

STABLE PERIODIC ORBITS OF THE EQUAL-MASS ZERO ANGULAR MOMENTUM THREE-BODY PROBLEM

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The Structure of Non-Hierarchical Triple System Stability Regions

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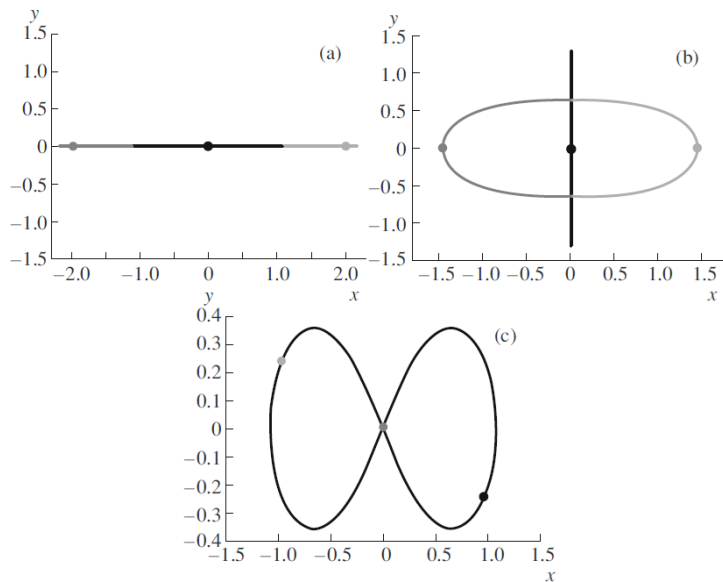


Fig. 1. Three stable periodic orbits in planar equal-mass three-body problem: (a) Schubart orbit [11]; (b) Broucke orbit [12]; (c) "eight-like" orbit [13, 16].



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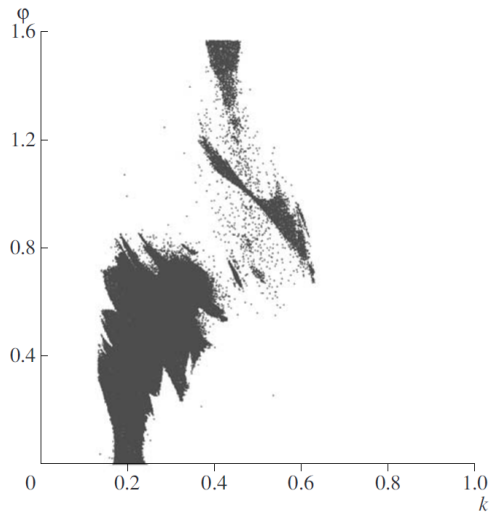
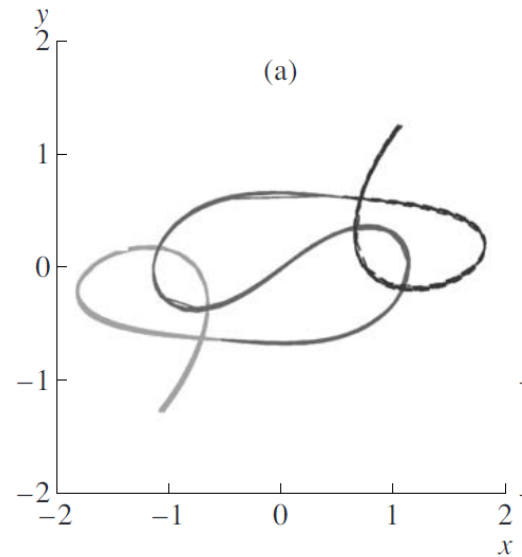


Fig. 2. Stability regions on the diagram (k, φ) at $t > 10000\tau$.



