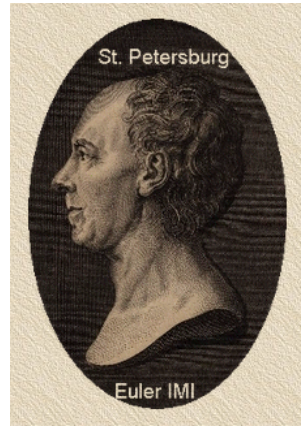




Analytic bounds of the distance between two elliptic orbits

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Introduction

Computing the distance between two confocal elliptic orbits $\mathcal{E}_1$ and $\mathcal{E}_2$ is one of the important problems of modern celestial mechanics. This problem arises, for example, when computing the MOID (Minimum Orbital Intersection Distance) in the context of the asteroid hazard.

The problem has no general analytic solution in explicit functions of orbital elements, so numerical methods are required.

The existence of a relatively simple lower bound of the distance can significantly accelerate the selection of close orbits.

The use of a lower bound becomes practically meaningful only if it remains positive whenever the orbits do not intersect.

Introduction

The main idea: to construct in the plane of $\mathcal{E}_k$ a geometrically simple two-dimensional set $\mathcal{H}_k$, entirely containing $\mathcal{E}_k$, and then to calculate $\rho(\mathcal{H}_1, \mathcal{H}_2)$.

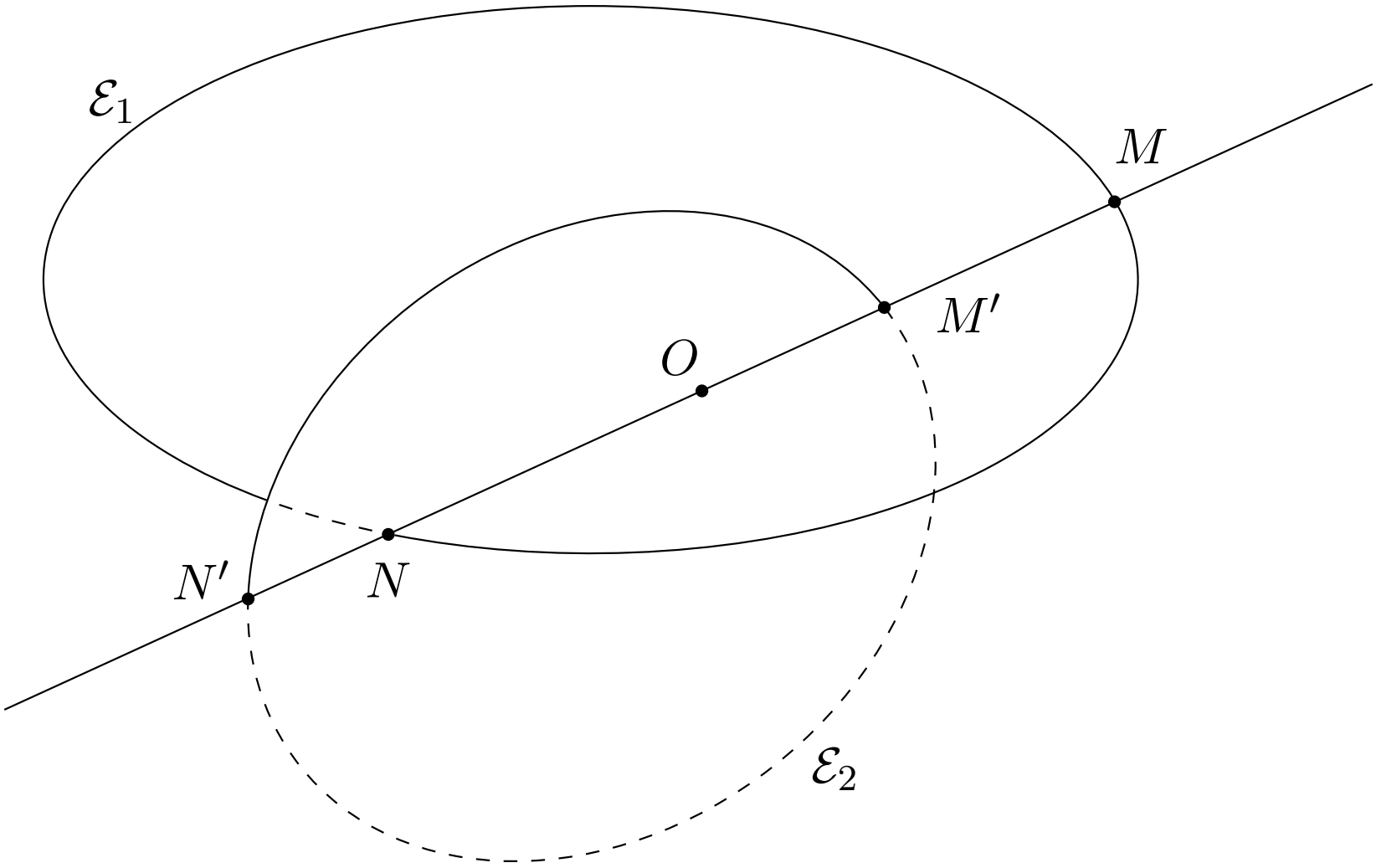
The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are bounded only by line segments and rays. This allows one to avoid dealing with the difficult problem of computing the distance between second-order curves.

The distance $\rho(\mathcal{H}_1, \mathcal{H}_2)$ between $\mathcal{H}_1$ and $\mathcal{H}_2$ turns out to be equal to the distance between two skew lines. It vanishes if and only if the orbits intersect.

As a result, we obtain a lower bound of the distance between the orbits $\rho(\mathcal{E}_1, \mathcal{E}_2) \geqslant \rho(\mathcal{H}_1, \mathcal{H}_2)$.

