

On the solution of the special BVP related to a full set of geodetic and gravimetric data

A. E. Dergileva V. V. Popadyev

PLC Roskadastr

Introduction

- ▷ **1945** – M. S. Molodensky “Basic problems of geodetic gravimetry”: a boundary-value problem was formulated and posed to determine the anomalous potential not specifically on the surface of the geoid, which was emphasized before, but on a fairly arbitrary physical surface, which is formed by measuring points.
- ▷ Knowledge of the potential difference (H^γ) at a point or its spatial position (H) leads to the two main types of free-air gravity reduction: Δg и δg .
- ▷ In this connection, geodetic boundary value problems arise for determining the anomalous potential or its transformants on or off the Earth's surface.
- ▷ When the anomalous potential is found from the solution of one boundary value problem, its transformation to a gravity reduction corresponding to another boundary value problem is possible.

Main types of gravity reductions

Depending on which surface the calculated value of normal gravity is referred to, a distinction is made between:

- ▷ gravity anomalies — when the surface of the telluroid is known

$$\Delta g = g(H) - \gamma(H^\gamma),$$

associated with the anomalous potential using the boundary condition

$$\frac{\partial T}{\partial \rho} + 2 \frac{T}{\rho} = -\Delta g, \quad (1)$$

corresponding solution

$$T = \frac{1}{4\pi R} \oint\!\!\!\oint_{\Omega} \Delta g S(\psi) d\Omega, \quad (2)$$

where $S(\psi)$ – Stokes' function

$$S(\psi) = \operatorname{cosec} \frac{\psi}{2} + 1 - 6 \sin \frac{\psi}{2} - \cos \psi \left[5 + 3 \ln \left(\sin \frac{\psi}{2} + \sin^2 \frac{\psi}{2} \right) \right]; \quad (3)$$

Main types of gravity reductions

- ▷ gravity disturbance — when the Earth's surface is known

$$\delta g = g(H) - \gamma(H),$$

associated with the anomalous potential using the boundary condition

$$\frac{\partial T}{\partial \rho} = -\delta g, \quad (4)$$

corresponding solution

$$T = \frac{1}{4\pi R} \oint\!\!\!\oint_{\Omega} \delta g K(\psi) d\Omega, \quad (5)$$

where $K(\psi)$ – Neumann-Koch function

$$K(\psi) = \operatorname{cosec} \frac{\psi}{2} - \ln \left(1 + \operatorname{cosec} \frac{\psi}{2} \right). \quad (6)$$

- ▶ Theoretically and practically, a situation is possible when H and H^γ are simultaneously known, and gravimetric determinations have also been made.
- ▶ As a result of such complex of measurements, it is possible to calculate separately the Δg и δg .
- ▶ Formally, calculations of the elements of the anomalous field can be performed based on the solution of the first boundary value problem.
- ▶ However, gravimetric observations are more closely spaced than benchmarks, and the normal height at benchmarks always contains a systematic error accumulated within the leveling network, distorting the anomalous potential by the same amount.
- ▶ Such an anomalous potential cannot be considered accurate, but it can be used for gravity reduction.
- ▶ It is generally believed that calculations using δg lead to a more accurate result due to the known boundary surface.
- ▶ Although the use of potential difference instead of spatial position is more justified from a physical point of view.

Spherical solution

- ▶ **1974** – L. P. Pellinen and O. M. Ostach proposed to use another type of anomalies corresponding to a special reduction

$$\Pi g = \Delta g + \frac{3}{2} \frac{T}{\rho} = \delta g - \frac{1}{2} \frac{T}{\rho}, \quad (7)$$

- ▶ with a boundary condition on the Earth's surface

$$\frac{\partial T}{\partial \rho} + \frac{1}{2} \frac{T}{\rho} = -\Pi g, \quad (8)$$

- ▶ solution at an external point in the spherical approximation

$$T = \frac{1}{4\pi} \oint\!\!\!\oint_{\Omega} \frac{2}{r} \Pi g d\Omega, \quad (9)$$