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Three Classes of Newtonian Three-Body Planar Periodic Orbits

Milovan Šuvakov* and V. Dmitrašinović

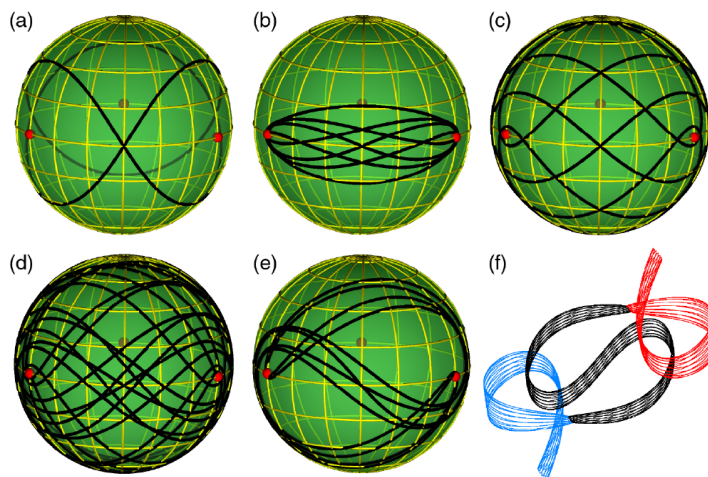
Institute of Physics Belgrade, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

(Received 2 November 2012; published 14 March 2013)

We present the results of a numerical search for periodic orbits of three equal masses moving in a plane under the influence of Newtonian gravity, with zero angular momentum. A topological method is used to classify periodic three-body orbits into families, which fall into four classes, with all three previously known families belonging to one class. The classes are defined by the orbits' geometric and algebraic symmetries. In each class we present a few orbits' initial conditions, 15 in all; 13 of these correspond to distinct orbits.

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Class, number, and name	$\dot{x}_1(0)$	$\dot{y}_1(0)$	T	Free group element
I.A.1 butterfly I	0.30689	0.12551	6.2356	$(ab)^2(AB)^2$
I.A.2 butterfly II	0.39295	0.09758	7.0039	$(ab)^2(AB)^2$
I.A.3 bumblebee	0.18428	0.58719	63.5345	$(b^2(ABab)^2A^2(baBA)^2ba)(B^2(abAB)^2a^2(Baba)^2BA)$
I.B.1 moth I	0.46444	0.39606	14.8939	$ba(BAB)ab(ABA)$
I.B.2 moth II	0.43917	0.45297	28.6703	$(abAB)^2A(baBA)^2B$
I.B.3 butterfly III	0.40592	0.23016	13.8658	$(ab)^2(ABA)(ba)^2(BAB)$
I.B.4 moth III	0.38344	0.37736	25.8406	$(babABA)^2a(abaBAB)^2b$
I.B.5 goggles	0.08330	0.12789	10.4668	$(ab)^2ABBA(ba)^2BAAB$
I.B.6 butterfly IV	0.350112	0.07934	79.4759	$((ab)^2(AB)^2)^6A((ba)^2(BA)^2)^6B$
I.B.7 dragonfly	0.08058	0.58884	21.2710	$(b^2(ABabAB))(a^2(BabaBA))$
II.B.1 yam	0.55906	0.34919	55.5018	$(babABabaBA)^3$
II.C.2a yin-yang I	0.51394	0.30474	17.3284	$(ab)^2(ABA)ba(BAB)$
II.C.2b yin-yang I	0.28270	0.32721	10.9626	$(ab)^2(ABA)ba(BAB)$
II.C.3a yin-yang II	0.41682	0.33033	55.7898	$(abaBAB)^3(abaBabab)(ABabab)^3(AB)^2$
II.C.3b yin-yang II	0.41734	0.31310	54.2076	$(abaBAB)^3(abaBabab)(ABabab)^3(AB)^2$



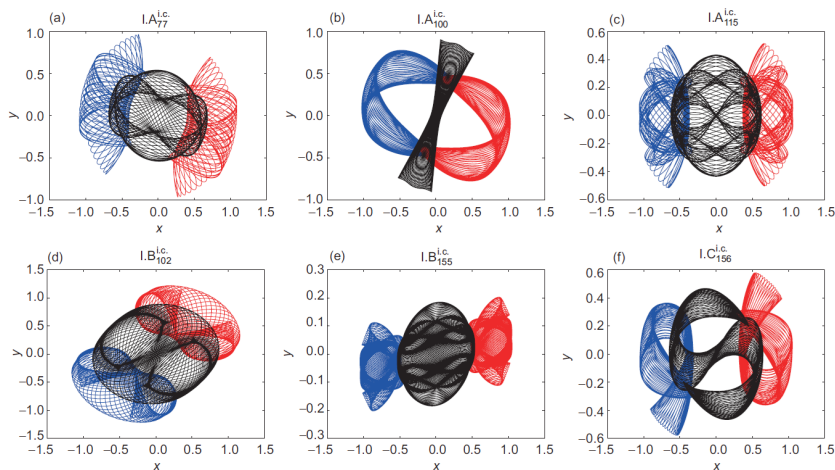
More than six hundred new families of Newtonian periodic planar collisionless three-body orbits