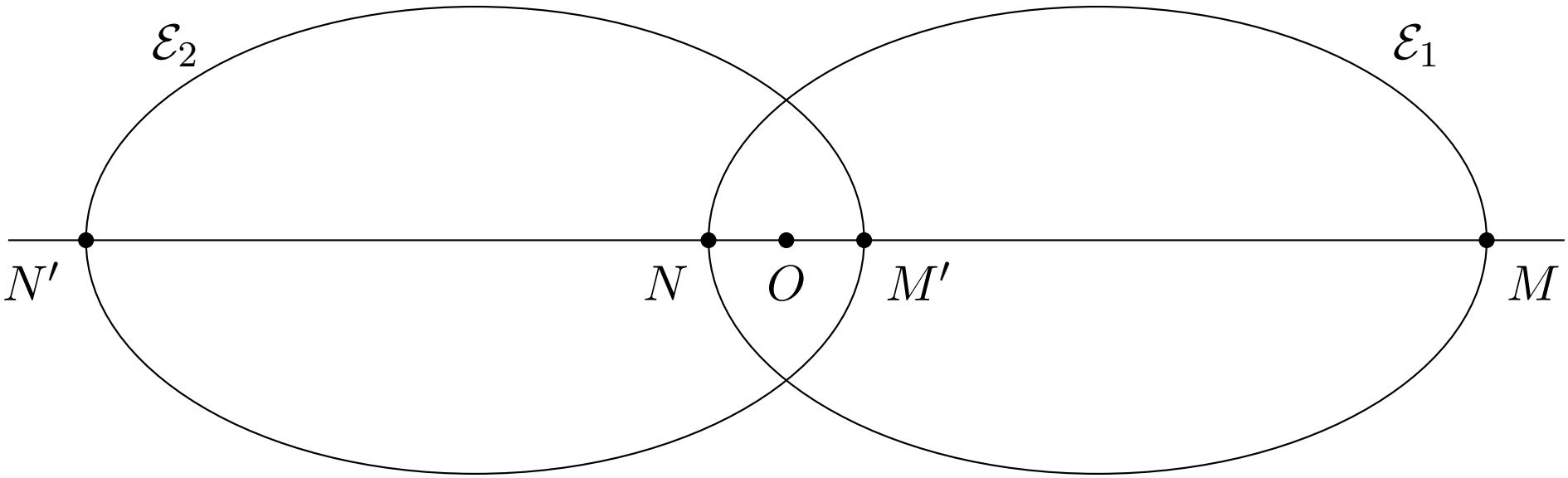
We consider only noncoplanar configurations.

Mathematical setting



A pair of noncoplanar confocal ellipses with common focus O. The orbits intersect if and only if $\min\{MM', NN'\} = 0$.

Mathematical setting



Two equal coplanar ellipses with oppositely directed lines of apses. A small value of $\min\{MM', NN'\}$ is not required for the orbits to be close.

$$\begin{aligned} MN' &= 2a(1+e), \\ MM' = NN' &= 2ae, \\ NM' &= 2a(1-e). \end{aligned}$$

Mathematical setting

A simple upper bound of the distance:

$$\rho(\mathcal{E}_1, \mathcal{E}_2) \leqslant \sigma = \sigma(\mathcal{E}_1, \mathcal{E}_2) = \min\{MM', NN', MN', NM'\}.$$

Objective: to construct a lower bound of the form

$$\rho(\mathcal{E}_1, \mathcal{E}_2) \geqslant \tau(\mathcal{E}_1, \mathcal{E}_2).$$

Here the function τ of the ten-dimensional argument $(\mathcal{E}_1, \mathcal{E}_2)$ has the form

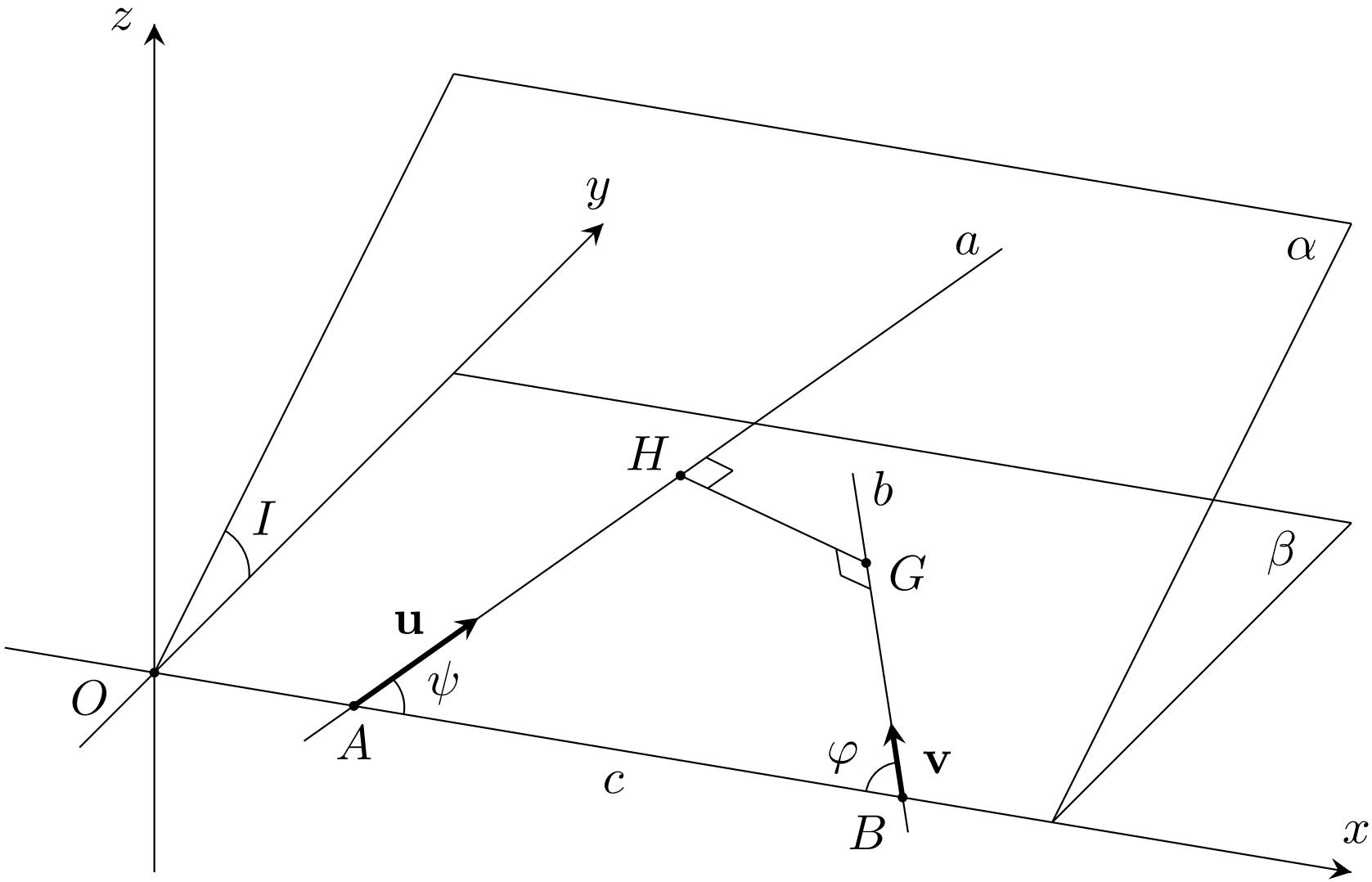
$$\tau(\mathcal{E}_1, \mathcal{E}_2) = C(e, e', I)\sigma(\mathcal{E}_1, \mathcal{E}_2),$$

