Birkhoff conjecture and finite energy foliation in Hill's lunar problem

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Analytical methods of celestial mechanics (AMCM′2026)

August 18th, 2026, EIMI, St.Petersburg

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The planar circular restricted three-body problem

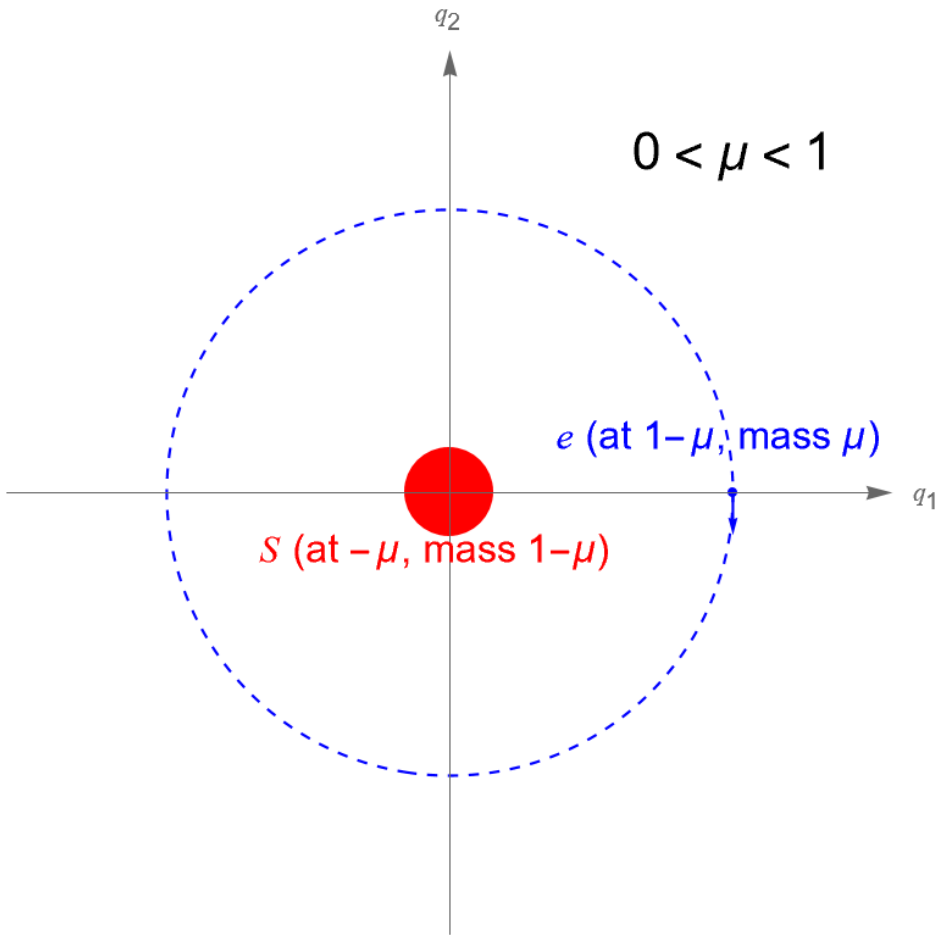


Figure: $H_\mu(p,q) \overset{\text{in rotat. coor.}}{=\!=\!=} \frac{(p_1-q_2)^2}{2} + \frac{(p_2+q_1)^2}{2} - \frac{\mu}{|q-(1-\mu)|} - \frac{1-\mu}{|q+\mu|} - \frac{|q|^2}{2}$

The Hamiltonian H_μ admits five Lagrange (critical) points $l_1, \cdots, l_5 \in \mathbb{C}^2$ with Lagrange (critical) value $L_i := H_\mu(l_i),\ \forall i$. Assume $L_1 < L_2 \leq L_3 < L_4 = L_5$. In this system, the moon is very close to the Earth but still influenced by the Sun.

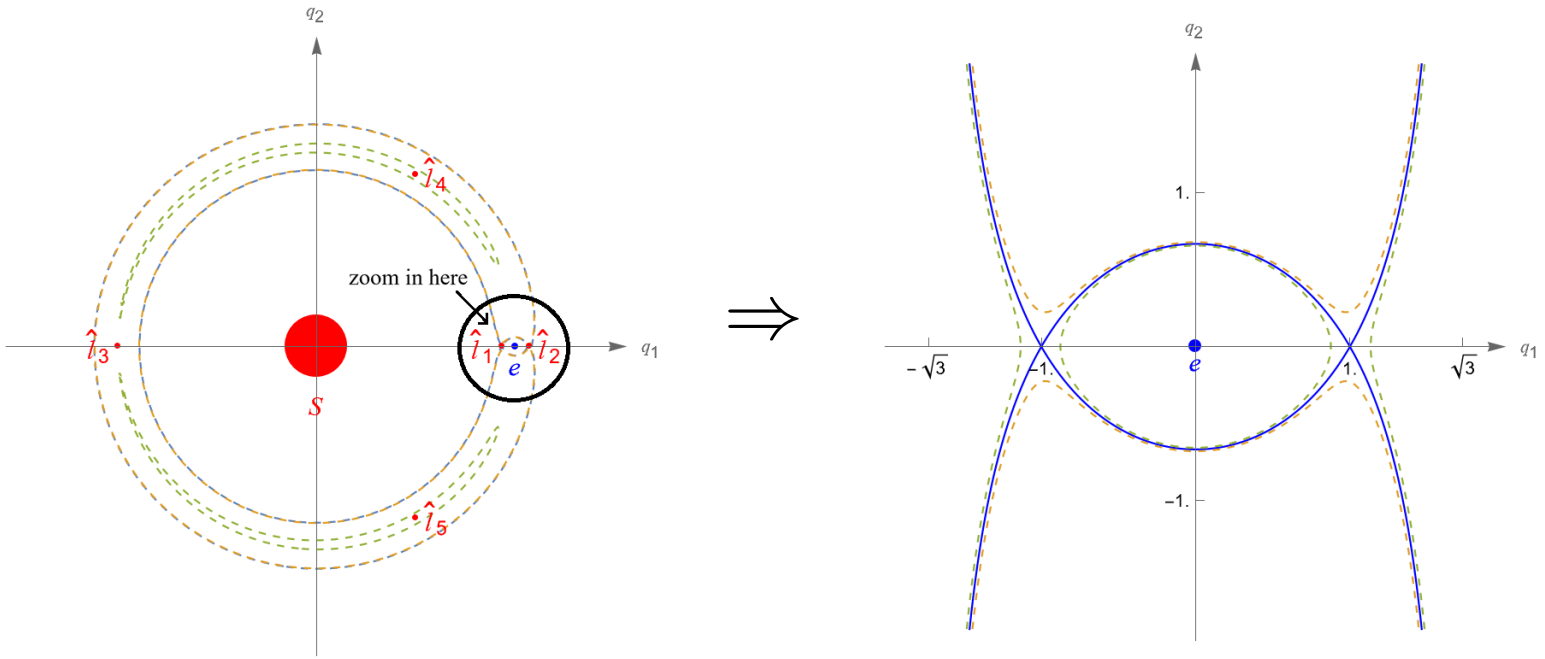


Figure: The Lagrange points $l_i = (-i\hat{l}_i, \hat{l}_i) \in \mathbb{C}^2$ and Lunar problem.

On 17 April 1766, Johann Albrecht Euler, son of Leonhard Euler, posed a problem:

- *"To determine the motion of the moon making its revolutions around the Earth in the plane of the ecliptic, without eccentricity, while the Sun moves uniformly in a circle around the Earth."*

The assumption of a circle for the orbit of the Sun or Earth implies that the Moon's mass is being taken as negligible.

- *"However chimerical this question may appear, I dare assert that if one succeeded in finding a perfect solution of it, one would hardly find any further difficulty in determining the true movement of the actual Moon."*

A hundred years later, George W. Hill discovered a model of the Lunar problem in his celebrated memoirs of 1877 and 1878 as part of Hill's program to obtain a precise analytical description of the motion of the Moon.

Note: See Brown (1896), Wintner (1941), Ligon (2025) for some further developments.