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Linear stability of periodic three-body orbits with zero angular momentum and topological dependence of Kepler's third law: a numerical test

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Geometric phase and periodic orbits of the equal-mass, planar three-body problem with vanishing angular momentum

Danya Rose

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the requirements for the degree of
Doctor of Philosophy

School of Mathematics and Statistics
University of Sydney



February 2016



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- To present high precision results for the decay times in the domain.
- To try to gather all known linearly stable collisionless periodic orbits from the considered domain in one place and to conduct an extensive long periods' search for such orbits in order to expand the list of known ones.
- To present the initial conditions and the periods of the found linearly stable orbits with many correct digits.
- To perform a high precision study of the linear stability and provide the eigenvalues of the monodromy matrices with many correct digits.
- To check if the non-resonance condition from the KAM-theory is satisfied for all found orbits.
- To establish a correspondence between the different stability regions found and a certain pattern in the syzygies sequences of the stable periodic orbits in these regions.



Differential equations

The differential equations for the three-body problem are derived from Newton's second law and Newton's law of gravity:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1, j \neq i}^3 G m_i m_j \frac{(\mathbf{r}_j - \mathbf{r}_i)}{\|\mathbf{r}_j - \mathbf{r}_i\|^3}, i = 1, 2, 3.$$

We consider normalization $G = m_1 = m_2 = m_3 = 1$ and planar motion. We solve the system numerically in the following first order form:

$$\dot{\mathbf{x}}_i = v\mathbf{x}_i, \dot{\mathbf{y}}_i = v\mathbf{y}_i$$

$$v\dot{\mathbf{x}}_i = \sum_{j=1, j \neq i}^3 \frac{(\mathbf{x}_j - \mathbf{x}_i)}{\|\mathbf{r}_j - \mathbf{r}_i\|^3}, v\dot{\mathbf{y}}_i = \sum_{j=1, j \neq i}^3 \frac{(\mathbf{y}_j - \mathbf{y}_i)}{\|\mathbf{r}_j - \mathbf{r}_i\|^3}, i = 1, 2, 3$$

So we have a vector of 12 unknown functions:

$$\mathbf{X}(t) = (\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_3, \mathbf{y}_3, v\mathbf{x}_1, v\mathbf{y}_1, v\mathbf{x}_2, v\mathbf{y}_2, v\mathbf{x}_3, v\mathbf{y}_3)^\top$$

The model treats the bodies as point masses.



Initial conditions

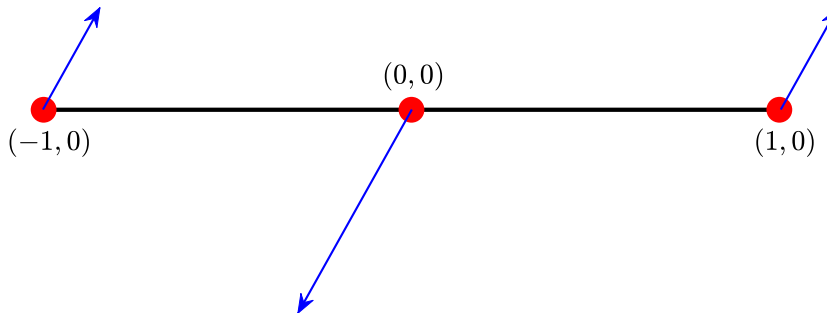
The initial conditions are the following symmetrical collinear with parallel velocities conditions with two parameters $v_x > 0$, $v_y > 0$:

$$(x_1(0), y_1(0)) = (-1, 0), (x_2(0), y_2(0)) = (1, 0)$$

$$(x_3(0), y_3(0)) = (0, 0)$$

$$(vx_1(0), vy_1(0)) = (vx_2(0), vy_2(0)) = (v_x, v_y)$$

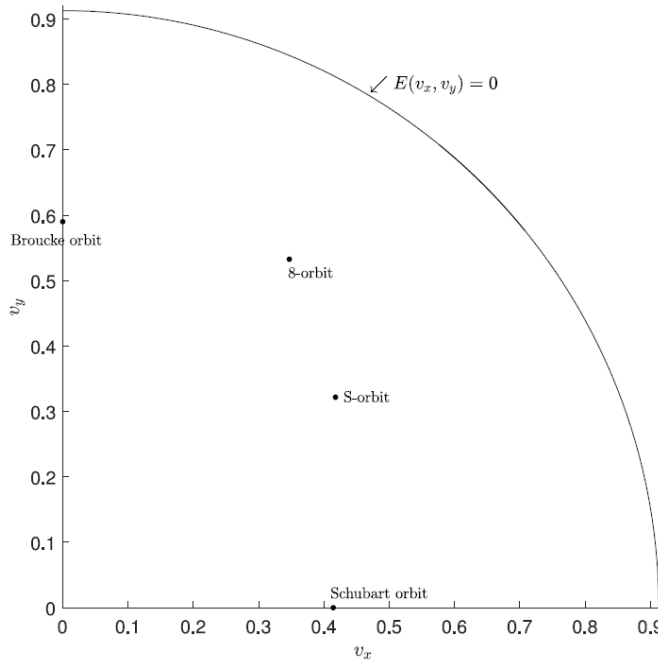
$$(vx_3(0), vy_3(0)) = -2(vx_1(0), vy_1(0)) = (-2v_x, -2v_y)$$



For initial positions $x_i(0), y_i(0)$ fixed as above, the two-dimensional space of velocities is precisely the space of all velocities for which the linear and angular momentum are zero and for which the moment of inertia $I = x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2$ has an extremum at $t = 0$.



Initial conditions: The 2D search domain



As only negative energies (only bounded motions) have to be considered, the 2D search domain is actually those bounded by $v_x = 0$ and $v_y = 0$ axis and $E = 0$ curve. $E = -2.5 + 3(v_x^2 + v_y^2) = 0$.



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How do we search for periodic orbits?

Newton's method is used to capture and compute the periodic orbits with high accuracy.

The **grid-search method** is used to find initial approximations for Newton's method.

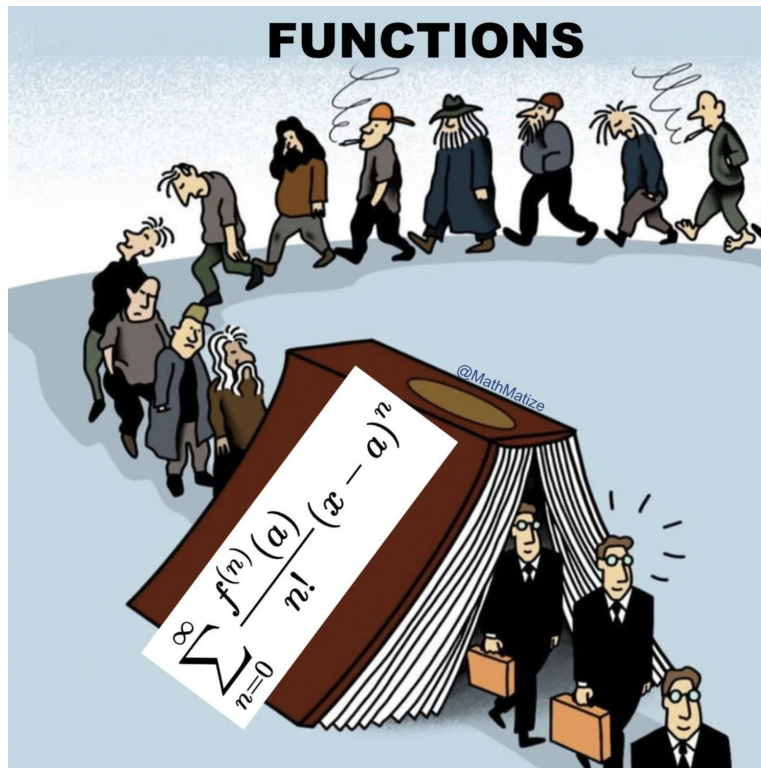
The choice for ODE Solver is crucial. As an ODE solver we use the **high-order Taylor Series Method** implemented with high precision floating-point arithmetic.

