

Set-theoretic distance and critical points of the surface distances on the example of asteroid 2102 Tantalus

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2. Abstract

In this paper, the evolution of the two-dimensional surface of the distances between the Keplerian orbits of asteroids and the Earth is investigated by long-term numerical integration of the equations of motion. The distance surface is considered as a smooth function on the torus of eccentric anomalies, the critical points of which determine local minima, maxima and saddles, including the value of the minimum distance between orbits (MOID). The model dynamics of MOID on the distance surface and the relevance of this status to different local minima during evolution over long time intervals are investigated.

3. Some data about asteroids

As of 13.08.2026, it is known that:

- 1 555 938 - the total number of asteroids (numbered and unnumbered).
- 42129 - "near-Earth asteroids"

It is believed that the threat to Earth may come from "near-Earth asteroids" with the perihelion distance $q < 1.3$ A.U.

- 11683 - the diameter is larger 140 m
- 874 - the diameter is larger 1000 m
- 2500 - "potentially hazardous asteroids"

Among near-Earth asteroids, the following are distinguished "potentially hazardous asteroids". Their MOID (the minimum distance between the asteroid's orbit and Earth's orbit) is less than 0.05 A.U., absolute magnitude $H < 22^m$ (the diameter is larger 140 m).

- 2181 - possible close approaches and collisions with Earth in the next 100 years have been identified for these asteroids.

4. Introduction

Determining the proximity of Keplerian trajectories is a fundamental task of celestial mechanics, which is of critical importance for assessing the risks of near-Earth asteroids (NEOs) or potentially dangerous objects (PHOs) colliding with our planet. From the point of view of mathematics, this problem is formulated as the search for the minimum distance between points of two one-dimensional sets in three-dimensional Euclidean space, that is, the search for the minimum distance between orbits (MOID). Historically, MOID search algorithms have evolved from pure analytical elliptical cases to more general configurations.

5. Introduction (continued)

In the classical work of K.V.Kholshevnikov and N.N.Vasiliev [2] the problem of determining critical points for two ellipses, it was reduced to finding the roots of a trigonometric polynomial of the 8th degree. Modern numerical methods make it possible to find all stationary points of the distance function by solving an algebraic equation of the 16th degree, ensuring high accuracy and reliability of calculations. The importance of finding all critical points (minima, maxima, and saddle points), not just the global minimum, is dictated by the topology of the distance function. For two ellipses, this function is defined on a two-dimensional torus, and according to Morse theory [3] in the non-degenerate case it must have at least four critical points.

6. Introduction (continued)

Despite the availability of efficient algorithms for static computing MOID, the question of how these critical points move across the distance surface under the influence of secular perturbations remains less studied. Most existing methods focus on instantaneous distance values. In this paper, the evolution of the surface is investigated. distances over long time intervals. We analyze the trajectories of critical points along the surface of the distance, which allows us to better understand the mechanisms of changing the conditions of convergence of celestial bodies. Special attention is paid to the movement of points over significant time intervals, when a change in the elements of the orbits leads to topological rearrangements of the distance surface.

7. Set-theoretic distance and the critical points of the distance surface

To find the distance between two points in elliptical orbits, we use the function given in the article by K.V.Kholshevnikov and N.N.Vasilyev [2]. In the same article, it is shown that the problem of finding the minimum distance between two points in elliptical orbits is reduced to finding the roots of a trigonometric polynomial of the 8th degree, which is equivalent to an algebraic polynomial of the 16th degree. We also note that our distance function, which depends on two eccentric anomalies, is defined on the torus.

8. Set-theoretic distance and the critical points of the distance surface (continued)

Hence, the number of critical points found according to Morse theory [3] satisfies the following equality:

$$N_{\text{saddle}} = N_{\text{min}} + N_{\text{max}},$$

where N_{saddle} is the number of saddle tokens,

N_{min} is the number of points of minimum distances,

N_{max} is the number of points of maximum distances.

This is a very important point in the context of MOID, because as will be shown later, it is not a specific point on the surface, but only a status that can be transferred between local minima. To assess the topological state of the interaction of two orbits, the value of the gearing coefficient is used, the calculation method of which is given in the textbook the manual [1]. According to him, this value for two elliptical orbits is calculated much more simply than the minimum distance between orbits (MOID).