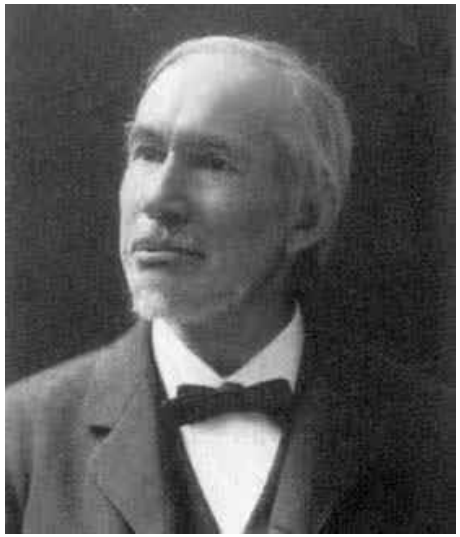


Figure: George W. Hill

Hill's lunar problem

To obtain the Hill's model, we need the following steps:

(1) $\tilde{H}_\mu(p,q) = H_\mu(p - i(1-\mu), q + 1 - \mu)$ (shift the origin to the earth);

(2) $\hat{H}_\mu(p,q) = (\frac{\mu}{3})^{-\frac{2}{3}} \tilde{H}((\frac{\mu}{3})^{\frac{1}{3}} p, (\frac{\mu}{3})^{\frac{1}{3}} q) + 1 - \mu)$ (re-scale the coordinates);

(3) $H(p,q) := \lim_{\mu \to 0^+} \hat{H}_\mu(p,q)$ (push to the limit);

Eventually, we obtain the Hamiltonian of the Hill's lunar problem:

$$H(p,q) = \frac{1}{2}((p_1 - q_2)^2 + (p_2 + q_1)^2) \underbrace{- \frac{3}{|q|} - \frac{3}{2} q_1^2}_{U(q)}.$$

The effective potential $U(q)$ admits a critical level $E_c = -9/2$ with two critical points $\bar{S}_1 = (-1, 0), \bar{S}_2 = (1, 0)$, which are projections of the equilibriums.

Topology of the energy surfaces

To see the topology of the energy surface, we observe that

(i) If $E < E_c$, $H^{-1}(E) = \Sigma_E \cup \Sigma_u$, where $\Sigma_E + S^1 \overset{Levi-Civita}{\Longrightarrow} \mathcal{M}_E \cong \mathbb{R}P^3$.

(ii) If $E = E_c$, then Σ_E meets Σ_u at $S_1 = (0, 1, -1, 0)$ and $S_2 = (0, -1, 1, 0)$.

(iii) If $E > E_c$, then Σ_E, Σ_u are connected through two neck regions.

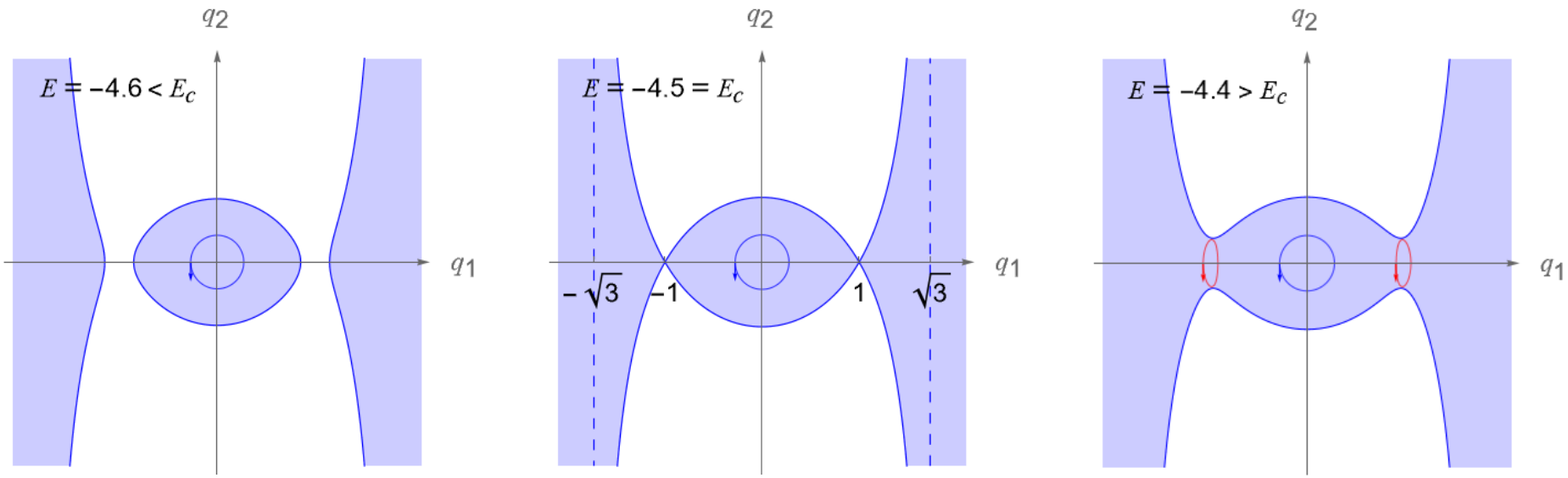


Figure: Hill regions are projection of the energy surface

Regularization

To regularize the singularity at $q = 0$, there is a standard approach:

(1) $\hat{K}_c(u, v) = 4|v|^2(H(\frac{u}{2\bar{v}}, v^2) + c)$ (Levi-Civita coordinate $(p, q) = (\frac{u}{2\bar{v}}, v^2)$)

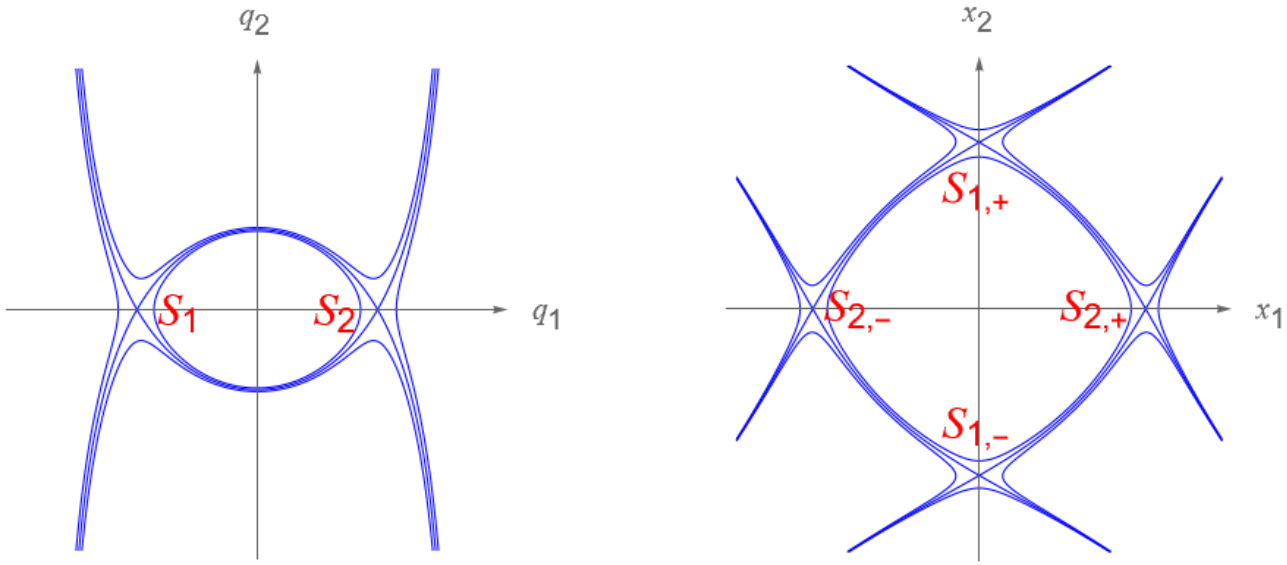
(2) $K(y, x) = c^{-\frac{3}{2}}(\hat{K}_0(c^{\frac{3}{4}}y, c^{\frac{1}{4}}x) + 12)$ (remove the parameter $c = -E$)

Finally, we obtain the new regularized Hamiltonian

$$K(y,x) = \underbrace{\frac{(y_1 - 2|x|^2x_2)^2}{2} + \frac{(y_2 + 2|x|^2x_1)^2}{2}}_{\frac{1}{2}|y+F(x)|^2} + \underbrace{2|x|^2(2 - 3(x_1^2 - x_2^2)^2)}_{V(x)}.$$

Note: K is a magnetic-mechanical system (magnetic field F and potential V).

