 = \alpha_2 \rightarrow (e\alpha_2^{(1)}[\#] + e^2\alpha_2^{(2)}[\#]\&).$$

Collecting coefficients of the first power of e , using **Series[#, {e, 0, 2}]&** and **Coefficient[eqs, e, k]** ($k=1, 2$) we obtain the following system of linear differential equations of equal powers of e

$$\begin{aligned} \alpha_1^{(1)''} - d_2\alpha_1^{(1)''} + 2(d_2 - 1)\sin \vartheta &= 0, \\ \alpha_2^{(1)''} - d_1\alpha_1^{(1)''} + 2(d_1 - 1)\sin \vartheta &= 0, \end{aligned} \left(\text{Here } d_1 = \frac{B_1 + Ma_1^2}{Ma_1a_2}, d_2 = \frac{B_2 + Ma_2^2}{Ma_1a_2} \right). \quad (10)$$

and of equal powers of e^2

$$\begin{aligned} \alpha_1^{(2)''} - d_2\alpha_2^{(2)''} - \alpha_2^{(1)''}d_2 \cos \vartheta + \alpha_1^{(1)''} \cos \vartheta + \\ + 2(1 - \sin \vartheta)\alpha_1^{(1)'} + 2\alpha_2^{(1)'}d_2 \sin \vartheta + \cos \vartheta &= 0, \\ \alpha_2^{(2)''} - d_1\alpha_1^{(2)''} - \alpha_1^{(1)''}d_1 \cos \vartheta + \alpha_2^{(1)''} \cos \vartheta + \\ + 2(1 - \sin \vartheta)\alpha_2^{(1)'} + 2\alpha_1^{(1)'}d_1 \sin \vartheta + \cos \vartheta &= 0. \end{aligned} \quad (11)$$

3.5 Periodic solutions

Solving equations (10-11) with initial conditions

$$\alpha_1^{(1)}(0) = \alpha_2^{(1)}(0) = 0,$$

$$\alpha_1^{(2)}(0) = \alpha_2^{(2)}(0) = 0,$$

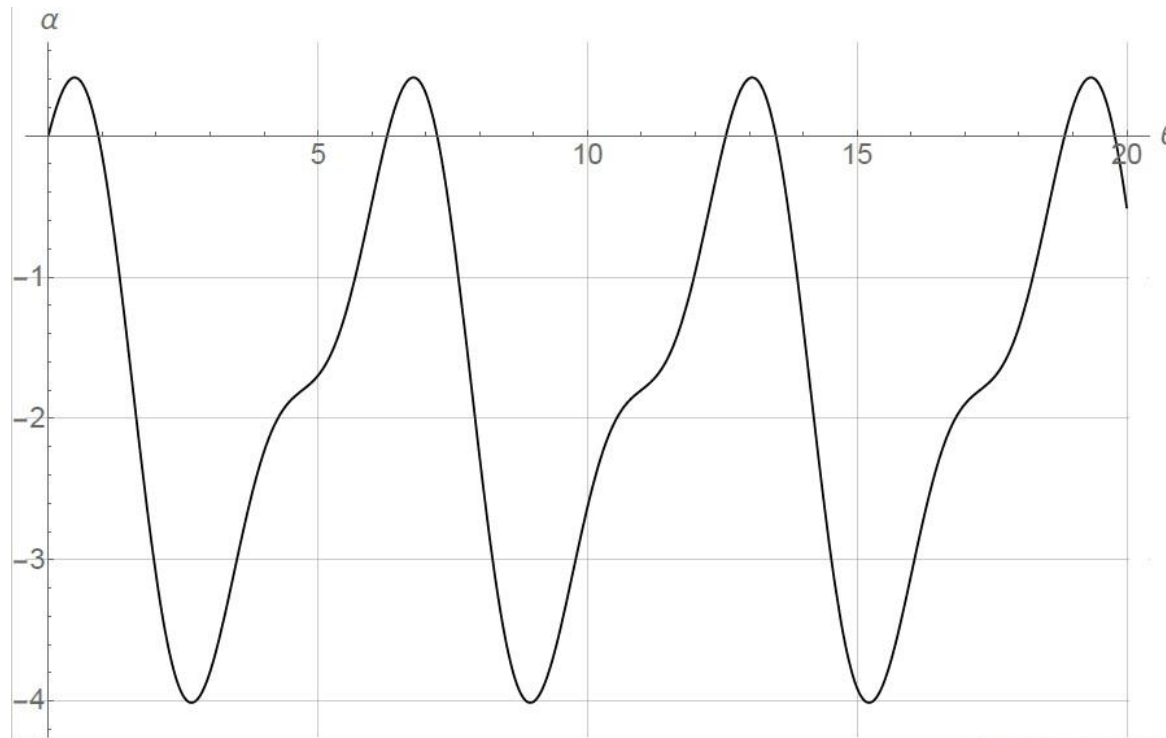
we obtain the corresponding periodic solutions.

We have done the calculations up to second order in e , and the corresponding periodic solutions are given by

$$\begin{aligned}\alpha_1(\vartheta) &= -2e \sin \vartheta + e^2 \left(\frac{3(1+d_2)}{d_1 d_2 - 1} (\cos \vartheta - 1) - \frac{3}{4} \sin 2\vartheta \right), \\ \alpha_2(\vartheta) &= -2e \sin \vartheta + e^2 \left(\frac{3(1+d_1)}{d_1 d_2 - 1} (\cos \vartheta - 1) - \frac{3}{4} \sin 2\vartheta \right).\end{aligned}\tag{12}$$

3.6 Periodic solutions

Example of the form of quadratic approximation (solutions of (11))
for $d_1=2$, $d_2=3$



4.1 Stability Analysis

The existence of a periodic solution to the equations of motion (8) means that for a given value of the parameter e one can choose initial conditions such that the system remains in a state of dynamical equilibrium, with the bodies oscillating near the zero stationary position (5).