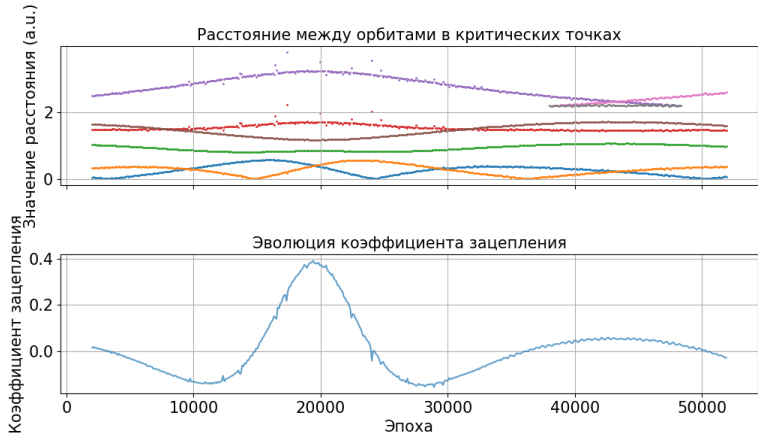
9. Set-theoretic distance and the critical points of the distance surface (continued)

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10. The results of research using the example of asteroid 2102 Tantalus (1975 YA)

Let's consider the results of the study using the example of asteroid 2102 Tantalus from the Apollo group. During the simulation, the following phenomenon is observed over a time interval of 50,000 years: the inconstancy of the number of critical points. The observed change in their number ($6 \rightarrow 8 \rightarrow 6$) is interpreted as a result of bifurcations of the distance function that occur when the parameters of the orbits change.

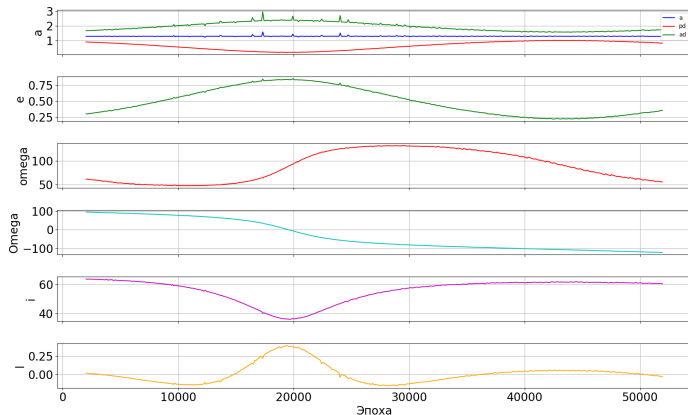
11. Evolution of the critical points of asteroid 2102 Tantalus over a time interval of 50,000 years



12. Evolution of the critical points of asteroid 2102 Tantalus over a time interval of 50,000 years (continued)

As a result of the evolution of the distances between the orbits, a characteristic structure appears at critical points, superficially resembling a catastrophe of the type a "swallow tail" from the theory of catastrophes, which is formed as a result of the appearance of two critical points: a local maximum and a saddle point. One of the local maxima increases and becomes global, which is confirmed by comparing the values of the distance function at all critical points, but its predecessor in this status merges with the saddle point and disappears.

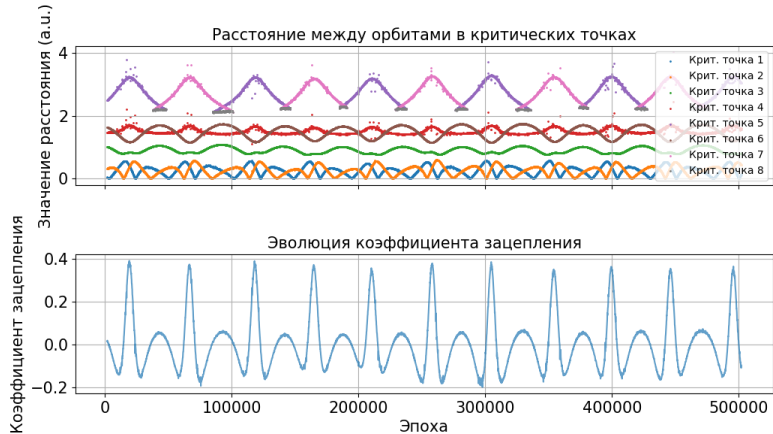
13. The evolution of the orbital elements of asteroid 2102 Tantalus over a time interval of 50,000 years



14. The evolution of the orbital elements of asteroid 2102 Tantalus over a time interval of 50,000 years (continued)

In all numerical experiments, as a result of evolution, a consistent behavior of the distances between orbits at critical points, perihelion and aphelion distances, eccentricity and inclination with a gearing coefficient is observed. The high peak of the gearing coefficient coincides with the peak of the global maximum, and the low peak coincides with the moment of the transition of the global status between the maxima in the "swallow's tail".