For high-precision arithmetic, we use **the GMP library (The GNU Multiple Precision Arithmetic Library)**. The length of the mantissa can be chosen to be arbitrarily long. The exponent is fixed at 64 bits, which is much more than the 11 bits for standard double-precision numbers.



Taylor Series Method (TSM)

Joseph-Louis Lagrange termed Taylor's theorem
"The main foundation of differential calculus."



Taylor Series Method (TSM)

For the initial value problem $\dot{\mathbf{u}}(t) = \mathbf{f}(\mathbf{u}, t)$, $\mathbf{u}_0 = \mathbf{u}(0)$, the N-th order Taylor series method for finding an approximate solution $\mathbf{U}(t) \approx \mathbf{u}(t)$ is given by:

$$\mathbf{U}(t + \tau) = \sum_{i=0}^N \mathbf{U}^{[i]} \tau^i, \quad \mathbf{U}^{[i]} = \frac{1}{i!} \frac{d\mathbf{U}^{(i)}(t)}{dt^i},$$

where the coefficients $\mathbf{U}^{[i]}$ are called normalized derivatives.

We use the strategy of choosing the time stepsize τ , the order N and the precision ε from:

Jorba, Angel, and Maorong Zou. "A software package for the numerical integration of ODEs by means of high-order Taylor methods." *Experimental Mathematics* 14.1 (2005): 99-117

$$\tau = \frac{e^{-0.7/(N-1)}}{e^2} \min \left\{ \left(\frac{1}{\|\mathbf{U}^{[N-1]}\|_{\infty}} \right)^{\frac{1}{N-1}}, \left(\frac{1}{\|\mathbf{U}^{[N]}\|_{\infty}} \right)^{\frac{1}{N}} \right\}$$

Taylor Series Method (TSM)

The order N of the method controls the local error and is chosen according to the absolute local error tolerance ε by the formula:

$$N = \lceil -0.5 \ln(\varepsilon) + 1 \rceil,$$

where $\lceil \cdot \rceil$ is the ceiling function. If we take ε to be the precision corresponding to **64k** bits of mantissa, we obtain:

$$N = \lceil -0.5 \ln(2^{-64k}) + 1 \rceil = \lceil \approx 22.1807k + 1 \rceil$$

In our computations we use the dependence $N = 22k$.

Normalized derivatives are computed with the rules of automatic differentiation! The formulas for automatic differentiation can be found in the previous work:

Hristov, I., Hristova, R., Puzynin, I., Puzynina, T., Sharipov, Z., Tukhliev, Z. "Newton's method for computing periodic orbits of the planar three-body problem." *16th Annual Meeting of the Bulgarian Section of SIAM, December, 2021, Sofia, Bulgaria*



Linear stability study

We study the linear stability study, by computing the monodromy matrices for of the periodic solutions and then computing their eigenvalues. Eight of the eigenvalues are equal to **1**. The other four determine the linear stability.

The computation of the monodromy matrices does not create additional technical difficulties because they are of the same Jacobian type as the matrices in Newton's method and are computed in the same way - through TSM.

For computing the eigenvalues we use a Multiprecision Computing Toolbox **Advanpix** for MATLAB ®.