where the function $C(e, e', I)$ is always positive.

As a result, we obtain a two-sided estimate of the distance

$$\tau(\mathcal{E}_1, \mathcal{E}_2) \leqslant \rho(\mathcal{E}_1, \mathcal{E}_2) \leqslant \sigma(\mathcal{E}_1, \mathcal{E}_2)\,.$$

Basic geometric constructions



The distance $\rho(a, b)$ between the lines a and b depends linearly on c.

Basic geometric constructions

The distance $\rho(a,b)$ between the lines a and b is

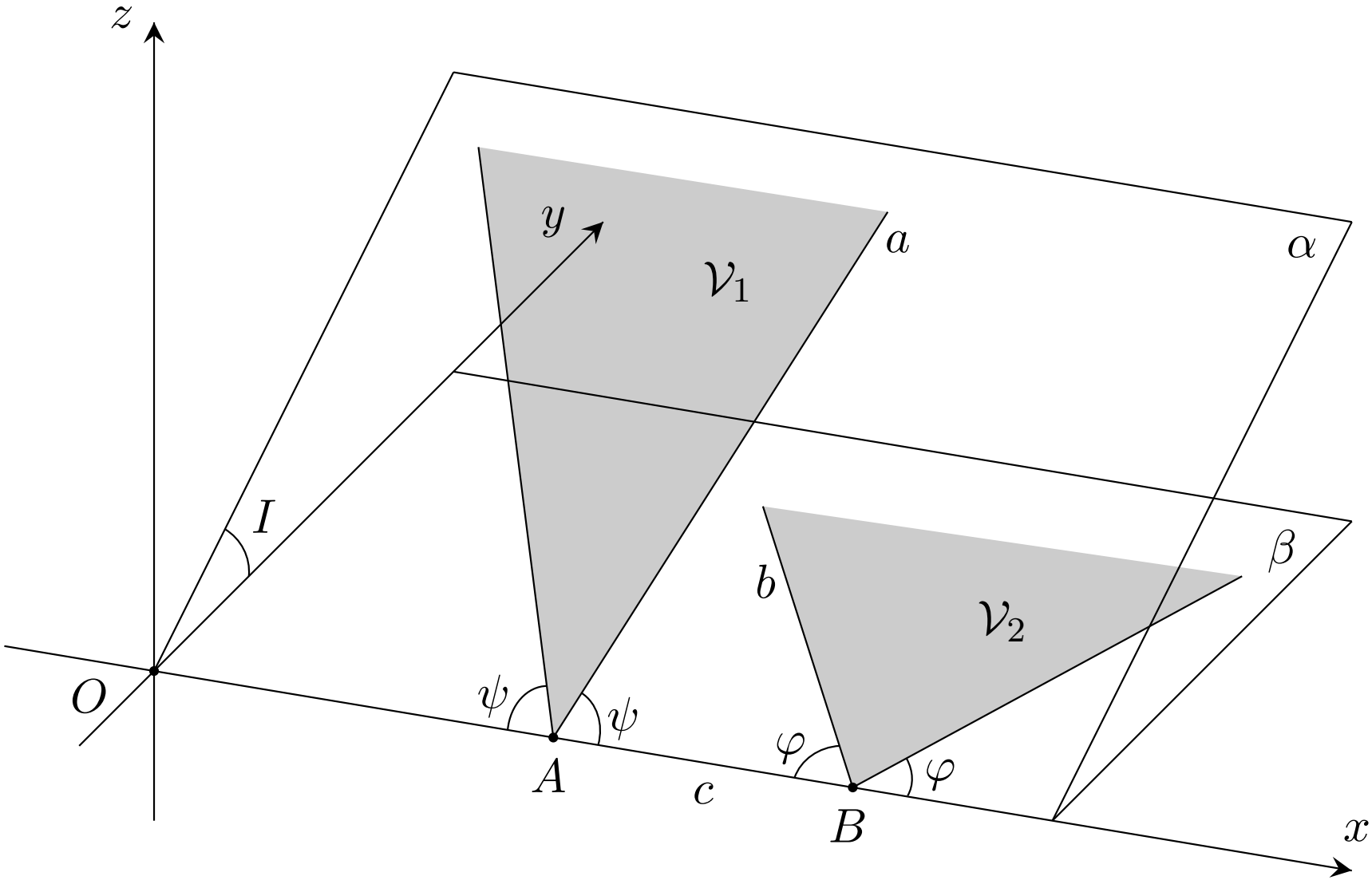
$$\rho(a,b) = K(\varphi,\psi,I)\,c,$$

where $K(\varphi,\psi,I) =$

$$= \frac{\sin\varphi\sin\psi\sin I}{\sqrt{\sin^2\psi(\sin^2 I + \cos^2\varphi\cos^2 I) + \cos^2\psi\sin^2\varphi + \dfrac{\sin 2\varphi\sin 2\psi\cos I}{2}}}.$$