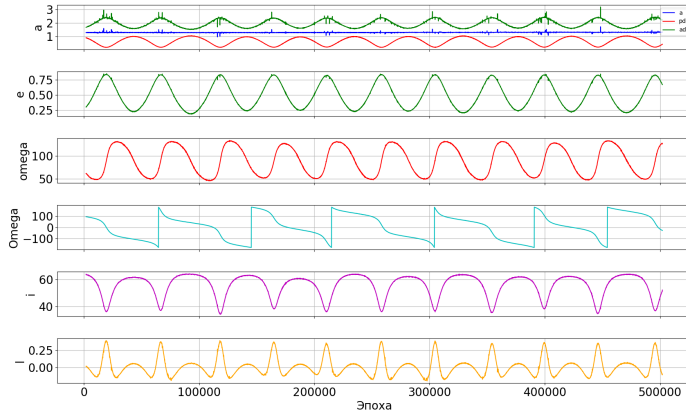
15. Evolution of the critical points of asteroid 2102 Tantalus over a time interval of 500,000 years



16. Evolution of the critical points of asteroid 2102 Tantalus over a time interval of 500,000 years (continued)

It is this apparent consistency in behavior that has become the motivation for research on the 500,000 year interval, where a long-period structure is observed. Similarly, the "highs of the maximum" occur at high peaks of the gearing coefficient, and the moments with the appearance of new critical points at low ones. At the same time, we note, that the positions of the "caps" of both types of peaks exceed 0 in values, which corresponds to the absence of engagement.

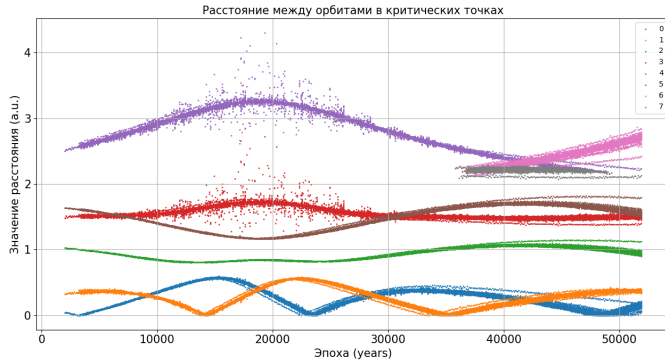
17. The evolution of the orbital elements of asteroid 2102 Tantalus over a time interval of 500,000 years



18. The evolution of the orbital elements of asteroid 2102 Tantalus over a time interval of 500,000 years (continued)

Moreover, if the evolution of the semimajor axis, perihelion and aphelion are compared with the evolution of the gearing coefficient, it becomes clear that at the moments of "finding at low peaks of the gearing coefficient and the appearance of new critical points, the asteroid's orbit becomes external to the Earth, and at the moments of "being at high peaks" and "maximum value". On the contrary, the Earth's orbit is external to the asteroid. This corresponds to the transition between classification groups Cupid and Apollo.

19. The impact of initial data errors on the evolution of the orbital elements of asteroid 2102 Tantalus over a long time interval



20. The impact of initial data errors on the evolution of the orbital elements of asteroid 2102 Tantalus over a long time interval (continued)

During modeling for such long time intervals, the question arises about the stability and impact of errors in the initial data. The task is purely a model one and uses only Newtonian mechanics, but it is still interesting to find out how the influence of errors in the initial data will affect the results of the study. It turned out that the tube of trajectories when varying the Tantalus elements according to known errors, it has a small width for them. 50 experiments were conducted, and deviations were generated according to a normal distribution.

21. The impact of initial data errors on the evolution of the orbital elements of asteroid 2102 Tantalus over a long time interval (continued)

It has been numerically established that despite the change in the distances between the orbits at critical points, the general structure of evolution with a "swallow tail" remains: somewhere it is formed earlier, somewhere later. This fact indicates the structural stability of the observed effect.

22. Conclusion)

In the course of the work, studies were conducted on the evolution of the distances between the orbits of the asteroid 2102 Tantalus and the Earth. The equations of motion were integrated for different time intervals from 5 to 50,000 years, and graphs of the evolution of the Keplerian elements of the orbit and the distances between the orbits were studied in all variants. Based on the results of the study, the following conclusions can be drawn::

- changes in the orbital elements lead to bifurcations of the distance function, accompanied by the birth and disappearance of pairs of critical points;
- there is a consistent behavior between the evolution of a part of the elements the asteroid's orbit, the gearing coefficient, and the appearance or disappearance of critical points;

23. Conclusion (continued)

- it is shown that the minimum distance between orbits (MOID) is not associated with a fixed critical point of the distance function, but can switch between different local minima during the evolution of the orbital elements;
 - it is shown that when the elements are varied in accordance with the errors in their definition, the pattern of changing orbits and the appearance/disappearance of critical points persists.
- The results obtained indicate the need to take into account the full the structures of the critical points of the distance function in the analysis of the long-term dynamics of the approaches of celestial bodies.

24. References

- [1] K.V. Kholoshevnikov, V.B. Titov "The two-body problem" Saint Petersburg State University, 2007.
- [2] Kholoshevnikov K.V., Vasiliev N.N. "On the distance function between two Keplerian elliptical orbits". Celestial mechanics and dynamic astronomy. 75, 75-83 (1999).
- [3] Shintaro Fushida-Hardy. The Morse theory. 381B Sloan Hall, Stanford University, California.

Thanks you very match!