

Birkhoff conjecture and finite energy foliations in Hill's lunar problem

Lei Liu and Pedro A. S. Salomão

Abstract. We prove Birkhoff's retrograde orbit conjecture in Hill's lunar problem by showing that the retrograde orbit bounds a disk-like global surface of section for every energy below the critical value. We also obtain a global description of the dynamics through the critical energy level by constructing finite energy foliations for energies slightly above it. The binding of these foliations consists of the retrograde orbit together with the Lyapunov orbits near the critical points. As a consequence, there exist infinitely many periodic orbits and infinitely many trajectories asymptotic to the Lyapunov orbits. The proof combines pseudo-holomorphic curve techniques with a new convexity theorem for Hill's lunar problem. More precisely, we construct an explicit global symplectic change of coordinates under which the bounded regularized component becomes strictly convex up to the critical energy level. This convexity implies strong dynamical consequences, including lower bounds for the Conley-Zehnder indices of periodic orbits, and allows the application of the Hofer-Wysocki-Zehnder theory of finite energy foliations. As a result, we obtain disk-like global surfaces of section below the critical level and $2 - 3 - 2$ foliations for energies slightly above it.

Introduction

The restricted three-body problem has a central place in celestial mechanics since the works of Euler, Lagrange, Poincaré, and Birkhoff. In its planar circular version, one studies the motion of a massless body under the gravitational influence of two primaries moving on circular orbits around their center of mass. Despite its simple formulation, the problem has a wide variety of dynamical phenomena, including periodic orbits, homoclinic intersections, chaotic scattering, and transport through the neck regions near the Lagrange equilibrium points.

One of the most influential models arising from the restricted three-body problem is Hill's lunar problem. Introduced by George W. Hill (1877,1878), the

model was developed as part of Hill's program to obtain a precise analytical description of the motion of the Moon under the perturbative influence of the Sun. Rather than studying the full three-body problem directly, Hill derived a limiting system describing the dynamics near the Earth after a suitable rescaling procedure. The resulting equations contain the principal nonlinear effects of the solar perturbation while presenting a considerably simpler structure.

Hill's work represents one of the major achievements of nineteenth-century celestial mechanics. Using Fourier series and numerical calculations, he obtained highly accurate approximations of lunar periodic motions and set the foundations of a new lunar theory. His methods influenced generations of mathematicians and astronomers, and his lunar theory remained the standard reference for many years. See Brown(1896) and Wintner(1941) for historical accounts and further developments. A modern discussion of Hill's original lunar theory, including the convergence of the associated series expansions, was given by Ligon(2025).

For much of the twentieth century, progress on Hill's lunar problem was driven primarily by perturbative methods, normal form techniques, and numerical investigations. Important advances were obtained through the study of invariant manifolds, periodic trajectories, and chaotic dynamics near the equilibrium points. In particular, the numerical investigations of Simó and Stuchi(2000) revealed a rich global structure involving Lyapunov orbits, stable and unstable manifolds, homoclinic and heteroclinic trajectories, and complicated transport phenomena near the critical energy level. The problem is known to be non-integrable Meletlidou, Ichtiaroglou, and Winterberg(2001), Morales-Ruiz, Simó, and Simon(2005) and transverse intersections of invariant manifolds produce chaotic recurrent dynamics in the sense of Poincaré. See Conley(1968), McGehee(1969), Simó and Stuchi(2000) and references therein.

From the perspective of Hamiltonian and symplectic dynamics, Albers, Frauenfelder, Paternain and van Koert(2012) proved that the regularized energy surface below the critical value is fiberwise star-shaped. In particular, the regularized dynamics naturally defines a Reeb flow on a contact three-manifold. Later, Lee(2017) proved fiberwise convexity of the regularized energy surface below the critical value, relating the dynamics to a Finsler geodesic flow on the two-sphere. Quantitative aspects of the dynamics for very negative energies were studied in Grotta-Ragazzo, L. and Salomão(2025) using Birkhoff normal forms and symplectic techniques. Also, recent works Aydın(2023,2024) combined numerical investigations with techniques from symplectic dynamics to study bifurcating families of periodic solutions in the spatial Hill problem.