- ▶ in the form of a series

$$T = R \sum_{n=0}^{\infty} \frac{\Pi g_n}{n + 1/2}. \quad (10)$$

When representing the anomalous potential T using the density of a simple layer μ

$$T = \oiint_S \frac{\mu}{r} dS, \quad (11)$$

the following integral equations can be obtained

$$\frac{1}{2\rho} \oiint_S \mu \frac{\rho'^2 - \rho^2}{r^3} dS - 2\pi\mu \cos \alpha = -\Pi g, \quad (12)$$

$$\frac{1}{2\rho} \oiint_{\Sigma} \mu \frac{\rho'^2 - \rho^2}{r^3} d\Sigma + \frac{3}{2\rho} \oiint_{\Sigma} \frac{\mu}{r} d\Sigma - 2\pi\mu \cos \alpha = -\Delta g, \quad (13)$$

$$\frac{1}{2\rho} \oiint_S \mu \frac{\rho'^2 - \rho^2}{r^3} dS - \frac{1}{2\rho} \oiint_S \frac{\mu}{r} dS - 2\pi\mu \cos \alpha = -\delta g. \quad (14)$$

Influence of the “zero-zone”

- ▷ Attraction of a circle

$$T_{0-zone} \approx \Pi g \cdot r_{0-zone}. \quad (15)$$

- ▷ Attraction of a spherical trapezium

$$\begin{aligned} T_{0-zone} \approx \Pi g \frac{R}{2\pi} & \left[\ln \left(\alpha_2 + \sqrt{\alpha_2^2 + \sin^2 \theta_0 \cdot \beta^2} \right) \sin \theta_0 \cdot \beta + \right. \\ & + \alpha_2 \ln \left(\sin \theta_0 \cdot \beta + \sqrt{\alpha_2^2 + \sin^2 \theta_0 \cdot \beta^2} \right) - \ln \left(\alpha_1 + \sqrt{\alpha_1^2 + \sin^2 \theta_0 \cdot \beta^2} \right) \sin \theta_0 \cdot \beta - \\ & \left. - \alpha_1 \ln \left(\sin \theta_0 \cdot \beta + \sqrt{\alpha_1^2 + \sin^2 \theta_0 \cdot \beta^2} \right) \right]_{\beta_1}^{\beta_2}, \quad (16) \end{aligned}$$

где $\alpha = \theta - \theta_0$, $\beta = \lambda - \lambda_0$.

"Flattened" solution

Spheroidal coordinates u, v, w are related to Cartesian coordinates X, Y, Z by

$$\begin{aligned} X &= c \operatorname{ch} w \cos u \cos v, \\ Y &= c \operatorname{ch} w \cos u \sin v, \\ Z &= c \operatorname{sh} w \sin u. \end{aligned} \quad (17)$$

The normal gravity $\Gamma = -\frac{1}{h_w} \frac{\partial U}{\partial w}$ can be represented as

$$\Gamma = \frac{GM}{c^2} \mathcal{F}_1 - \frac{1}{2} \omega^2 c \mathcal{F} - \frac{2}{3} \frac{\omega^2 a^2}{cm} \mathcal{F}_2, \quad (18)$$

where

$$\begin{aligned} \mathcal{F} &= \frac{\cos^2 u \operatorname{sh} 2w}{\sqrt{\operatorname{ch}^2 w - \cos^2 u}}, \quad \mathcal{F}_1 = \frac{1}{\operatorname{ch} w \sqrt{\operatorname{ch}^2 w - \cos^2 u}}, \\ \mathcal{F}_2 &= \mathcal{F}_1 (3 \operatorname{sh} w \operatorname{ch}^2 w \operatorname{arctg} \operatorname{sh} w - 3 \operatorname{sh}^2 w - 2) P_2(\sin u), \\ m &= \left(3 \frac{b^2}{c^2} + 1 \right) \operatorname{arctg} \frac{c}{b} - 3 \frac{b}{c}. \end{aligned}$$

The boundary condition for gravity anomalies Δg

$$\frac{1}{h_w} \frac{\partial T}{\partial w} \Big|_S - \left(\frac{1}{h_w} \frac{1}{\Gamma} \frac{\partial \Gamma}{\partial w} \right)_{\Sigma} T \Big|_S = -\Delta g, \quad (19)$$

corresponds to the integral equation

$$\begin{aligned} \frac{\text{th } w}{h_w} \oint\!\!\!\oint_S \mu' \frac{c^2}{r^3} \left(\frac{\cos^2 u' + \text{sh}^2 w' + \cos^2 u + \text{sh}^2 w}{2} + \right. \\ \left. + \frac{\text{sh } w' \sin u' \sin u}{\text{sh } w} - \text{ch}^2 w \right) dS + \frac{3}{2} \frac{\text{th } w}{h_w} \oint\!\!\!\oint_S \frac{\mu'}{r} dS - 2\pi\mu \cos \alpha = -\Delta g. \end{aligned} \quad (20)$$