For every $E < 0$, the energy level $K^{-1}(h = 12|E|^{-3/2})$, projects to a regularization of the energy surface $H^{-1}(E)$ under the antipodal map.



Note: For every $h < h(E_c) = \frac{8\sqrt{2}}{9}$, $K^{-1}(h)$ contains a S^3-component $\mathcal{K}_E$, whose projection is the regularized component of $H^{-1}(E)$, named $\mathcal{M}_E \cong \mathbb{R}P^3$.

We aim to study the dynamics on the energy surface $H^{-1}(E)$.

The retrograde orbits (against the rotating frame)

Birkhoff (1915) first proved the existence of retrograde orbits $P_{3,E}, E < E_c$ for the Hill's problem using his shooting method. Then a general argument applies to the planar circular restricted three-body problem for every $0 < \mu < 1$.

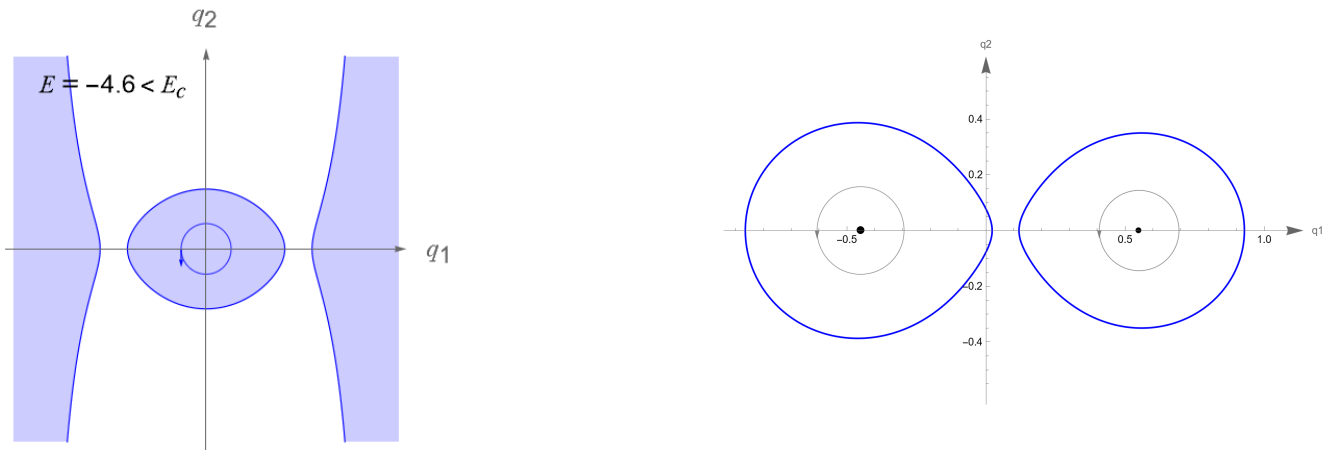


Figure: The retrograde orbits in Lunar problem and PCR3BP

Note: The moon is actually a direct orbit (following the rotating frame).

In 20th century, most of the progress on Hill's lunar problem involved:

perturbative methods, normal form techniques, and numerical investigations.

Important advances were the study of invariant manifolds, periodic trajectories, and chaotic dynamics near the equilibrium points.

- Simó and Stuchi (2000): revealed Lyapunov orbits, stable/unstable manifolds, homoclinic/heteroclinic trajectories, and complicated transport phenomena near the critical level.

Later, Meletlidou, Ichtiaroglou and Winterberg (2001) and Morales-Ruiz, Simó and Simon (2005): non-integrability.

From the perspective of Hamiltonian and symplectic dynamics:

- Albers, Frauenfelder, Paternain and van Koert (2012): $\mathcal{M}_E, \forall E < E_c$ is fiberwise star-shaped. (Ham flow $\Rightarrow$ Reeb flow)
- Lee (2017): $\mathcal{M}_E, \forall E < E_c$ is fiberwise convex. (Ham flow $\Rightarrow$ Finsler geodesic flow on S^2).
- Grotta-Ragazzo, L. , Salomão (2025): $\mathcal{M}_E$ admits a non-resonant Hopf link for $E \ll E_c$. (non-resonance + HMS2015 $\Rightarrow \infty$ many periodic orbits)
- Aydin, Batkhin (2024) and the references therein: $\exists$ bifurcating families of periodic solutions in the (spatial) Hill's lunar problem.

Birkhoff's conjecture

After proving the retrograde orbits $P_{3,E}$, Birkhoff (1915 Pg: 238) raised a question:

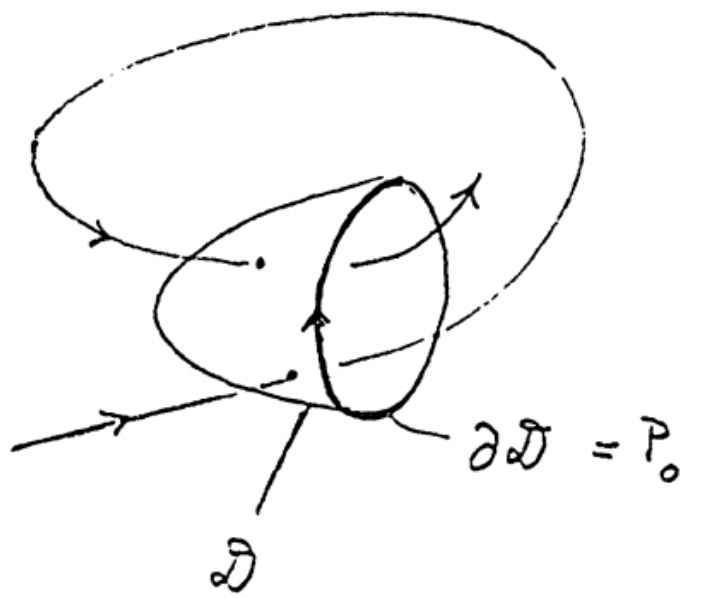
B: *This state of affairs seems to me to make it probable that the restricted problem of three bodies admit of reduction to the transformation of a discoid into itself as long as there is a closed oval of zero velocity about Jupiter, and that in consequence there always exists at least one direct periodic orbit of simple type.*

Note: Jupiter is the Earth in our setting.

Birkhoff's conjecture

In modern language, Birkhoff asked the following question:

Q: For every $E < E_c$, if the double cover of $P_{3,E}$ bounds a disk-like global surface of section on $\mathcal{M}_E \cong \mathbb{R}P^3$?



Q: How to find the global surface of section on $\mathcal{M}_E$ for $E < E_c$?

A: J-holomorphic curves + convexity of $\mathcal{M}_E$.

The global surface of section enable us to encode the global dynamics on $\mathcal{M}_E$ into an area-preserving surface map given by the first return map. However, finding a global surface of section is in general very difficult.

The work by Poincaré, Conley, MeGehee and Kummer are more perturbative. The theory of pseudo-holomorphic curves in symplectization developed by Hofer (1993) and Hofer-Wysocki-Zehnder (1995-1998) gives more light to this problem.

Pseudo-holomorphic curves in symplectization

$(\mathcal{M}, \xi = \ker \alpha)$: a contact manifold (e.g. convex S^3); $\mathcal{R}$: the Reeb vector field determined by $\iota_{\mathcal{R}} d\alpha = 0, \alpha(\mathcal{R}) = 1$; (Σ, j): a Riemann surface (with nonempty boundary); J: an almost complex structure on $\mathbb{R} \times \mathcal{M}$ ($\xi \to \xi$ and $\partial_a \mapsto \mathcal{R}$).