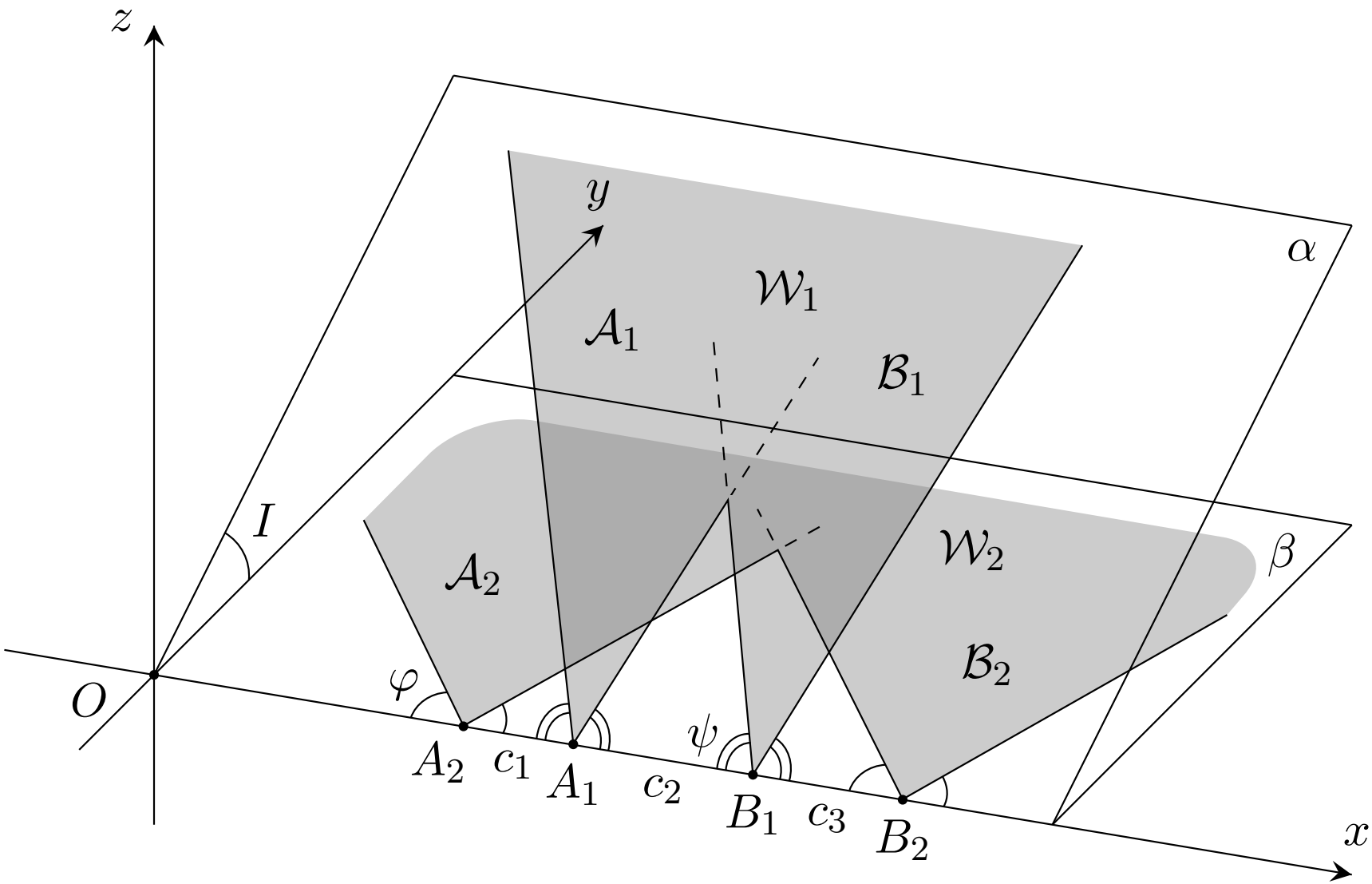
The expression under the square root is always positive, since by assumption $0 < \varphi, \psi < \pi/2$ and $0 < I \leqslant \pi/2$. The symmetry of the function $K(\varphi,\psi,I)$ with respect to φ and ψ is easily verified.

Basic geometric constructions. V-sets



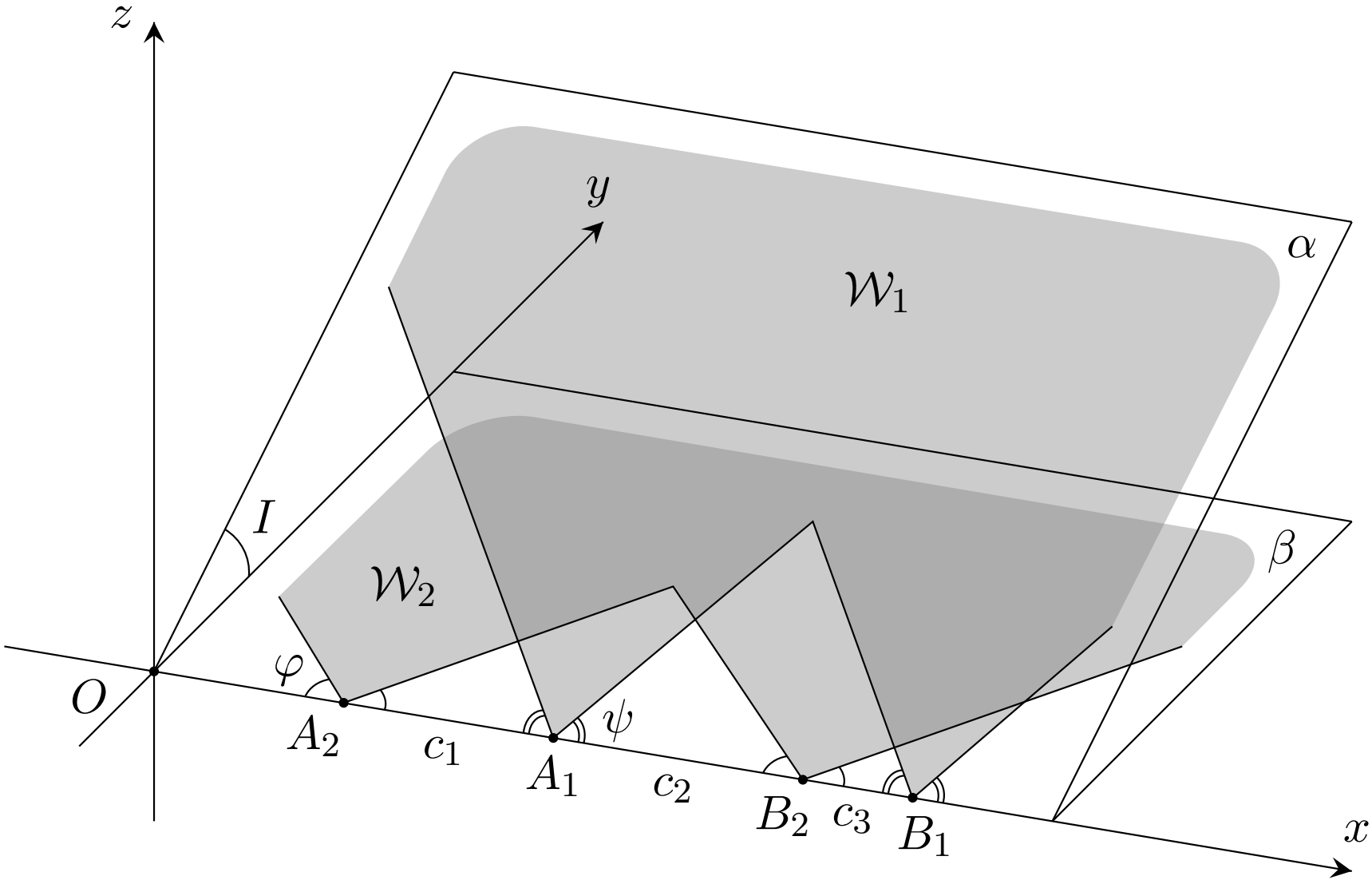
The distance $\rho(\mathcal{V}_1, \mathcal{V}_2)$ between $\mathcal{V}_1$ and $\mathcal{V}_2$ depends linearly on c, since $\rho(\mathcal{V}_1, \mathcal{V}_2) = \rho(a, b)$.

