

Analytic bounds of the distance between two elliptic orbits

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Abstract. In this talk we consider simple estimates of the Euclidean distance between two elliptic orbits, expressed as explicit functions of osculating elements. The primary focus is on the lower bound, as in practice the lower bound is more often required than the upper one. The practical convenience of the constructed lower bound τ is that it is nonnegative and vanishes only when the orbits intersect.

Introduction

The problem of computing the Euclidean distance function ρ between two confocal elliptic orbits $\mathcal{E}$ and $\mathcal{E}'$ arises in many areas of modern celestial mechanics. This function is now often referred to as the MOID (Minimum Orbital Intersection Distance). Unfortunately, no general analytic solution to this problem exists, and in practice it is solved numerically (see e.g. [1, 2]). As a result, processing a large number of orbital pairs can become computationally expensive.

In practice, however, the MOID usually needs to be computed only for orbits that are close to each other. Therefore, it is of great interest to construct a lower bound τ of the distance ρ that is an explicit nonnegative function of the osculating elements and vanishes only when the orbits intersect. Indeed, in practice the notion of closeness is quantified by some threshold value $\delta > 0$. If $\tau > \delta$, then $\rho > \delta$ as well, and hence computing ρ , which is much more computationally expensive than finding τ , is unnecessary for the given pair of orbits. The value of ρ is computed only when $\tau \leq \delta$. This leads to a significant acceleration of the construction of a catalog of close orbits, since sufficiently distant orbits are filtered out rather quickly.

In the present talk, we consider one of the simplest such estimates for the case of noncoplanar orbits. In practice, this estimate can always be applied, since strictly coplanar orbits do not exist. However, it should be noted that the efficiency of this estimate decreases as the orbits approach a coplanar configuration.

Mathematical setting

In the inertial Euclidean space $\mathbb{R}^3$ consider noncoplanar bounded Keplerian orbits $\mathcal{E}$ and $\mathcal{E}'$ with a common focus O . Let $\mathbf{Z}$ and $\mathbf{Z}'$ be unit vectors directed along the angular momentum vectors of the orbits $\mathcal{E}$ and $\mathcal{E}'$, respectively. The vector $\mathbf{w} = \mathbf{Z} \times \mathbf{Z}'$ defines the line of mutual nodes c , i.e., the line along which the orbital planes of $\mathcal{E}$ and $\mathcal{E}'$ intersect. Denote by e , e' , and I the eccentricities of the orbits $\mathcal{E}$, $\mathcal{E}'$ and the angle between $\mathbf{Z}$ and $\mathbf{Z}'$, respectively. By assumption, $0 \leq e, e' < 1$, $\sin I > 0$.

The point O divides the mutual nodal line into two rays. On the ray directed along the vector $\mathbf{w}$, we mark the points $M \in \mathcal{E}$ and $M' \in \mathcal{E}'$, and on the other ray, the points $N \in \mathcal{E}$ and $N' \in \mathcal{E}'$. The points M, M', N, N' always exist and are unique (although some of them may coincide).

Put

$$\sigma_1 = r - r', \quad \sigma_2 = R - R', \quad \sigma_3 = r + R', \quad \sigma_4 = R + r',$$

where $r = OM$, $r' = OM'$, $R = ON$, $R' = ON'$, and define the functions

$$\begin{aligned} \ell &= \ell(\mathcal{E}, \mathcal{E}') = \sigma_1 \sigma_2, \\ \sigma &= \sigma(\mathcal{E}, \mathcal{E}') = \min\{|\sigma_1|, |\sigma_2|, |\sigma_3|, |\sigma_4|\}. \end{aligned} \quad (1)$$

Now for the distance function $\rho = \rho(\mathcal{E}, \mathcal{E}')$ we can write the obvious upper bounds

$$\rho^2 \leq |\ell|, \quad \rho \leq \sigma \leq |\ell|^{1/2}. \quad (2)$$

Both functions ℓ and σ of the ten-dimensional argument $(\mathcal{E}, \mathcal{E}')$ are continuous, vanish if and only if the orbits intersect, and are expressed explicitly in terms of orbital elements. In the definition of σ , we take the absolute value of σ_3 and σ_4 to make formula (1) more symmetric.

The properties of the function ℓ , called the linking coefficient, are discussed in detail in [3]. That paper also raises the question of finding a positive function $C'(e, e', I)$ such that the inequality

$$\rho^2 \geq C'(e, e', I)|\ell|$$

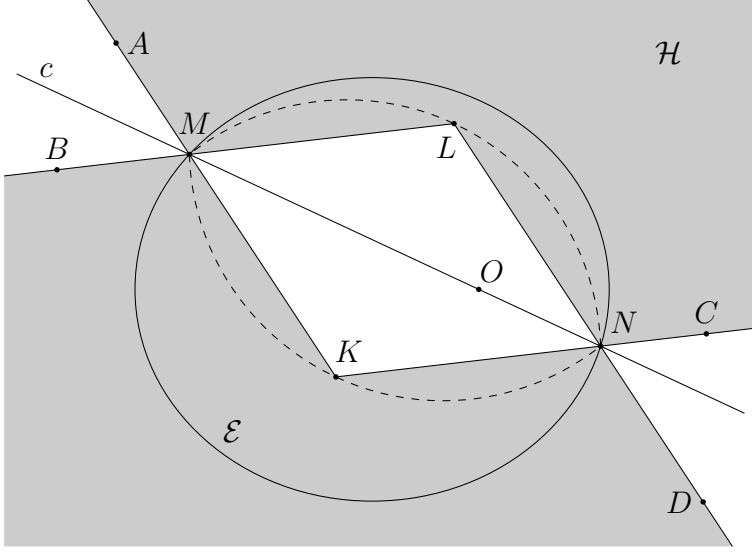
holds.