A finite energy J-holomorphic curve in symplectization is a map

$$\tilde{u} = (a, u) : \dot{\Sigma} = (\Sigma, j) \setminus \Gamma \text{ (punctures)} \to (\mathbb{R} \times \mathcal{M}, J)$$

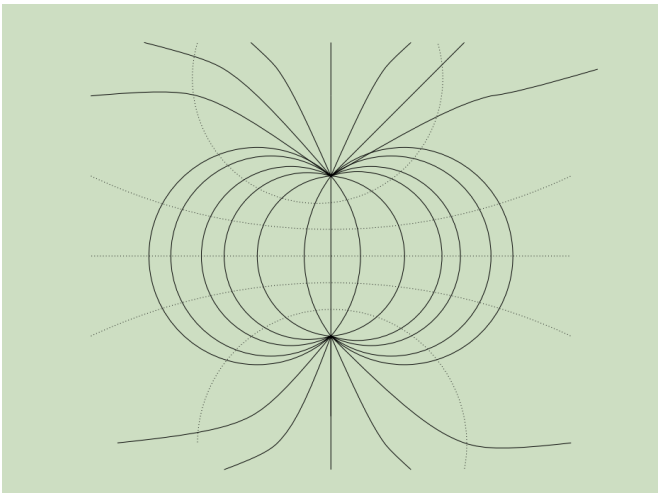
so that $\bar{\partial}\tilde{u} := d\tilde{u} + J \circ d\tilde{u} \circ j = 0$ and with finite Hofer energy,

$$0 < E(\tilde{u}) := \sup_{\phi \in \Lambda} \int_{\Sigma} \tilde{u}^* d(\phi\, \alpha) < +\infty, \quad \Lambda := \{\phi : \mathbb{R} \to [0, 1], \phi' \geq 0\}.$$

Theorem 2.1 (Hofer–Wysocki–Zehnder 1998)

If $(\mathcal{M} \cong S^3, \alpha)$ is dynamically convex, then α admits an index-3 periodic orbit bounding an open book decomposition whose pages are disk-like global surfaces of section. This open book arises as the projection of a finite energy foliation.

(By Franks (1992, 1996), there are 2 or ∞ many periodic orbits.)



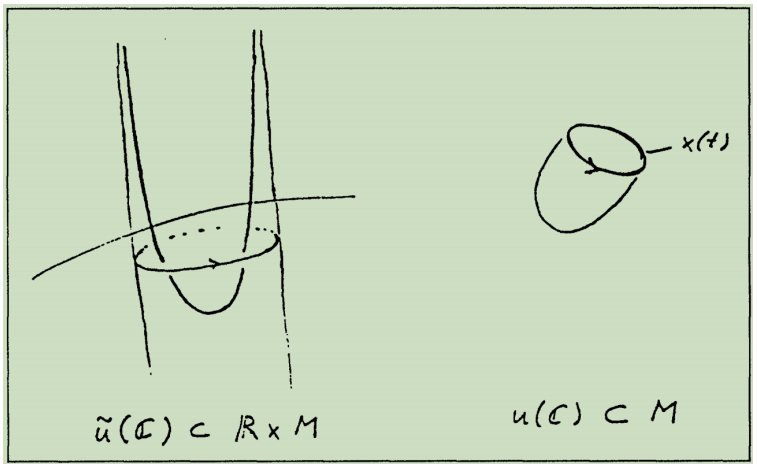


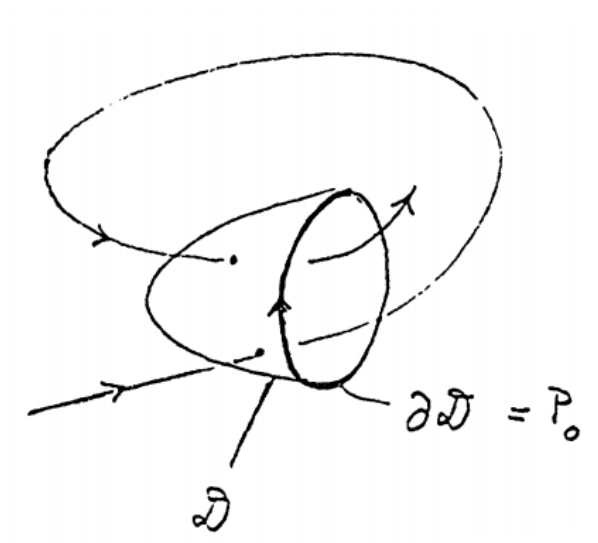
Figure: View S^3 as a $\mathbb{R}^2 \cup \infty$ and each line is a J-holomorphic plane.

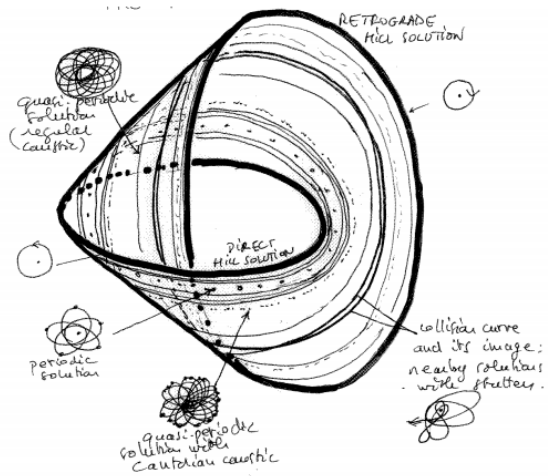
Under the dynamical convexity

Hryniewicz-Salomão (2016)/Hryniewicz-Salomão-Wysocki (2022) proved that:

- For $(\mathcal{M} = S^3$ or $\mathbb{R}P^3, \alpha)$, every Hopf fiber binds an open book decomposition, whose pages are disk-like global surfaces of section, and
- every Hopf link $P \cup P'$ bounds an annulus-like global surface of section.

Note: 1 Hopf link = 2 simply-linked Hopf fibers (unknotted + self-linking -1)





Main result 1

Theorem 2.2 ($\Rightarrow$ Birkhoff's conjecture)

For every $E < E_c$, $\mathcal{M}_E$ is dynamically convex and Birkhoff's conjecture holds.

Proof: $P^2_{3,E}$ is a Hopf fiber + $\mathcal{M}_E$ is "strictly convex" ($\Rightarrow$ dynamical convexity).

For $E \leq E_c$, we prove for its double cover $\mathcal{K}_E \subset K^{-1}(h(E))$ that:

Proposition 2.1 (Strict convexity)

There exists a smooth symplectic shear $\Phi : \mathbb{R}^4 \to \mathbb{R}^4((y,x) \mapsto (y + \nabla G_0(x), x))$ so that $\Phi^{-1}(\mathcal{K}_E)$ is strictly convex for every $E \leq E_c$, where the generating function is $G_0(x) := \frac{2}{3}(x_1^2 - x_2^2)x_1x_2, x \in \mathbb{R}^2$.

Proof of strict convexity

(1) After a certain simplification, we obtain a new Hamiltonian:

$$\hat{K}(y,x) = \underbrace{\frac{(y_1 - x_2^3)^2}{2} + \frac{(y_2 + x_1^3)^2}{2}}_{F(x)=(-x_2^3, x_1^3)} + \underbrace{\frac{9}{32}(x_1^2 + x_2^2)(2 - 3(x_1^2 - x_2^2)^2)}_{V(x)}.$$

(2) For local convexity of $\hat{K}^{-1}(h)$, we established a general criterion formula:

Theorem 2.3 (L.-Salomão (2025))

Let $M = K^{-1}(h)$. Then $\nabla^2 K|_{TM} > 0 \Leftrightarrow \det U_W > 0$ on M, where

$$U_W(\theta, x_1, x_2) := r^2(\cos\theta \nabla^2 f_1 + \sin\theta \nabla^2 f_2) + r\nabla^2 V + r^{-1}\nabla V \otimes \nabla V,$$

is a 2×2-matrix. Here, $r = |y_F|$ and θ is the argument of $y_F := y + F$.