Basic geometric constructions. W-sets



The vertices of a set $\mathcal{W}_1$ both lie between the vertices of a set $\mathcal{W}_2$.
The distance $\rho(\mathcal{W}_1, \mathcal{W}_2)$ tends to zero with $\min\{c_1, c_3\}$.

Basic geometric constructions. W-sets



One vertex of a set $\mathcal{W}_1$ lies between the vertices of a set $\mathcal{W}_2$, while the other does not. The distance $\rho(\mathcal{W}_1, \mathcal{W}_2)$ tends to zero as $\min\{c_1, c_2, c_3\} \to 0$.

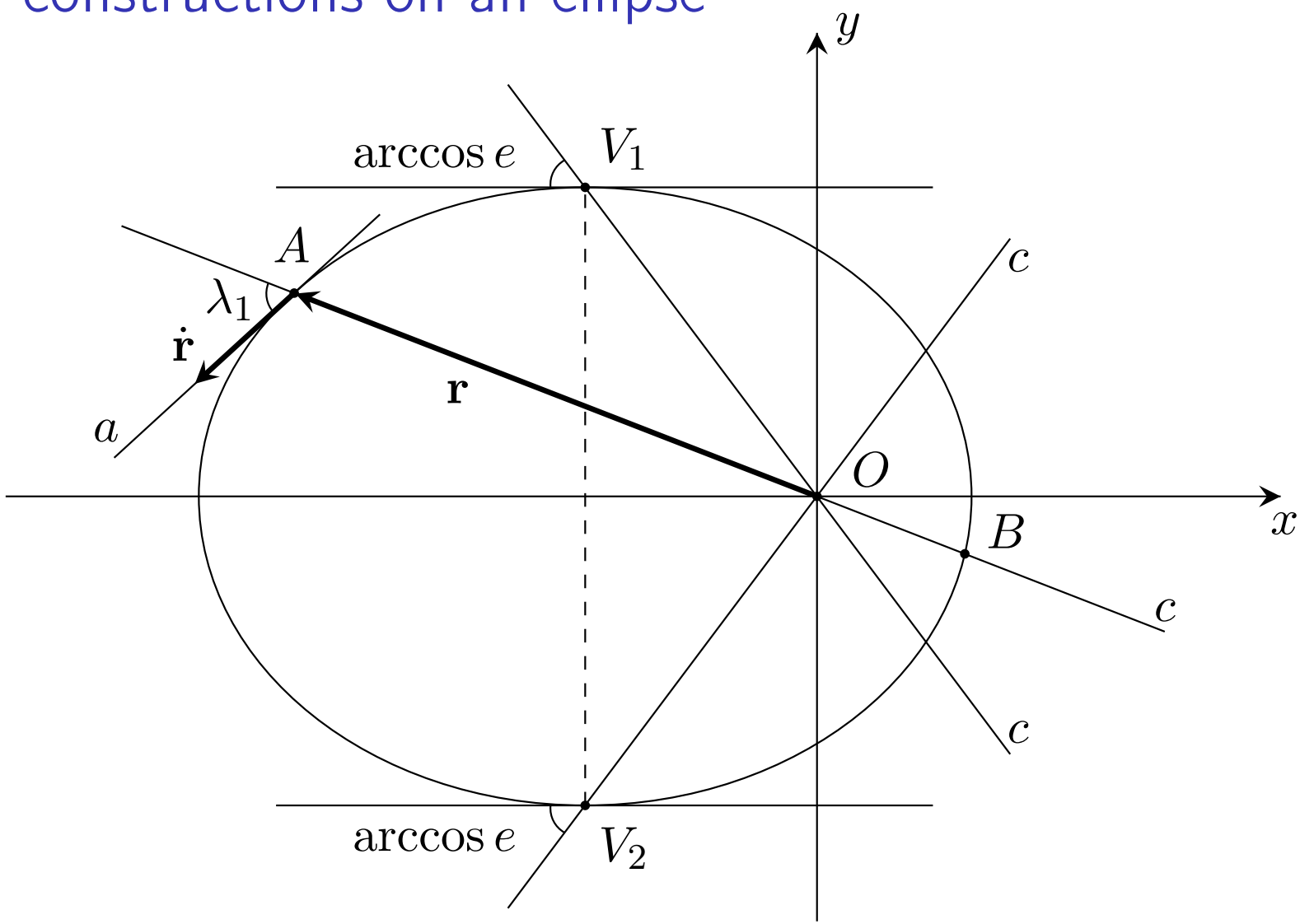
Basic geometric constructions. W-sets

For any relative position of the sets $\mathcal{W}_1$ and $\mathcal{W}_2$, the general formula is

$$\rho(\mathcal{W}_1, \mathcal{W}_2) = K(\psi, \varphi, I) \min\{A_1A_2, B_1B_2, A_1B_2, A_2B_1\}.$$

If any two of the three angles ψ, φ, I are fixed and the third tends to zero, then $\rho(\mathcal{W}_1, \mathcal{W}_2) \to 0$.