For gravity disturbance δg satisfying the boundary condition

$$\frac{1}{h_w} \frac{\partial T}{\partial w} \Big|_S = -\delta g, \quad (21)$$

the following equation holds true

$$\begin{aligned} \frac{\text{th } w}{h_w} \oint\!\!\!\oint_S \mu' \frac{c^2}{r^3} \left(\frac{\cos^2 u' + \text{sh}^2 w' + \cos^2 u + \text{sh}^2 w}{2} + \right. \\ \left. + \frac{\text{sh } w' \sin u' \sin u}{\text{sh } w} - \text{ch}^2 w \right) dS - \frac{1}{2} \frac{\text{th } w}{h_w} \oint\!\!\!\oint_S \frac{\mu'}{r} dS - 2\pi\mu \cos \alpha = -\delta g. \end{aligned} \quad (22)$$

If we correct

- ▷ the gravity anomalies Δg by

$$+ \frac{3}{2} \frac{\text{th } w}{h_w} \cdot T \quad \text{или} \quad + \left(\frac{3}{2} + 2\delta_{\Delta g} \right) \frac{\text{th } w}{h_w} \cdot T, \quad (23)$$

where

$$\delta_{\Delta g} = \frac{1}{2} \frac{\cos^2 u}{\text{ch}^2 w} + \frac{4}{15} \frac{\omega^2 a^2 c}{GMm} \frac{P_2(\sin u)}{\text{sh}^2 w} + \frac{3}{2} \frac{\omega^2 c^3}{GM} \cos^2 u \text{ch}^3 w. \quad (24)$$

By separating the telluroid Σ and the Earth's surface S , this correction can be updated to the form:

$$+ 2 \left(1 + \delta_{\Delta g} \right) \frac{\text{th } w}{h_w} \Big|_{\Sigma} \cdot T - \frac{1}{2} \frac{\text{th } w}{h_w} \Big|_S \cdot T; \quad (25)$$

- ▷ the gravity disturbances δg by

$$- \frac{1}{2} \frac{\text{th } w}{h_w} \cdot T; \quad (26)$$

we get the corresponding equation for gravity in special reduction Πg

$$\frac{\text{th } w}{h_w} \oint_S \mu' \frac{c^2}{r^3} \left(\frac{\cos^2 u' + \text{sh}^2 w' + \cos^2 u + \text{sh}^2 w}{2} + \frac{\text{sh } w' \sin u' \sin u}{\text{sh } w} - \text{ch}^2 w \right) dS - 2\pi\mu \cos \alpha = -\Pi g, \quad (27)$$

which corresponds to the boundary condition

$$\frac{1}{h_w} \frac{\partial T}{\partial w} + \frac{1}{2} \frac{\text{th } w}{h_w} T \Big|_S = -\Pi g. \quad (28)$$

- ▶ The transition to these anomalies can be achieved by introducing the necessary corrections into the primary types of free-air gravity anomalies.
- ▶ One needs to first solve the boundary value problem for Δg or δg , followed by further refinement of the solution based on the Πg anomalies in accordance with equation (27).

Numerical experiments

- ▶ Numerical validation of the obtained theoretical results should be performed under conditions where all quantities used in the work are known with a sufficient degree of confidence.
- ▶ Simulating both input and target quantities is required. This approach eliminates the effect of real-world data noise on the algorithm under test, thereby focusing exclusively on evaluating its inherent error within a given model.
- ▶ Two Earth models have been created. Models Sources: PZ-90.11, TM-60 point mass system, TM-360 point mass system, *ETOPO1*.
- ▶ Two groups of experiments were conducted.

Problem

Let the surface S of the real Earth be known: it is defined by points with known geodetic heights, for which the values of gravity anomalies Δg are also given. It is required to determine the anomalous potential at these points.

Solution of the Problem

- 1 Computation of the zeroth approximation of the anomalous potential T_0 using equation

$$T_0 = \frac{R}{4\pi} \iint_{\omega} G_0 [S(\psi) - 1] d\omega, \quad (29)$$

where $G_0 = \Delta g$.