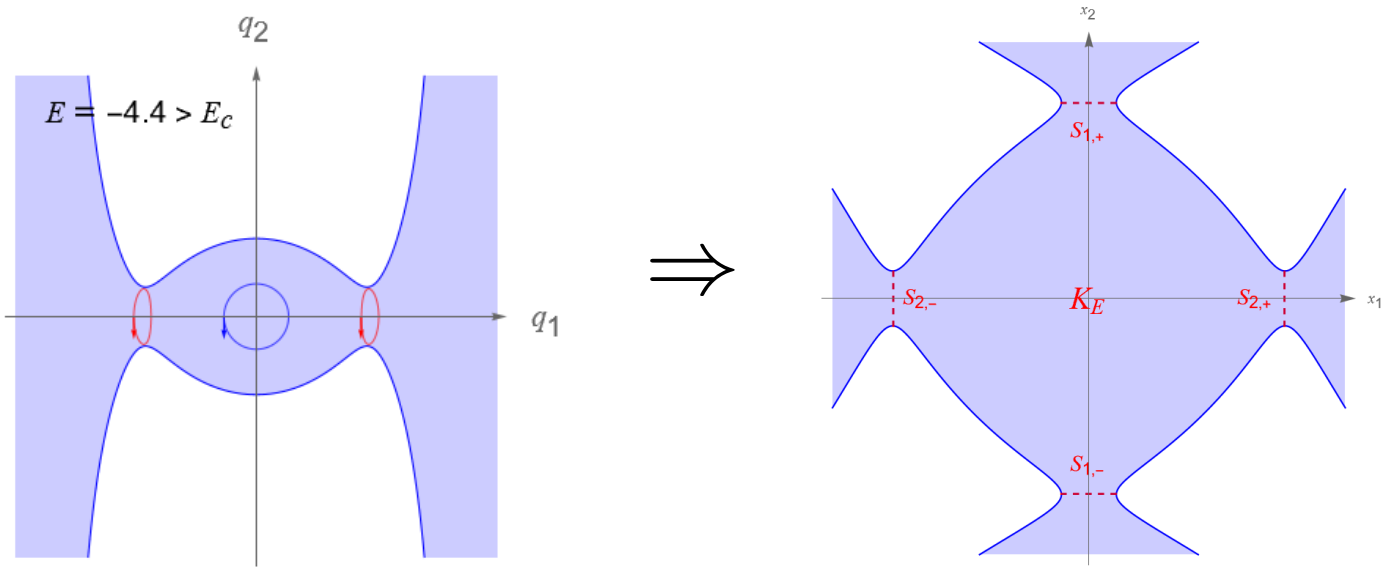
(3) A local-to-global convexity argument.

For higher energies

Q: How to study the global dynamics on $H^{-1}(E)$ for $E > E_c$?

A: Inspired by HWZ (2003), we aim to establish a 'finite energy foliations'.

It is nature to consider the dynamics between the two neck regions near S_1, S_2.



Note: We need 2 splitting spheres (or 4 splitting spheres in its double cover)

Finite energy foliation

$(\mathcal{M}, \alpha)$: a contact manifold (may have boundary); $\mathcal{R}$: the Reeb vector field; $\mathcal{P}$: an orbit set; J: an almost complex structure on $\mathbb{R} \times \mathcal{M}$ ($\xi = \ker \alpha \to \xi$ and $\partial_a \mapsto \mathcal{R}$).

Definition 3.1

A finite energy foliation $\tilde{\mathcal{F}}$ of $\mathbb{R} \times \mathcal{M}$ adapted to α, $\mathcal{P}$ and J is a regular foliation, which consists of the following leaves:

- *trivial cylinders over the orbits in $\mathcal{P}$.*
- *the image of nicely embedded J-holomorphic curves (the projection to $\mathcal{M}$ is also embedded) with uniformly bounded Hofer energies.*

Note: $\tilde{\mathcal{F}}$ is $\mathbb{R}$-invariant and project to a 'transverse singular foliation' $\mathcal{F}$ on $\mathcal{M}$.

Known orbits: The retrograde/Lyapunov orbit

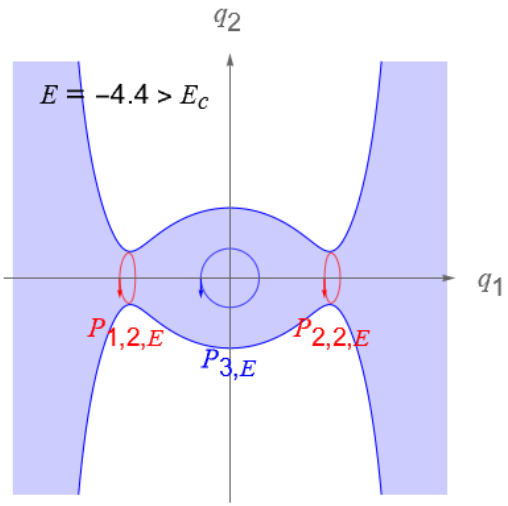
1. The retrograde orbits $P_{3,E}$: By Birkhoff's shooting method, we can prove:

Proposition 3.1 (Retrograde orbits for higher energies)

There exists a continuous family of retrograde orbits $P_{3,E} \subset \mathcal{M}_E \setminus \partial\mathcal{M}_E$ so that the projection to the q-plane is a simple closed curve circling the origin and symmetric with respect to both axes. Moreover, $P_{3,E} \overset{C^\infty}{\to} P_{3,E_c}$ as $E \to E_c$.

Proof: Shooting method + A local analysis near S_1 + symmetry of S_1, S_2.

2. Lyapunov orbit $P_{i,2,E}$: generated from S_i for every E slightly above E_c.



Construction of a contact structure

Note: To adapt the theory of J-holomorphic curves, we need to establish a contact structure on $\mathcal{M}_E$ for E slightly above E_c.

Using a strategy of Albers-Frauenfelder-van Koert-Paternain (2012). It is sufficient to construct a transverse Liouville vector field $\hat{K}^{-1}(h(E_c+\epsilon))$:

(i) Choose Liouville vector fields $Y_0 = y_1\partial_{y_1} + y_1\partial_{y_1} + x_1\partial_{x_1} + x_2\partial_{x_2}$ (centering at the origin) and $Y_1 = \frac{1}{2}(3w_1\partial_{w_1} + w_2\partial_{w_2} - w_3\partial_{w_3} + w_4\partial_{w_4})$ (centering at $S_{1,-}$).

(ii) Establish an interpolating Liouville vector field $\underbrace{Y_1 + Y_\epsilon + Y_0}_{\text{near } S_{1,-}} \Rightarrow Y_E \pitchfork \mathcal{K}'_E$.

Note: Then $\alpha_E = \iota_{Y_E}\omega_0$ is the induced contact form, where $L_{Y_E}\omega_0 = \omega_0$.