The development of pseudo-holomorphic curve techniques by Hofer(1993) and Hofer, Wysocki and Zehnder(1995-2023) introduced powerful new methods for studying global properties of three-dimensional Hamiltonian dynamics. In particular, finite energy foliations organize the dynamics of Reeb flows through families of pseudo-holomorphic curves in symplectizations. These methods led to the construction of global surfaces of section and finite energy foliations in several situations, with strong dynamical consequences.

One of the central open questions in Hill's lunar problem concerns the existence of global surfaces of section. Birkhoff(1915) conjectured that the retrograde orbit bounds a disk-like global surface of section on the regularized energy surface below the critical value. In particular, it provides a complete reduction of the dynamics on the energy surface to a two-dimensional conservative system and implies the existence of a "direct" periodic orbit as a fixed point of the return map.

Conclusion

In this presentation, we will resolve Birkhoff's conjecture and gives the first global geometric description of Hill's lunar problem through the critical energy level. Our first main result proves that the retrograde orbit bounds a disk-like global surface of section for every energy below the critical value. Our second main result concerns energies slightly above the critical value. In this case, we construct $2 - 3 - 2$ finite energy foliations whose binding consists of the retrograde orbit together with the Lyapunov orbits near the critical points. These foliations provide the first global description of the dynamics in a neighborhood of the critical energy level. In particular, they organize the dynamics by families of planes and cylinders transverse to the flow and imply the existence of infinitely many periodic orbits and infinitely many trajectories asymptotic to the Lyapunov orbits.

The results in this paper complete a program initiated by Birkhoff more than a century ago and further demonstrate the effectiveness of the Hofer-Wysocki-Zehnder theory of pseudo-holomorphic curves and finite energy foliations in the study of classical problems from celestial mechanics.

0.1. Ingredients of the proofs

The key ingredient in the proofs is a convexity theorem for the bounded regularized component of the energy surface. More precisely, we show that after a suitable symplectic change of coordinates, the bounded component becomes strictly convex up to the critical energy level. This convexity implies lower bounds for the Conley-Zehnder indices of periodic orbits and provides the dynamical conditions required for the theory of pseudo-holomorphic curves. The resulting finite energy curves give rise to open book decompositions by disk-like global surfaces of section below the critical level, and $2 - 3 - 2$ foliations for energies slightly above it, see Figure 1. To obtain such a foliation for the energy surfaces, we apply the a theorem from [1], which is a general criterion for the existence of finite energy foliations with prescribed binding orbits.

The convexity properties established in this paper appear to have applications beyond Hill's lunar problem. In particular, the symplectic coordinates constructed here naturally extend to the regularized planar restricted three-body problem for sufficiently small mass ratio μ . This suggests a possible approach to constructing finite energy foliations for energy levels ranging from the first to slightly above the second Lagrange value. Such foliations would provide a global description of the dynamics across the neck regions and will be investigated in future work.

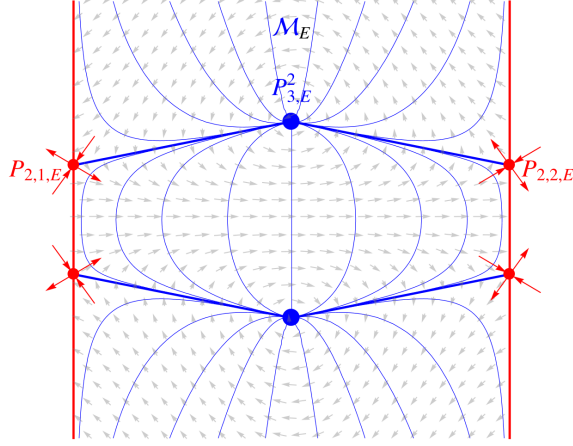


FIGURE 1. The 2 – 3 – 2 foliation in the Hill's lunar problem

References

- [1] L. Liu, Pedro A. S. Salomão, Finite energy foliations and global dynamics in the restricted three-body problem, arXiv:2506.17867v2
- [2] L. Liu, Pedro A. S. Salomão, Birkhoff conjecture and finite energy foliations in the Hill's Lunar problem, arXiv:2606.12912v1.

Lei Liu
 School of Mathematics
 Shandong University
 Jinan, China
 e-mail: liulei30@email.sdu.edu.cn

Pedro A. S. Salomão
 Shenzhen International Center for Mathematics
 Southern University of Science and Technology (SUSTech) Shenzhen, China
 e-mail: psalomao@sustech.edu.cn