Basic constructions on an ellipse



The minimum value of the angle between the line c and the orbit is attained at vertices V_1 and V_2 of semi-minor axes (and only there) and equals $\arccos e$.

Basic constructions on an ellipse

Position and velocity on the orbit are given by

$$\mathbf{r} = \left(\frac{p\cos\theta}{1+e\cos\theta}, \quad \frac{p\sin\theta}{1+e\cos\theta} \right),$$
$$\dot{\mathbf{r}} = \left(-\frac{\varkappa}{\sqrt{p}}\sin\theta, \quad \frac{\varkappa}{\sqrt{p}}(e+\cos\theta) \right).$$

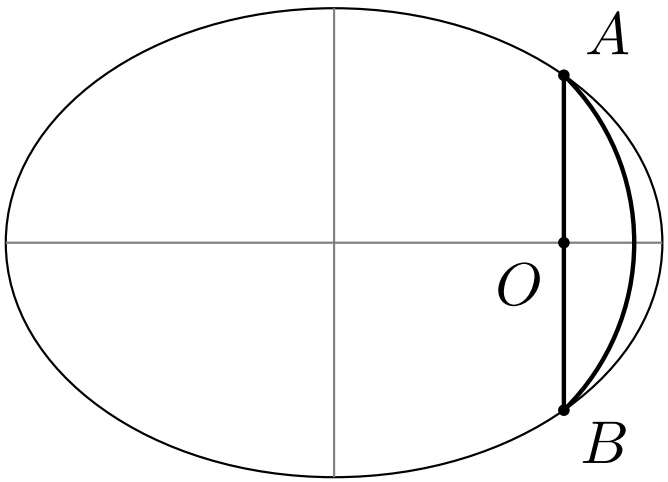
Define the function of the true anomaly

$$v(\theta) = \frac{\mathbf{r}\dot{\mathbf{r}}}{|\mathbf{r}||\dot{\mathbf{r}}|} = \frac{e\sin\theta}{\sqrt{1+2e\cos\theta+e^2}},$$

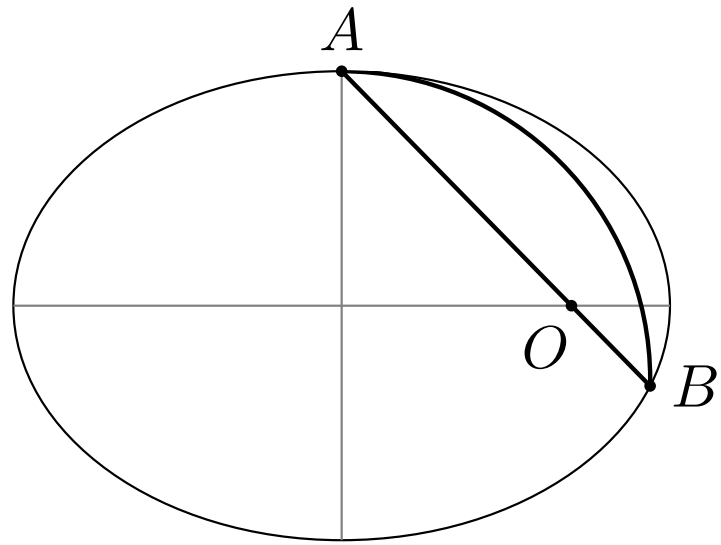
which is the cosine of the angle between the vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$, and calculate its derivative

$$v'(\theta) = \frac{e(e+\cos\theta)(1+e\cos\theta)}{(1+2e\cos\theta+e^2)^{3/2}}.$$

Basic constructions on an ellipse



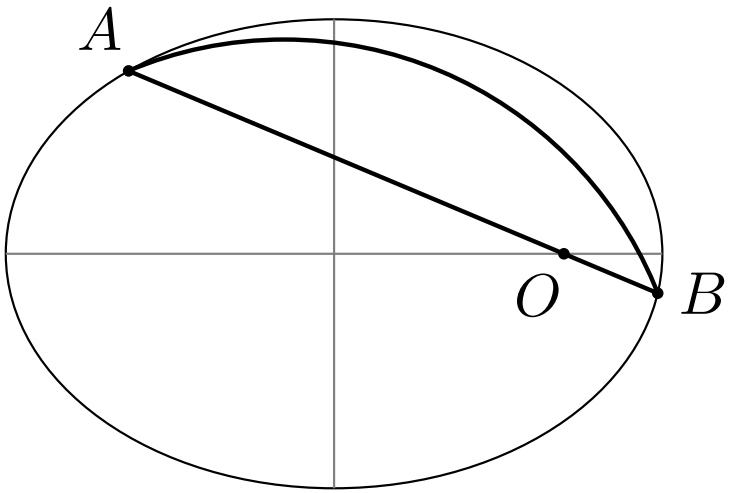
$\theta_A = \pi/2$

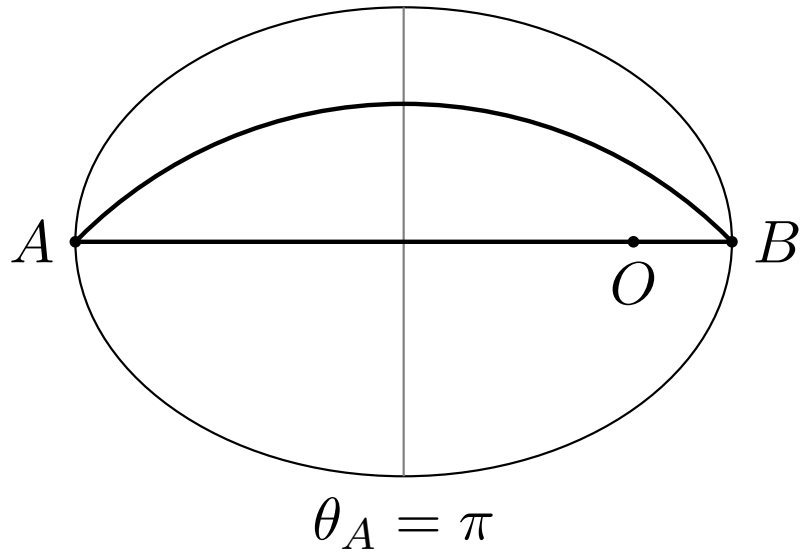


$\theta_A = \arccos(-e)$

For any orientation of the line c, both circular arcs lie wholly within the ellipse. Only one of the two circular arcs is shown on each ellipse. Both the circular arc and the chord AB are shown in bold.