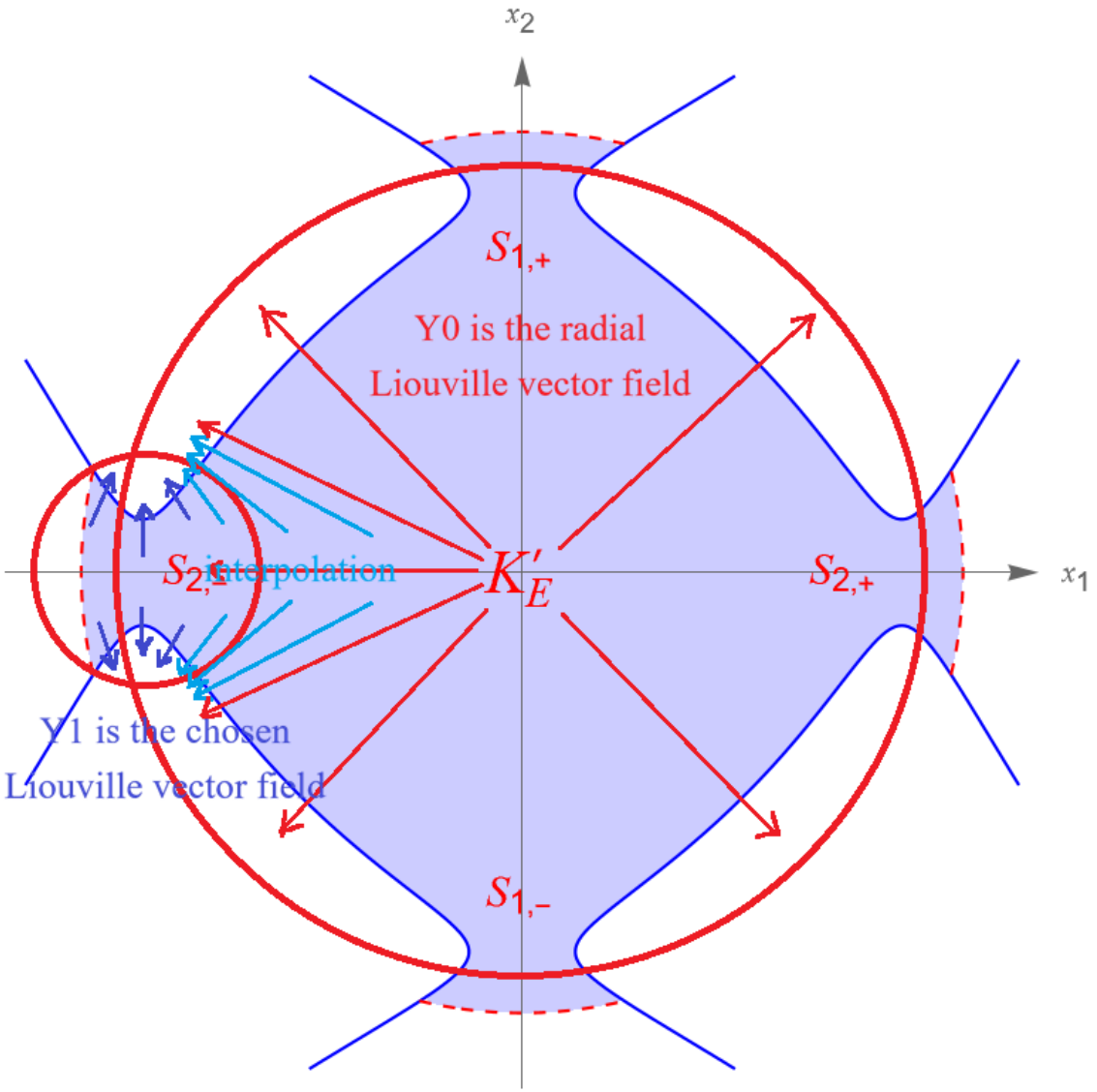


Figure: The interpolation between Y_0 and Y_1.

The choice of splitting spheres

In the neck regions, we need a good choice of splitting 2-sphere $\mathcal{S}_{i,\pm}$ near $S_{i,\pm}$ adapt to the Lyapunov orbit $P_{i,2,E}$:

Theorem 3.1 (Spherical shield)

For every E slightly above E_c, there exists an almost complex structure J_E on $\mathbb{R} \times \mathcal{M}_E$ adapted to α and two J_E-holomorphic planes asymptotic to $P_{i,2,E}$ and project to a C^1 2-sphere $\mathcal{S}_i = \mathcal{S}_{i,E}$ in $\mathcal{M}_E$. ($\mathrm{dist}(\mathcal{S}_{i,E}, S_i) \to 0$ *as* $E \to E_c^+$)

Proof: Solve C-R equation + 'transversality' of J-planes by Wendl (2010).

Note: The flow transverses to the hemispheres $\mathcal{S}_i \setminus P_{i,2,E}$ in opposite directions.

As a conclusion, we prove that:

Theorem 3.2 (Contact-type structure)

For every E slightly above E_c, there exists a compact subset $\mathcal{M}_E$ bounded by $\mathcal{S}_{1,E} \cup \mathcal{S}_{2,E} = -\mathcal{S}_{1,E}$, $\alpha = \alpha_E$ on $\mathcal{M}_E$, and J_E on $\mathbb{R} \times \mathcal{M}_E$ adapted to α so that

- *the Reeb vector field is $\mathcal{R}_\alpha = \frac{X_H}{\alpha(X_H)}$. Moreover, $\alpha_E \overset{C^\infty_{loc}}{\to} \alpha_{E_c}$.*
- *The boundary spheres $\mathcal{S}_{1,E} \cup \mathcal{S}_{2,E}$ uniformly to $\{S_1, S_2\}$ as $E \to E_c^+$.*
- *The hemispheres of $\mathcal{S}_j \setminus P_{j,2,E}$ are the projections to $\mathcal{M}_E$ of J_E-holomorphic planes asymptotic to $P_{j,2,E}$ through opposite directions.*

Note: Now $(\mathcal{M}_E \equiv \mathbb{R}P^3 \setminus \{\text{two open 3-balls}\}, \xi = \ker \alpha)$ is a contact manifold.

Main result 2

Now, we establish the following Finite energy foliation (FEF):

Theorem 3.3 ($2-3-2$ foliation on HLP)

For every E slightly above E_c, the flow on $\mathcal{M}_E$ admits a $2-3-2$ foliation whose binding orbits consist of retrograde orbit $P_{3,E}$ and Lyapunov orbit $P_{1,2,E}, P_{2,2,E}$.

Proof: strict convexity of $\mathcal{M}_{E_c}$ + an index estimate near S_i (L.-Salomão (2025)) + a criterion for FEF (L.-Salomão (2025)) (we introduced later).

Theorem 3.4 (a criterion for FEF)

Let $\mathcal{P} := \{P_{3,E}, P_{1,2,E}, P_{2,2,E}\}$. If $\nexists\, P' \subset \mathcal{M}_E \setminus (\partial\mathcal{M}_E \cup P_{3,E})$ satisfying

$$\rho(P') = 1, \quad \mathrm{link}(P', P^2_{3,E}) = 0 \quad \text{and} \quad \mathcal{A}(P') \leq \mathrm{T}(P^p_{3,E}) = \mathcal{S}(\mathcal{D})$$

"the linking condition", then $\mathcal{M}_E$ admits a $2-3-2$ foliation with binding $\mathcal{P}$.