We express cautious doubts about the existence of the aforementioned function $C'(e, e', I)$. Nevertheless, one can construct a positive function $C(e, e', I)$ satisfying the lower bound

$$\rho \geq \tau = \tau(\mathcal{E}, \mathcal{E}') = C(e, e', I)\sigma. \quad (3)$$

Basic geometric constructions

The main idea behind constructing the estimate τ is that the orbits $\mathcal{E}$ and $\mathcal{E}'$ are entirely embedded into two-dimensional sets $\mathcal{H}$ and $\mathcal{H}'$ of relatively simple geometric structure, after which the distance $\rho(\mathcal{H}, \mathcal{H}')$ between them is computed and taken as τ . The sets $\mathcal{H}$ and $\mathcal{H}'$ are simple in the sense that their boundaries

FIGURE 1. The set $\mathcal{H}$ is shaded in gray.

consist only of segments and rays. As a result, the distance $\rho(\mathcal{H}, \mathcal{H}')$ turns out to be equal to the distance between two skew lines. In this way, we avoid dealing with the difficult problem of computing the distance between two second-order curves.

Let us outline the construction of the set $\mathcal{H} \supset \mathcal{E}$. The set $\mathcal{H}' \supset \mathcal{E}'$ is constructed similarly.

Draw the tangent m to the orbit $\mathcal{E}$ at the point M . If we hold e fixed, the quantity $\lambda = \angle(c, m)$ is a function of the true anomaly θ of the point M . (By the angle between lines we mean the angle not exceeding $\pi/2$.) By continuity and periodicity, the function $\lambda(\theta)$ attains its maximum and minimum values. It is easy to prove that the minimum value is attained at vertices of the minor axis of the ellipse and equals $\arccos e$. (The maximum value $\pi/2$ is obviously attained at the apses for any e .)

Next, on the segment MN as a chord, let us construct a circle such that the tangents to it at the points M and N form an angle $\arccos e$ with the line c . In that way, two circles can be constructed. We only need their smaller arcs, which are shown in dashed lines in Fig. 1. Denote the midpoints of these arcs by K and L , and draw rays from these points toward M and N . As a result, the set $\mathcal{H}$ is defined by the infinite polygonal chains $AMLNC$ and $BMKND$ as in Fig. 1. One can prove that $\mathcal{E} \subset \mathcal{H}$ for any position of the line c .

Consider the basic properties of the set $\mathcal{H}$.

The set $\mathcal{H}$ and its boundary are pathwise connected sets. When c rotates about O , we obtain different sets $\mathcal{H}$. However, for a fixed e , they are all similar: only the distance between M and N changes. When $e = 0$, the rhombus $MLNK$

degenerates into a square inscribed in the circular orbit $\mathcal{E}$, while the orbit $\mathcal{E}$ itself can be considered as a result of the degeneration of the circular arcs (shown dashed in Fig. 1) into a union of two semicircles with common endpoints M and N .

Results and conclusions

The construction of the set $\mathcal{H}$ is discussed in more detail in [4]. It is also shown there that the distance $\rho(\mathcal{H}, \mathcal{H}') = C(e, e', I)\sigma$, where

$$C(e, e', I) = \sqrt{\frac{(1-e)(1-e')\sin^2 I}{(1-e)(1-e')\sin^2 I + 2(1+|\cos I|\sqrt{(1-e^2)(1-e'^2)} - ee')}} \quad (4)$$

and σ is defined by (1). Taking $\tau(\mathcal{E}, \mathcal{E}') = \rho(\mathcal{H}, \mathcal{H}')$, we finally conclude that the lower bound (3) holds, where the function $C(e, e', I)$ is given by (4). The function (4) is always positive and satisfies $C(e, e', I) = C(e', e, I)$.

The constructed estimate τ is most likely not sharp. For example, for circular perpendicular orbits, formulas (3) and (4) give $\tau = \sigma/\sqrt{3}$, whereas clearly $\rho = \sigma$ in this case. Combining (2) and (3), one obtains for the distance $\rho(\mathcal{E}, \mathcal{E}')$ an effective two-sided estimate

$$\tau(\mathcal{E}, \mathcal{E}') \leq \rho(\mathcal{E}, \mathcal{E}') \leq \sigma(\mathcal{E}, \mathcal{E}'), \quad (5)$$

where the functions τ , ρ , σ are either all zero ($\mathcal{E}$ and $\mathcal{E}'$ intersect) or all positive ($\mathcal{E}$ and $\mathcal{E}'$ do not intersect). Two-sided estimate (5) contains only simple and explicit functions of the osculating elements.

The practical efficiency of the estimate τ was studied in [4].

References

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