By utilizing the expressions for the kinetic energy of the system (1) and the force function (2) and performing a Legendre transformation, we define the Hamiltonian in case ($\alpha_i \ll 1$)

$$H(q_i, p_i, t) = \sum_{i=1}^2 p_i \dot{q}_i - L(q_i, \dot{q}_i, t) = p_1 \dot{\alpha}_1 + p_2 \dot{\alpha}_2 - L, \quad (\text{where } q_i = \alpha_i), \quad (13)$$

$$p_i = \frac{\partial L}{\partial \dot{\alpha}_i} = \omega_0^2 (1 + e \cos \vartheta)^4 \left((B_i + M a_i^2) \alpha'_i - M a_1 a_2 \alpha'_{i+1} + (B_i + M a_i^2 - M a_1 a_2) \right).$$

The equations of motion written in the Hamiltonian form are

$$\begin{aligned} \frac{\partial \alpha_1}{\partial \vartheta} &= \frac{\partial H}{\partial p_1}, & \frac{\partial p_1}{\partial \vartheta} &= -\frac{\partial H}{\partial \alpha_1}, \\ \frac{\partial \alpha_2}{\partial \vartheta} &= \frac{\partial H}{\partial p_2}, & \frac{\partial p_2}{\partial \vartheta} &= -\frac{\partial H}{\partial \alpha_2}. \end{aligned} \quad (14)$$

4.2 Stability Analysis

To investigate the periodic solution stability we introduce new canonical variables q_1, q_2, p_1, p_2 . Through a canonical transformation, the Hamiltonian is rewritten in terms of these variables as $\tilde{H}$. We can expand the Hamiltonian into a power series in terms q_1, q_2, p_1, p_2 up to second order inclusive, representing it in the form:

$$\tilde{H} = H_0 + H_1 + H_2, \quad (15)$$

where H_k are is the k th order homogeneous polynomial with respect to canonical variables, which are considered as small perturbations of periodic solution. The quadratic part H_2 in (15) determines the linearized equations of the perturbed motion which is convenient to write in the matrix form

$$\dot{x} = JH(\mathcal{J}, e)x, \quad J = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \quad (16)$$

where $x^T = (q_1, q_2, p_1, p_2)$ is a 4-dimensional vector, E_2 is the second-order identity matrix, and the elements of fourth-order matrix-function $H(\mathcal{J}, e)$ are obtained by differentiation of H_2 . The matrix $H(\mathcal{J}, e)$ is a periodic function, and so the perturbed motion of the system is described by the linear system of four differential equations with periodic coefficients.

4.3 Computing the Monodromy Matrix

The behavior of solutions to system (16) is determined by its characteristic multipliers which are the eigenvalues of the monodromy matrix $X(2\pi, e)$, where $X(\mathcal{G}, e)$, is a fundamental matrix for system (16) satisfying the initial condition $X(0) = E_4$. As periodic solution (12) is represented by power series in parameter e , the matrix $H(\mathcal{G}, e)$, can be also represented in the form of power series. The fundamental matrix $X(\mathcal{G}, e)$ can also be sought in the form of power series.

On substituting these power series into (16) and collecting coefficients of equal powers of e , we obtain the sequence of differential equations:

$$\begin{aligned}\dot{X}_0 &= JH_0 X_0(\mathcal{G}), \\ \dot{X}_k - JH_0 X_k &= \sum_{j=1}^k JH_j X_{k-j}(\mathcal{G}), \quad (k \geq 1).\end{aligned}\tag{17}$$

Finally, the monodromy matrix of the system (16) can be found in the form of power series. Characteristic multipliers for the system (16) are the eigenvalues of the monodromy matrix (17).

4.4 Characteristic Multipliers

Using the monodromy matrix which was computed up to the third order in parameter e , we can write the characteristic equation determining the characteristic multipliers for the system (16) $\det(X(2\pi, e) - \rho E_4)$.

Solving characteristic equation we obtain that two characteristic multipliers are equal to 1 ($\rho_{1,2} = 1$). The absolute value of the second couple of the characteristic multipliers $\rho_{3,4}$ is also equal to 1. They are complex conjugate and determine two purely imaginary characteristic exponents.

According to Lyapunov theory, four linearly independent solutions to system (16) with 2π -periodic matrix may be represented in the form

$$(\lambda_{3,4} = (1/2\pi)\log \rho)$$

$$x_1(t) = f_1(t), x_2(t) = f_2(t), x_3(t) = \exp(\lambda_3) f_3(t), x_4(t) = \exp(\lambda_4) f_4(t), \quad (18)$$

where $f_k(t)$, ($k = 1, 2, 3, 4$) are 2π -periodic vector-functions. Therefore, in case of $e > 0$ solutions (18) describe the perturbed motion of the system in the domain in the neighborhood of the periodic solution (12). It means this solution is stable in linear approximation.

5. Conclusion

- We have considered the third approximation of the planar oscillations of a system of two bodies connected by a spherical joint that moves along an elliptic orbit.
- We have found the expressions of the periodic motion of the system in the linear, quadratic and cubic approximation
- To find the periodic solutions, the polynomial algebra methods and computer algebra system *Wolfram Mathematica* were used.
- We have found the conditions under which the periodic motion of the system may be realized and proved its stability in linear approximation.
- Simulation of the system shows that this periodic motion is stable but its stability in Lyapunov sense still should be proved;

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