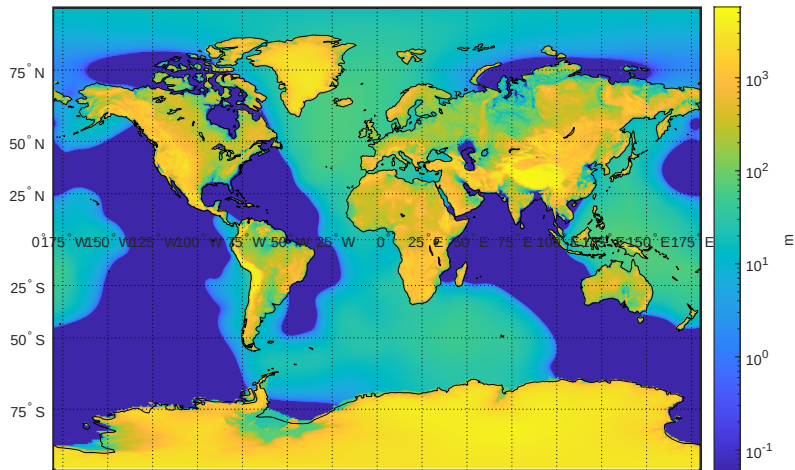
- 2 Computation of the auxiliary function χ for T using equation

$$\chi_n = \frac{G_n}{2\pi} + \frac{3}{(4\pi)^2} \iint_{\omega} G_n [S(\psi) - 1] d\omega.$$

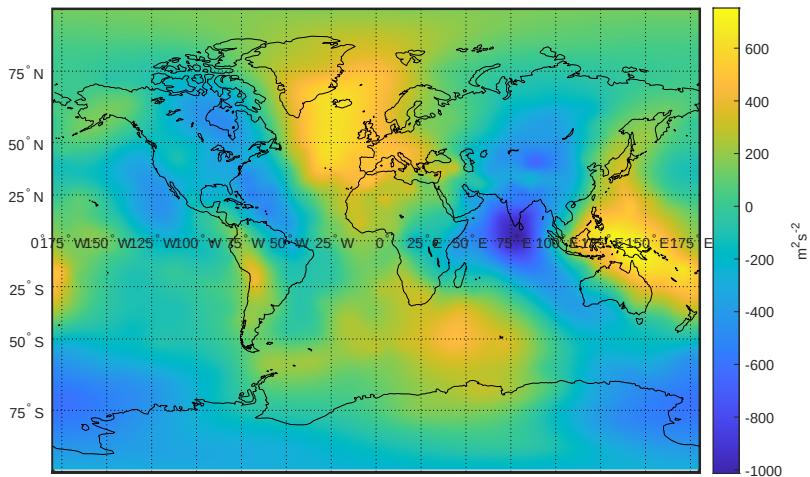
- 3 Computation of the zeroth approximation of the simple layer density using the formula

$$\mu_0 = \frac{R^2}{\rho^2} \chi_0 \cos \alpha.$$

Schematic map of heights of the second surface generated using *ETOPO1*



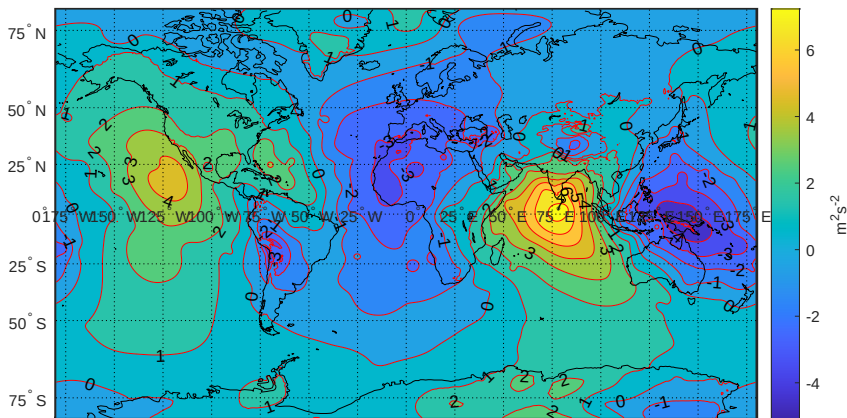
Schematic map of the anomalous potential generated on the second surface by the TM-320 point mass system