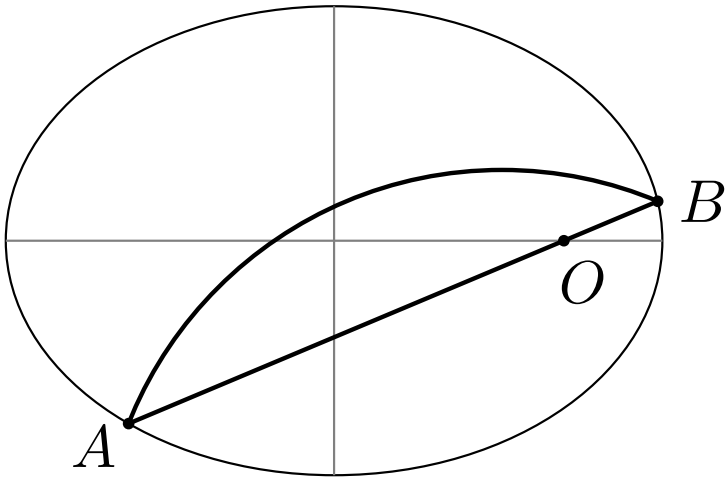
Basic constructions on an ellipse

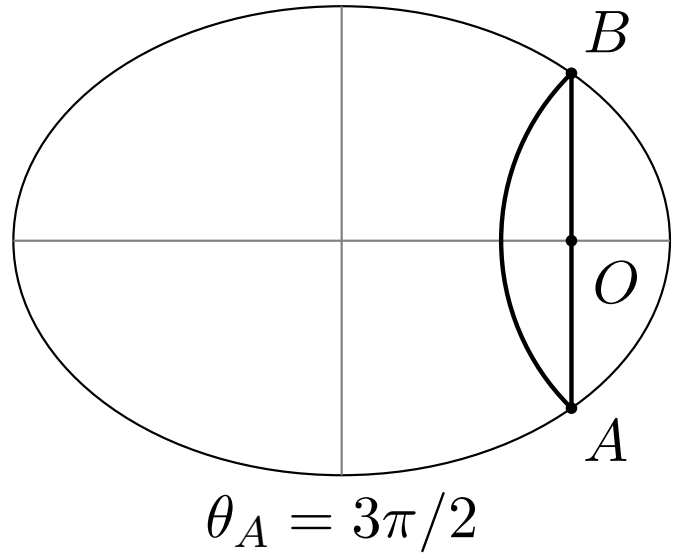




For any orientation of the line c, both circular arcs lie wholly within the ellipse. Only one of the two circular arcs is shown on each ellipse.