Numerical results

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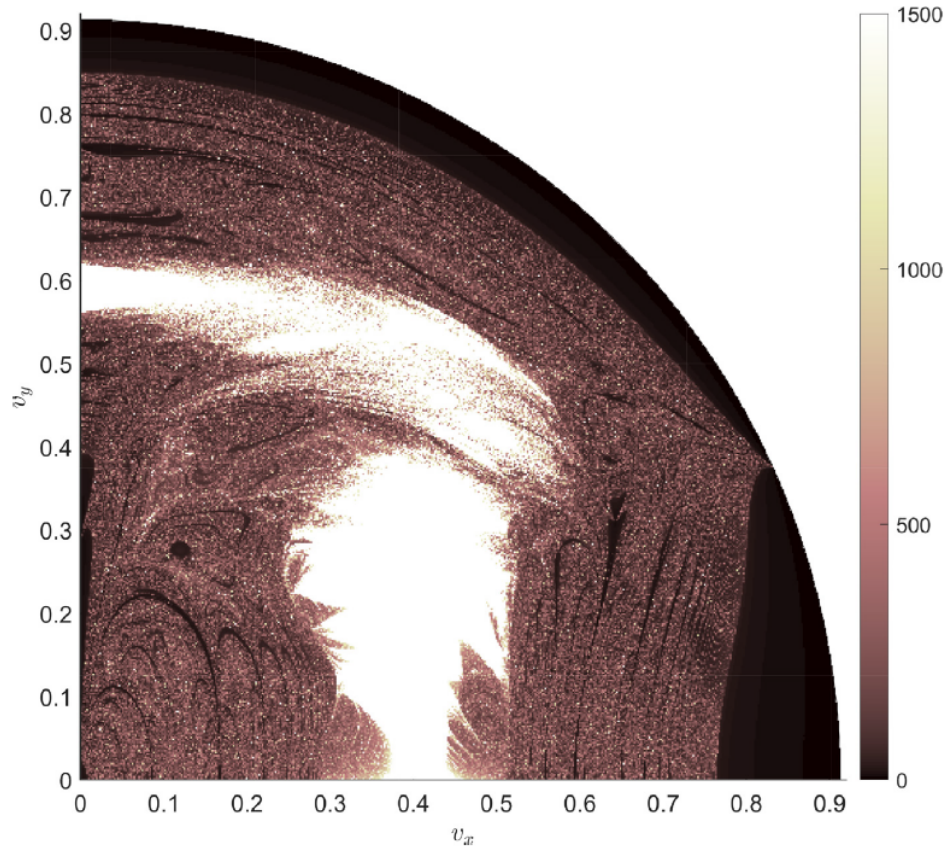


Fig. 3. A color map of the decay times.



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Numerical results

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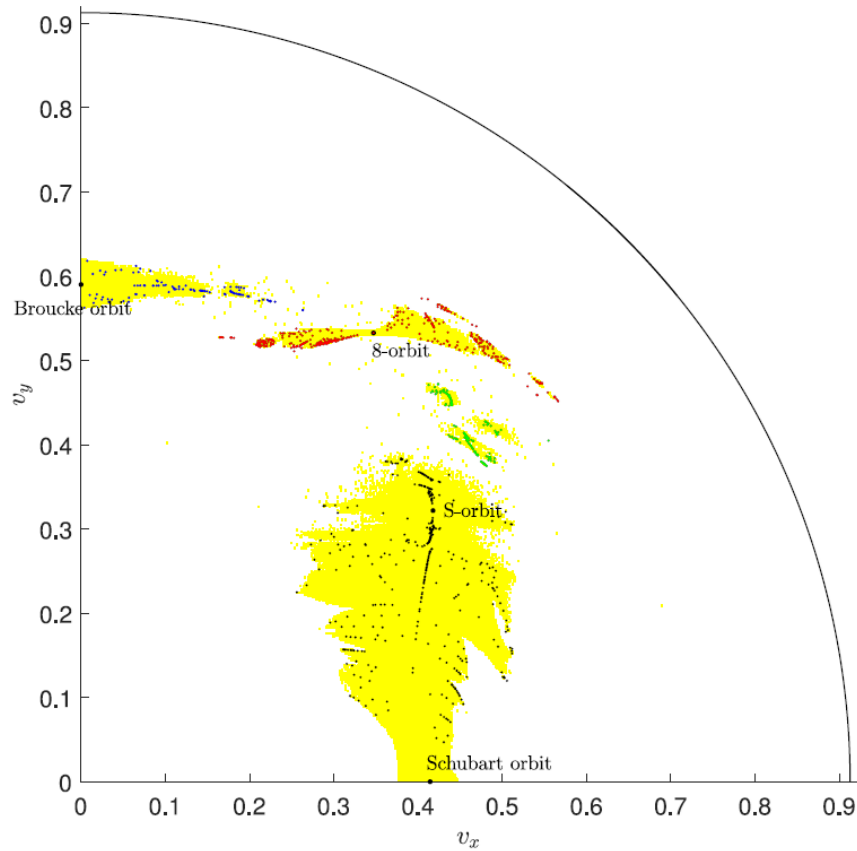


Fig. 4. Distribution of the stable i.c.s. The stability regions ($t_{decay}^* > 10000$) are in yellow.