$2-3-2$ foliation of HLP

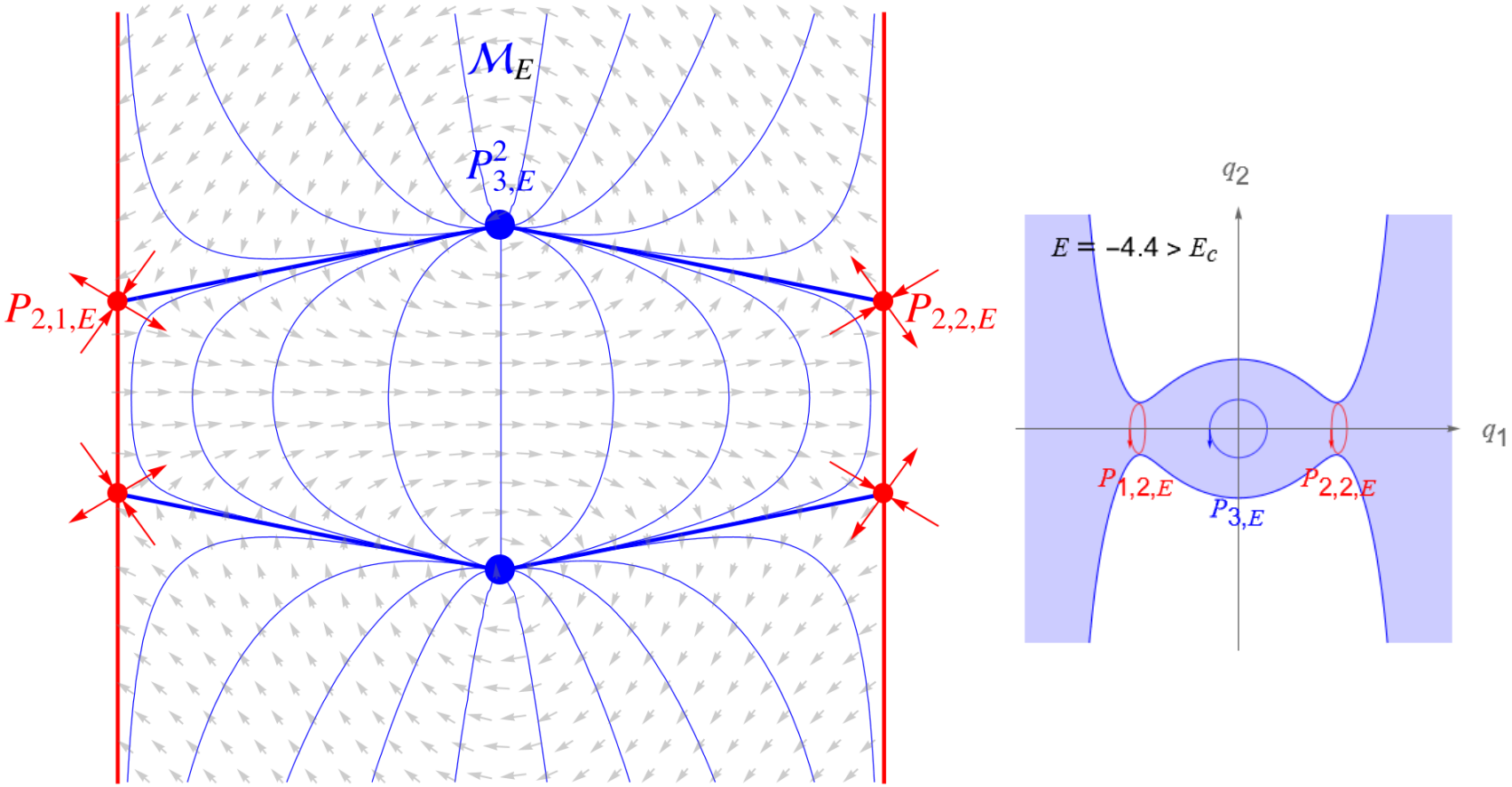


Figure: $2-3-2$ foliation on $\mathcal{M}_E$

As an immediate consequence, we have

Corollary 3.1 (Periodic and homoclinic orbits)

$\mathcal{M}_E$ has ∞ many periodic orbits and ∞ many homoclinic orbits connecting the set $P_{1,2,E} \cup P_{2,2,E}$. If their stable and unstable manifolds do not coincide, the flow on $\mathcal{M}_E$ has positive topological entropy.

Proof: Flowing the 'unstable manifold' of $P_{i,2,E}$ along the flow.

Note: The stable manifold intersects a plane (in FEF) at an analytic circle, and its forward image necessarily cause a self-intersection at this plane.

 L. Liu, Pedro A. S. Salomão,

Finite energy foliations and global dynamics in the restricted three-body problem, arXiv:2506.17867v2

 L. Liu, Pedro A. S. Salomão,

Birkhoff conjecture and finite energy foliations in the Hill's Lunar problem, arXiv:2606.12912v1.

Thank you for your attention!

Appendix: A criterion for FEF

$L(p,q) := \{z \in \mathbb{C}^2, |z| = 1\}/((z_1, z_2) \sim (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi q i}{p}} z_2))$ (lens space); $\xi_{\text{st}} = \ker \alpha$: standard contact structure induced from $\mathbb{S}^3$; $\mathcal{R}$: the Reeb vector field of α. We set:

(i) Topology: (l-holed lens space) $\mathcal{M} := L(p,q) \setminus \cup_{j=1}^{l} \mathcal{B}_j^3$ with boundary $\cup_j \mathcal{S}_j$.

(ii) Boundary conditions: Each $\mathcal{S}_j$ contains a hyperbolic index-2 orbit $P_{2,j}$, and the hemispheres in $\mathcal{S}_j \setminus P_{2,j}$ are transverse to $\mathcal{R}$ in opposite direction.

(iii) Special J: $\cup_j \mathcal{S}_j$ are spherical shields wrt J (prevent the J-curves escaping).

(iv) Special orbit: $\exists$ a p-unknotted periodic orbit P_3 with self-linking $-\frac{1}{p}$ and a p-disk $\mathcal{D}$, i.e. $u : \mathbb{D} \to \mathcal{M}$ (embedded interior and p-cover P_3 on $\partial\mathbb{D}$).

Let $\mathcal{P} = \{P_3, P_{2,1}, \cdots, P_{2,l}\}$ be an orbit set in $\mathcal{M}$.

Definition 4.1 (transverse singular foliation)

A transverse singular foliation $\mathcal{F}$ of $\mathcal{M}$ adapted to $\alpha, \mathcal{P}$, consists the follows:

- *$\mathcal{M} \setminus \mathcal{P}$ is foliated by regular surfaces (2-dim leaves) transverse to $\mathcal{R}$. Each regular leaf is a 2-sphere $\mathbb{C}P^1$ with finite punctures Γ, denoted by $\dot{\Sigma}$.*
- *Each puncture of $\dot{\Sigma}$ is asymptotic to $P_z \in \mathcal{P}$ (1-dim or singular leaves).*
- *The hemispheres in $\mathcal{S}_j \setminus P_{2,j}$ are also regular leaves of $\mathcal{F}$.*

Note: To obtain a transverse singular foliation, we first establish a FEF in $\mathbb{R} \times \mathcal{M}$.

Main result 3

Theorem 4.1 (+ *dyna. conv.* ⇒ Main result 2)

Assume $\mathcal{M}$, α, $\mathcal{P}$ *and* J *as before. Let* $\mathcal{P}' \subset \mathcal{P}_\alpha$ *be the set of contractible periodic orbits* $P' \subset \mathcal{M} \setminus (\partial\mathcal{M} \cup P_3)$ *satisfying "the linking condition"*

$$\rho(P') = 1, \quad \mathrm{link}(P', P_3^p) = 0 \quad \text{and} \quad \mathcal{A}(P') \leq \mathrm{T}(P_3^p) = \mathcal{S}(\mathcal{D}).$$

If $\mathcal{P}' = \emptyset$, *then* $\exists$ *a finite energy foliation* $\tilde{\mathcal{F}}$ *adapted to* α, $\mathcal{P}$, *and* J' *whose leaves are* J'*-planes and* J'*-cylinders, where* J' *is* C^∞*-close to* J *and coincides with* J *near* $\partial\mathcal{M}$. *The projection of* $\tilde{\mathcal{F}}$ *to* $\mathcal{M}$ *is a transverse singular foliation.*

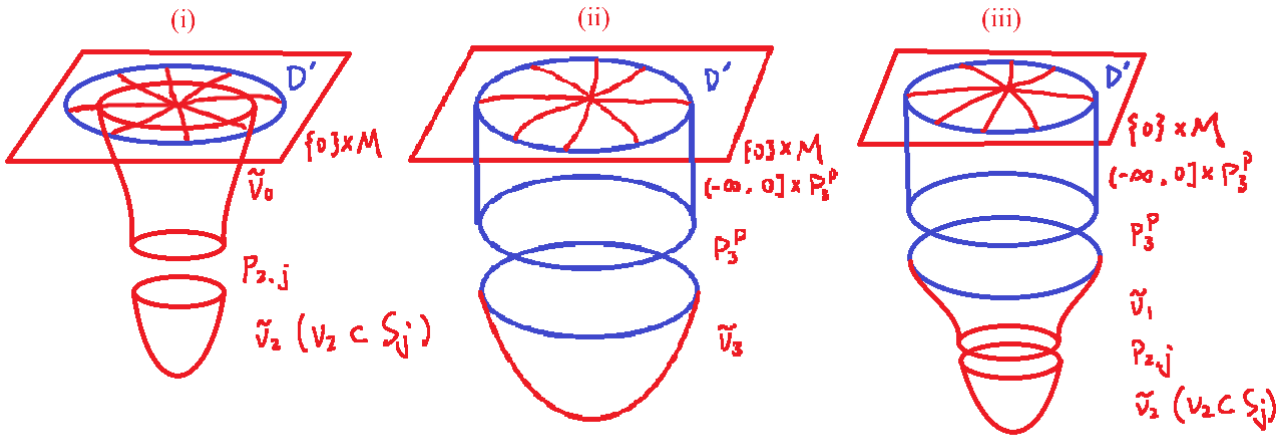
Note: Dynamical convexity ⇒ no orbit have $\rho = 1$ ⇒ the criterion holds.

