

Regularized figure-eight orbit

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Аннотация. The figure-eight orbit in the regularized shape space of the three-body problem is considered. The Lemaitre regularization is used: the shape sphere (or the corresponding extended complex plane) is mapped onto the regularized shape sphere. This mapping is four-sheeted. The mapping is constructed and its properties are studied.

1. Introduction

The shape space is the quotient space of $\mathbb{R}^n$ by translations and rotations. The figure-eight orbit is well known, it is obtained numerically with sufficient accuracy; it is stable and far from collisions. Nevertheless, its study in the regularized space allows one to look at this orbit from a different perspective. Since the masses of the bodies are comparable (equal), it is natural to use a global regularization, namely the Lemaitre regularization:

$$\zeta = \frac{z^3 + \sqrt{8}}{1 - \sqrt{8}z^3},$$

To represent the orbit, we will use its Fourier series expansion (with period 2π) as a solution in the plane $\mathbb{R}^2$:

$$\begin{aligned}x_1(t) &= 1.0958785 \sin t & -0.0252775 \sin 5t - 0.0058497 \sin 7t + \dots \\y_1(t) &= 0.3372826 \sin 2t & +0.0557118 \sin 4t - 0.0029908 \sin 8t + \dots \\x_2(t) &= x_1(t + 2\pi/3) \\y_2(t) &= y_1(t + 2\pi/3) \\x_3(t) &= x_1(t + 4\pi/3) \\y_3(t) &= y_1(t + 4\pi/3)\end{aligned}$$

Using the scaling symmetry, this solution can be easily transformed into a figure-eight solution with any period or any size.

2. The Figure-Eight Orbit in Shape Space

Factoring out translations and rotations, we pass to the shape space Ξ . The corresponding expansions are easily obtained:

$$\begin{aligned}
 & \text{Сдвиг на } 2\pi/3 \\
 \xi_1 = & 1.7446017 \cos(2t) + 0.2542500 \cos(4t) - 0.0137548 \cos(8t) \\
 & -0.0034641 \cos(10t) + 0.0002985 \cos(14t) + 0.0000878 \cos(16t) \\
 & -0.0000089 \cos(20t) - 0.0000029 \cos(22t) \\
 \xi_2 = & 1.7446017 \sin(2t) - 0.2542500 \sin(4t) - 0.0137548 \sin(8t) \\
 & +0.0034641 \sin(10t) + 0.0002985 \sin(14t) - 0.0000878 \sin(16t) \\
 & -0.0000089 \sin(20t) + 0.0000029 \sin(22t) \\
 \xi_3 = & -1.2672068 \sin(3t) + 0.0137536 \sin(9t) - 0.0003240 \sin(15t) \\
 & +0.0000102 \sin(21t)
 \end{aligned}$$

The expression for the moment of inertia (or the radius-vector (ξ_1, ξ_2, ξ_3)):

$$\begin{aligned}
 \rho = & 1.9777440 + 0.0043046 \cos(6t) - 0.0000416 \cos(12t) \\
 & + 0.0000007 \cos(18t) - 0.0000005 \cos(24t).
 \end{aligned}$$

The space Ξ is the cone $S^2 \times \mathbb{R}^+$. The singularities of double collisions are the points $(-1, 0)$, $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$. The preimage of the first point is $(\sqrt{2} - \sqrt{6})/2$ and $(\sqrt{2} + \sqrt{6})/2$; the preimages of the remaining points are obtained by rotations of 120° and 240° .

All non-singular points have four preimages. For any non-singular point ζ , they are easily obtained as the roots of the 4th-degree polynomial:

$$z^4 + \sqrt{8}\zeta z^3 + \sqrt{8}z - \zeta = 0$$

3. The Figure-Eight Orbit on the Regularized Shape Sphere

On the shape sphere S^2 or $\mathbb{Z}_\infty$, the figure-eight trajectory is a closed continuously differentiable curve $\zeta(t)$. We need to determine the corresponding continuous curves $z_i(t)$ on the regularized shape sphere ($i = 1, \dots, 4$).

Each curve $z_i(t)$ can be identified by its initial point $z_i(0)$. For all subsequent times t , from the four roots for a given curve z_k , one must choose the nearest one. Theoretically, there could be bifurcation points where the polynomial has multiple roots, but this is not the case for the figure-eight.

The figure shows the four preimages of the figure-eight orbit. The red, blue, and brown trajectories are rotated relative to each other by 120° ($e^{2\pi i/3}$). The cyan trajectory (drawn with a thick line) is close to the unit circle. All preimages pass through 6 Euler points, with a total of 12 such points, 3 on each sheet. Only on the cyan trajectory are the points evenly spaced, but if projected onto the sphere, they are all close to great circles; the cyan trajectory is close to the equator, while the planes of the others are inclined to the equator by approximately 71° .

It is not difficult to obtain expansions, for example, for the cyan curve.

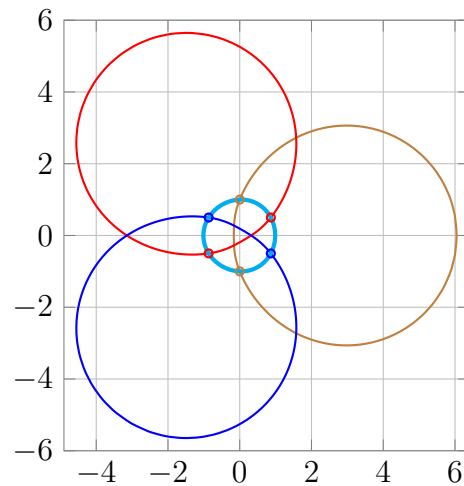


Рис. 1. Figure-eight on the regularized extended complex plane

4. Conclusion

The regularized figure-eight orbit appears simpler than the orbit in shape space. By choosing the preimage close to the unit circle (or to the equator on the shape sphere), we obtain a simple expression that can be used for further analysis.

Список литературы

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