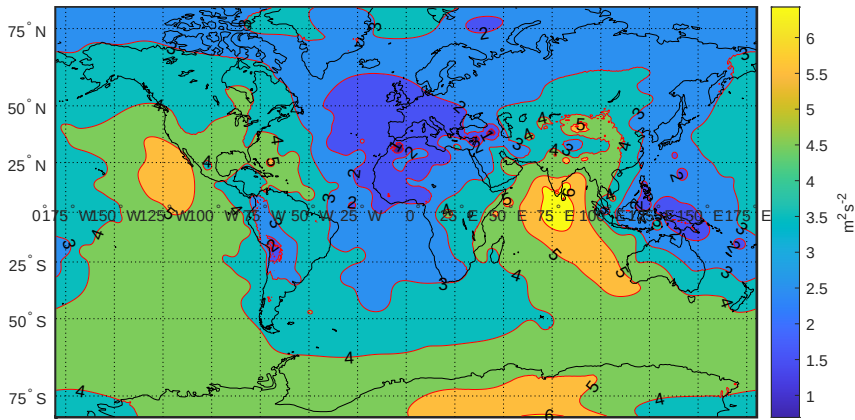
Integration results

Error	$T_0, \text{ m}^2 \text{ s}^{-2}$	$T_{\Pi g}^\mu, \text{ m}^2 \text{ s}^{-2}$	$T_{\delta g}^\mu, \text{ m}^2 \text{ s}^{-2}$
min	-5.1930	+0.2637	-1.1585
max	+7.4684	+6.8518	+7.5410
mean	+0.2910	+3.5900	+3.4072
MSE	1.7982	3.7284	3.7673
Number of iterations	—	2	5
Stopping criterion	—	Loss of convergence $\Delta(\Pi g) = 9.6583 \cdot 10^{-7}$	Loss of convergence $\Delta(\delta g) = 2.5104 \cdot 10^{-7}$
Time	—	$19^h 03^m 27^s$	$40^h 49^m 29^s$

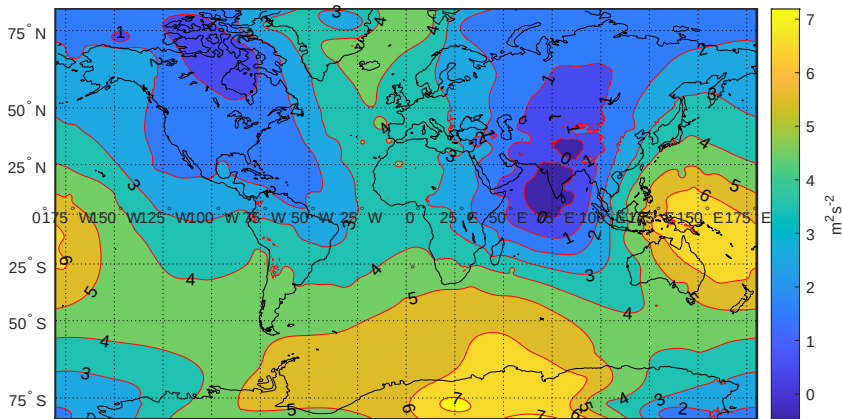
Schematic map of the differences between the calculated and model values of the anomalous potential using the spherical solution for Δg anomalies (second model)



Schematic map of the differences between the calculated and model values of the anomalous potential using the integral equation for Πg anomalies (second model, anomaly values being refined)



Schematic map of the differences between the calculated and model values of the anomalous potential using the integral equation for δg anomalies (second model, anomaly values being refined)



Conclusion

- ▶ The boundary condition in the spherical approximation is considered for solving the boundary value problem to determine the anomalous potential based on topographic gravity anomalies in a special reduction; the solution to the problem is obtained in the form of a spherical harmonic series and in an integral form.
- ▶ Algorithms have been developed to account for the effect of the “zero-zone” on the anomalous potential, represented both as a circle of the same area as the study area and as the attraction of a spherical trapezium.
- ▶ An integral equation has been derived for calculating the anomalous potential relative to the simple layer density, accounting for the flattening of the reference surface, and a relationship with other anomaly types has been established.
- ▶ Formulating the boundary condition in the spheroidal coordinate system for anomalies in a special reduction.
- ▶ Experimentally demonstrating the feasibility of using the considered method to solve the problem of determining the anomalous potential through the transition to gravity anomalies in a special reduction.

Thank you for your attention!