Proof of main result 3

To prove this, we follow the steps below assuming α is '$< \mathcal{S}(\mathcal{D})$' non-degenerate:

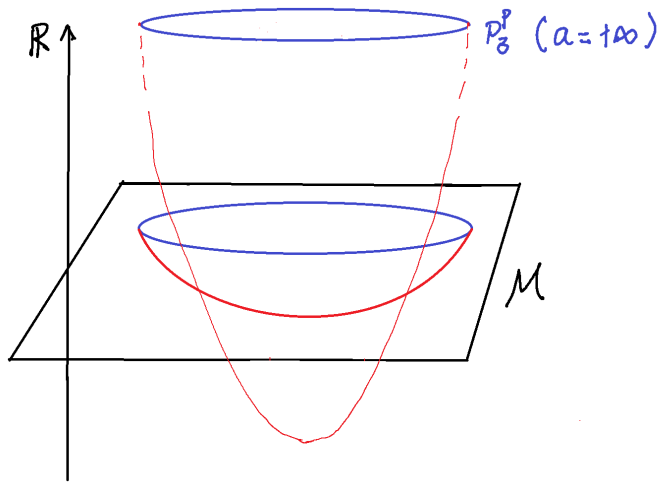
(1) Start a Bishop family $\{\tilde{u}_{\tau\in(0,\epsilon)} : \mathbb{D} \to \mathbb{R} \times \mathcal{M}\}$ on a perturbed p-disk $\mathcal{D}'$, due to Hofer (1993).

(2) By implicit function theorem, we obtain a maximal family $\{\tilde{u}_\tau, \tau \in (0,1)\}$. Since $\mathcal{P}' = \emptyset$, the SFT-limit $\lim_{\tau\to 1-} \tilde{u}_\tau$ has three generic possibilities:



(3) In case (i), by gluing two curves as Hutchings-Taubes (2007,2009), we obtain a new family $\{\tilde{u}_{\tau\in(1,1+\epsilon)} : \mathbb{D} \to \mathbb{R}\times\mathcal{M}\}$ such that $u_{\tau>1}(\partial\mathbb{D}) \subset \mathcal{D}'$ lies in the exterior of $u_1(\partial\mathbb{D})$ (monotonicity). $(\Rightarrow \tilde{u}_{(1,2)} \Rightarrow \tilde{u}_{(2,2+\epsilon)} \cdots \Rightarrow \tilde{u}_{(N-1,N)} \cdots)$

(4) The sequence $\{\tilde{u}_n\}_{n=1}^{+\infty}$ has a limit in case (ii) or (iii). In both case, we can obtain a J-plane $\tilde{v}_3$ asymptotic to P_3^p at ∞.



(5) Since the Fredholm index $\mathrm{ind}^{\delta}(\tilde{v}_3) = 2$, $\tilde{v}_3$ naturally generates a 1-dimensional family of planes $\tilde{u}'_{(0,\epsilon)}$ asymptotic to P_3^p at ∞. Then we extends this family to a maximal one $\tilde{u}'_{(0,1)}$. The SFT-limit $\lim_{\tau\to 1-}\tilde{u}'_\tau$ necessarily breaks into

$$\tilde{u}'_1 = j_1\text{-th cylinder} \ + \ \text{boundary hemisphere in } \mathcal{S}_{j_1}.$$

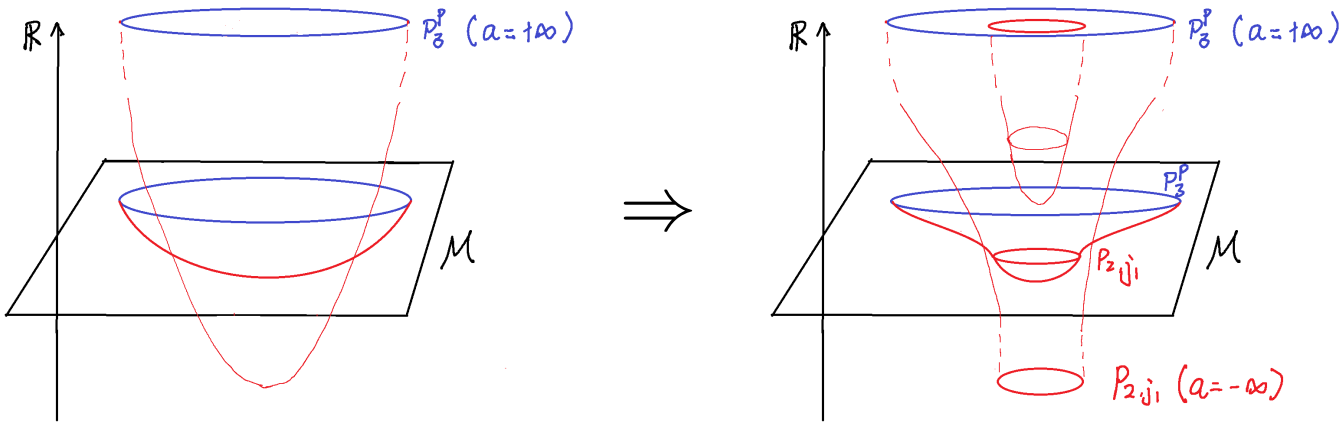
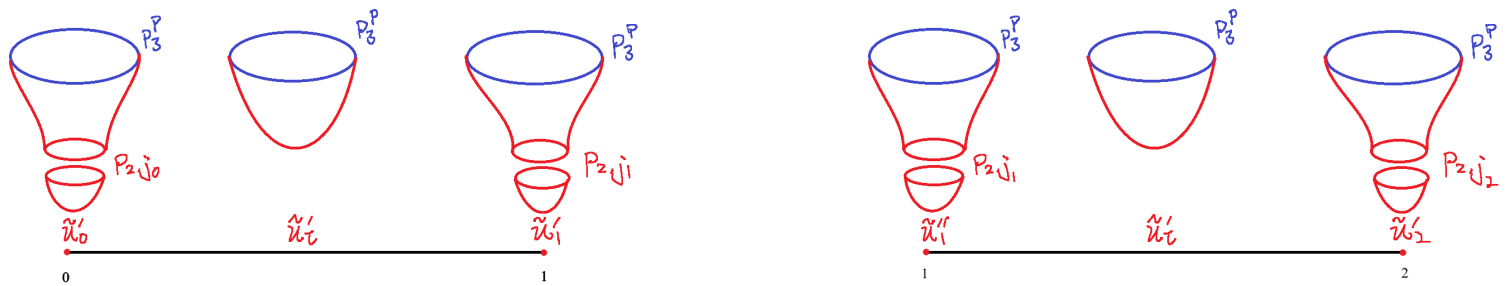


Figure: J-planes $\to$ a broken curve (SFT-limit).

(6) By gluing the j_1-th cylinder and the another hemisphere in $\mathcal{S}_{j_1} \setminus P_{2,j_1}$, we obtain a new family $\tilde{u}'_{(1,1+\epsilon)}$. Then, we further extend this family to a maximal one $\tilde{u}'_{(1,2)}$. $(\tilde{u}'_{(1,2)} \overset{S_{j_2}}{\Rightarrow} \tilde{u}'_{(2,2+\epsilon)} \cdots \overset{S_{j_2}}{\Rightarrow} \tilde{u}'_{(l,l+1)})$



(7) Since the broken curves $\tilde{u}'_n = j_n$-th cylinder + boundary hemisphere in S_{j_n} associate to pairwise distinct orbits $P_{2,j_n} \in \mathcal{S}_{j_n}$, then $\tilde{u}'_{(l,l+1)}$ must coincides with $\tilde{u}'_{(0,1)}$ up to a parametrization.

(8) For a general α, one can find a sequence of " $\leq \mathcal{S}(\mathcal{D})$" non-degenerate (α_k, J_k) that preserves $\mathcal{P}$ and the condition "$\mathcal{P}' = \emptyset$" also holds.

Finally, we establish a finite energy foliation on $\mathcal{M}$ adapted to $\alpha, \mathcal{P}$ and J':

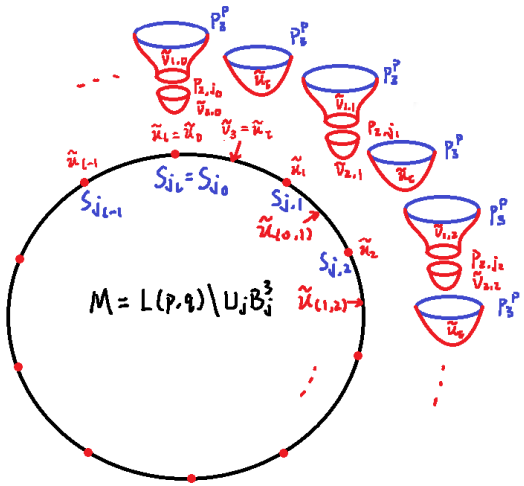


Figure: finite energy foliation on $\mathcal{M}$