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A correspondence between the stability regions found and a certain pattern in the syzygies sequences

The syzygy sequences in the region of Figure-8 are completely clear - they are of the form $(\mathbf{213})^{2k}, k \in \mathbb{N}$

The orbits in the Broucke region are characterized by containing the subsequence $(\mathbf{213})^2(\mathbf{231})^2$ in their syzygy sequences. The most well-known collisionless periodic orbit of this type is Bumblebee (Šuvakov, Dmitrašinović).

The orbits in the intermediate region with the islands (the green dots) are characterized by the condition that their syzygy sequences contain the subsequence $(\mathbf{213})^2\mathbf{2123}(\mathbf{123})^2$, considered cyclically, and do not contain $(\mathbf{213} \mathbf{2123} \mathbf{123})^2$. The most well-known collisionless periodic orbit here is Moth I (Šuvakov, Dmitrašinović).

All orbits of the largest region (the black dots) are characterized by the condition that their syzygy sequences contain $(\mathbf{213} \mathbf{2123} \mathbf{123})^2$ or $(\mathbf{2123} \mathbf{123})^2$, considered cyclically. The most well-known collisionless periodic orbit here is Butterfly III (Šuvakov, Dmitrašinović).



Numerical results

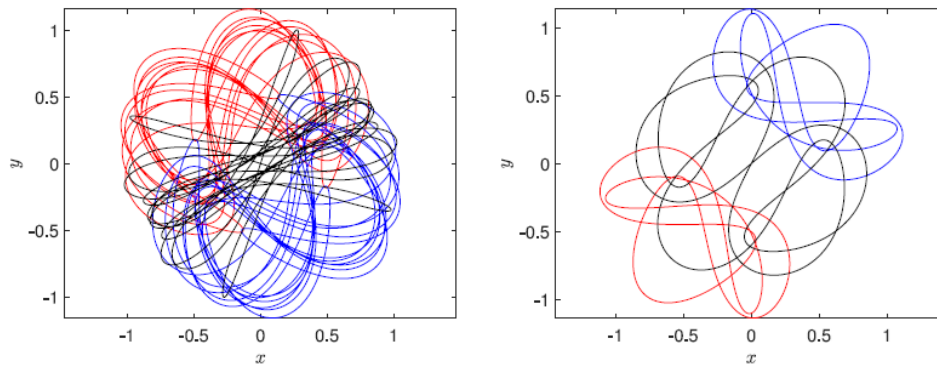


Fig. 5. Bumblebee (Šuvakov and Dmitrašinović, 2013) – left and Moth I (Šuvakov and Dmitrašinović, 2013) – right.

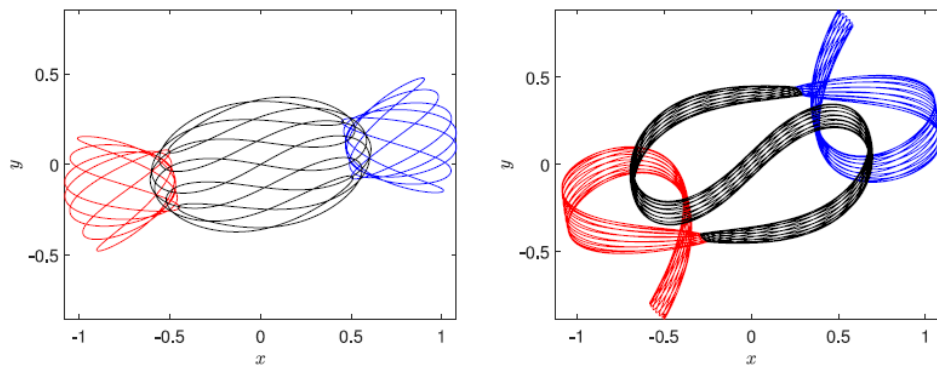


Fig. 6. Butterfly III (Šuvakov and Dmitrašinović, 2013) – left and a S-like orbit (Li and Liao, 2017) – right.



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The presentation is based on our recent work:

I. Hristov, R. Hristova, K. Tanikawa, "An extensive search for stable periodic orbits of the equal-mass zero angular momentum three-body problem" *New Astronomy*, 102528, 2026.

The high precision data is available here:

<https://db2.fmi.uni-sofia.bg/3bodystab971/>

THANK YOU FOR YOUR ATTENTION!