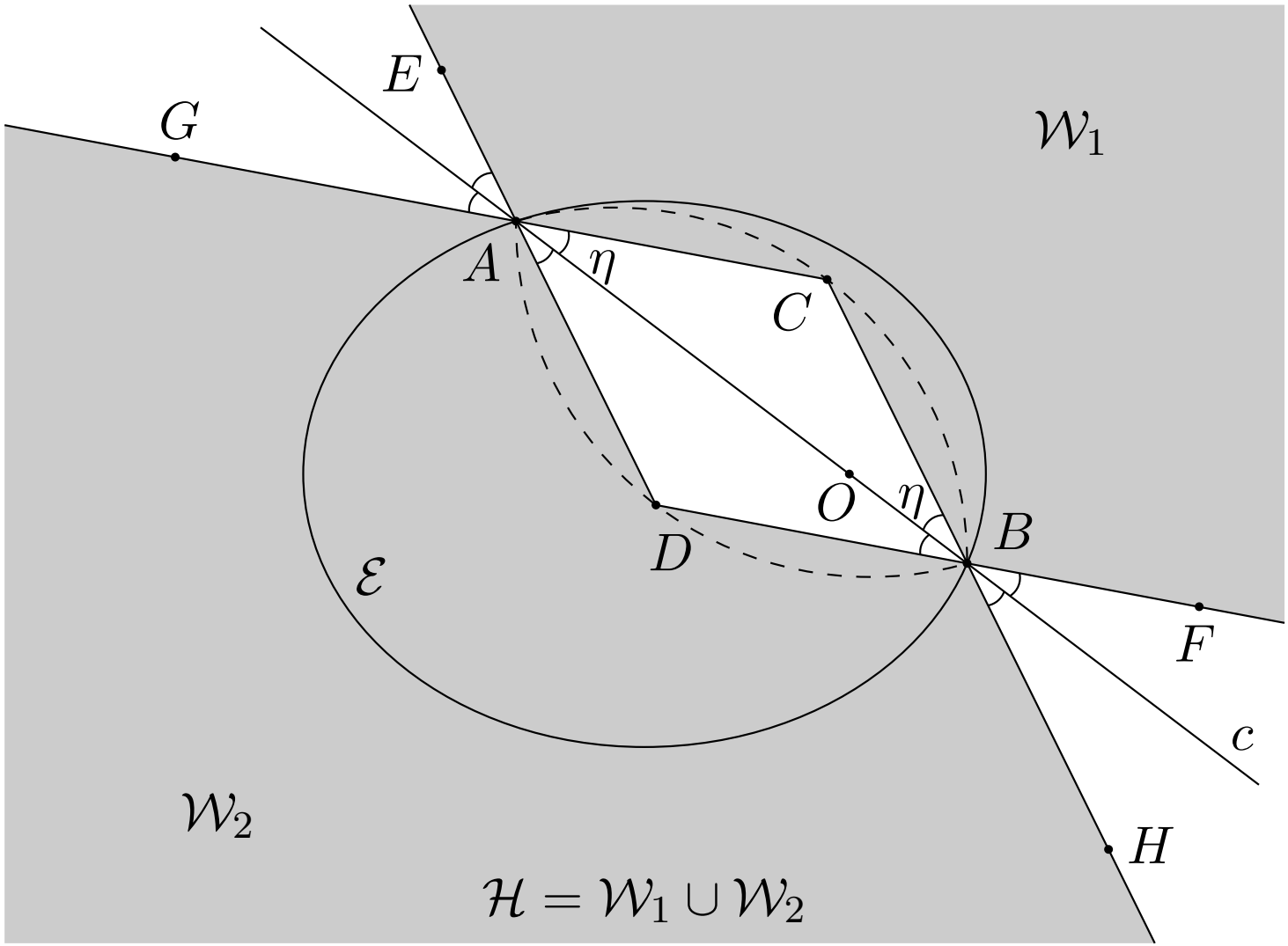
Basic constructions on an ellipse





For any orientation of the line c, both circular arcs lie wholly within the ellipse. Only one of the two circular arcs is shown on each ellipse.

Basic geometric constructions. H-sets



Polygonal chains $EACBF$ and $GADBH$ define two W-sets $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively, whose union $\mathcal{H}$ entirely contains the orbit $\mathcal{E}$ for any position of the line c.

Construction of a lower bound of the distance between orbits

Below we consider the sets $\mathcal{H} \supset \mathcal{E}$ and $\mathcal{H}' \supset \mathcal{E}'$.

Denote by c the mutual nodal line. In the plane of the orbit $\mathcal{E}$ define an H-set $\mathcal{H}$ with vertices M, N and exterior angle

$$\eta = \frac{\arccos e}{2}.$$

In the plane of the orbit $\mathcal{E}'$ define an H-set $\mathcal{H}'$ with vertices M, N and exterior angle

$$\eta' = \frac{\arccos e'}{2}.$$

The sets $\mathcal{H}$ and $\mathcal{H}'$ entirely contain the orbits $\mathcal{E}$ and $\mathcal{E}'$, respectively. Hence,

$$\rho(\mathcal{E}, \mathcal{E}') \geqslant \rho(\mathcal{H}, \mathcal{H}').$$

A lower bound τ of the distance between orbits

Next, we have

$$\rho(\mathcal{H},\mathcal{H}')=\rho(\mathcal{W},\mathcal{W}')=K(\eta,\eta',\min\{I,\pi-I\})\sigma(\mathcal{E},\mathcal{E}').$$

Consequently,

$$\rho(\mathcal{E},\mathcal{E}')\geqslant\tau(\mathcal{E},\mathcal{E}')=C(e,e',I)\sigma(\mathcal{E},\mathcal{E}'),\tag{1}$$

where

$$C(e,e',I)=K\Big(\frac{\arccos e}{2},\frac{\arccos e'}{2},\min\{I,\pi-I\}\Big).$$

After transformations we obtain $C(e,e',I)=$

$$=\sqrt{\frac{(1-e)(1-e')\sin^2 I}{(1-e)(1-e')\sin^2 I+2(1+|\cos I|\sqrt{(1-e^2)(1-e'^2)}-ee')}}.$$

A lower bound τ of the distance between orbits

Most likely, equality in the constructed estimate (1) is never attained (the estimate is not sharp). For example, for circular perpendicular orbits, inequality (1) gives

$$\rho \geqslant \tau = \frac{\sigma}{\sqrt{3}} \approx 0.577\sigma,$$

whereas clearly $\rho = \sigma$ in this case.

The notion of the distance between two skew lines is the foundation of all our constructions, so the appearance of $\sin I$ in the numerator of $C(e, e', I)$ is natural. It follows that the lower bound τ is heavily dependent on the mutual inclination and that the efficiency of the estimate $\rho \geqslant \tau$ decreases as the orbital configuration approaches the coplanar one.

Practical efficiency of the constructed estimate

Values of the functions $\tau \leqslant \rho \leqslant \sigma$ for some close orbits in the main asteroid belt.

Orbit pair $\mathcal{E}$ and $\mathcal{E}'$	$\tau(\mathcal{E}, \mathcal{E}')$, a.u.	$\rho(\mathcal{E}, \mathcal{E}')$, a.u.	$\sigma(\mathcal{E}, \mathcal{E}')$, a.u.
14 Irene – 32 Pomona	1121.1	9954.2	10824.1
4 Vesta – 17 Thetis	103.5	4216.3	4709.4
722 Frieda – 1218 Aster	123.3	1105.6	5697.9
946 Poesia – 954 Li	131.7	968.4	9370.8
704 Interamnia – 775 Lumiere	68.3	696.1	1000.4
1 Ceres – 512 Taurinensis	16.7	180.4	504.3
1333 Cevenola – 4699 Sootan	6.1	32.9	36.1

All values are multiplied by 10^{-6}.

References

The motivation originates from the paper
K.V. Kholshevnikov, and N.N. Vassiliev,
On linking coefficient of two Keplerian orbits,
Celestial Mechanics and Dynamical Astronomy (1999),
v. 75, pp. 67–74.

A more detailed discussion of the results presented herein
can be found in the article:
D.V. Mikryukov, and R.V. Baluev,
A lower bound of the distance between two elliptic orbits,
Celestial Mechanics and Dynamical Astronomy (2019),
v. 131, id. 28.

Conclusions

- In this talk we consider simple estimates of the Euclidean distance ρ between two elliptic orbits $\mathcal{E}_1$, $\mathcal{E}_2$, expressed as explicit functions of osculating elements.
- The primary focus is on the lower bound, since in practice the lower bound is more often required than the upper one.
- The main idea: to construct in the plane of $\mathcal{E}_k$ a geometrically simple two-dimensional set $\mathcal{H}_k$, entirely containing $\mathcal{E}_k$, and then to calculate $\rho(\mathcal{H}_1, \mathcal{H}_2)$.
- The practical convenience of the constructed lower bound τ is that it is nonnegative and vanishes only when the orbits intersect.

Thank you for your attention!
Спасибо за внимание!