



THE REGULARIZED FIGURE-EIGHT

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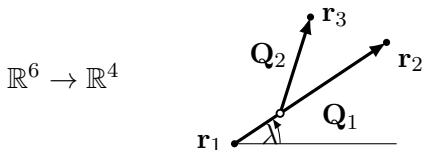
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Why it is interesting?

- «Figure-eight» exists!
- Periodic trajectory
- Angular momentum is zero!
- 6 Euler configurations per period!
- Symmetric (D_{12})!
- All three masses must be equal?!!
- Only one trajectory (up to a shift by $2\pi/3$)!
- Stable!!!
- Regularization.

Shape space, I. Reduction by translation ($\mathbb{R}^9 \rightarrow \mathbb{R}^6$)



$$\begin{aligned}\mathbf{Q}_1 &= \mathbf{r}_2 - \mathbf{r}_1, \\ \mathbf{Q}_2 &= \mathbf{r}_3 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2},\end{aligned}$$

$$m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2 + m_3 \mathbf{x}_3 = 0.$$

Reduction by rotation (Hopf transformation):

$$\begin{aligned}\mathbb{R}^4 &\rightarrow \mathbb{R}^3 : \mathcal{S}^1 \hookrightarrow \mathcal{S}^3 \rightarrow \mathcal{S}^2 \\ &(\mathcal{S}^3 \hookrightarrow \mathcal{S}^7 \rightarrow \mathcal{S}^4).\end{aligned}$$

$$\xi_1 = \mu_1 |\mathbf{Q}_1|^2 - \mu_2 |\mathbf{Q}_2|^2,$$

$$\xi_2 = 2\sqrt{\mu_1 \mu_2} \mathbf{Q}_1 \cdot \mathbf{Q}_2,$$

$$\xi_3 = 2\sqrt{\mu_1 \mu_2} \mathbf{Q}_1 \times \mathbf{Q}_2,$$

Moment of inertia:

$$\begin{aligned}I &= m_1 |\mathbf{r}_1|^2 + m_2 |\mathbf{r}_2|^2 + m_3 |\mathbf{r}_3|^2 = m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2 = \\ &= \mu_1 |\mathbf{Q}_1|^2 + \mu_2 |\mathbf{Q}_2|^2 = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.\end{aligned}$$

Expressions for mutual distances

$$r_{12}^2 = \frac{m_1 + m_2}{2m_1m_2} (\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} + \xi_1)$$

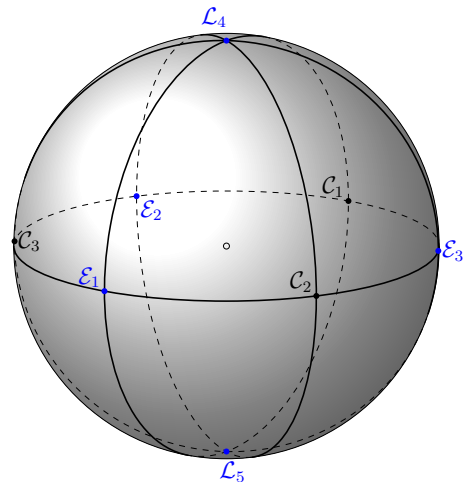
$$r_{13}^2 = \frac{m_1 + m_3}{2m_1m_3} \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \\ + \frac{m_2m_3 - m_1(m_1 + m_2 + m_3)}{2m_1m_3(m_1 + m_2)} \xi_1 + \frac{\sqrt{m_1m_2m_3(m_1 + m_2 + m_3)}}{m_1m_3(m_1 + m_2)} \xi_2$$

$$r_{23}^2 = \frac{m_2 + m_3}{2m_2m_3} \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \\ + \frac{m_1m_3 - m_2(m_1 + m_2 + m_3)}{2m_2m_3(m_1 + m_2)} \xi_1 - \frac{\sqrt{m_1m_2m_3(m_1 + m_2 + m_3)}}{m_2m_3(m_1 + m_2)} \xi_2$$

Shape space, II

Shape space, $\Xi = (\xi_1, \xi_2, \xi_3)$, is the space of congruent triangles.

Shape sphere, $S^2 \subset \Xi$, is the space of similar triangles.



If $\mathbf{r}_i(t)$, $i = 1, \dots, N$, is the solution of N body problem, then next expression yields the solution as well

$$\boldsymbol{\rho}_i(t) = \lambda \mathbf{r}_i(\lambda^{-3/2}t)$$

In addition

$$\begin{aligned}\dot{\boldsymbol{\rho}}_i &= \lambda^{-1/2} \mathbf{v}_i(\lambda^{-3/2}t) \\ h' &= h/\lambda\end{aligned}$$

Shape space, III: spherical coordinates

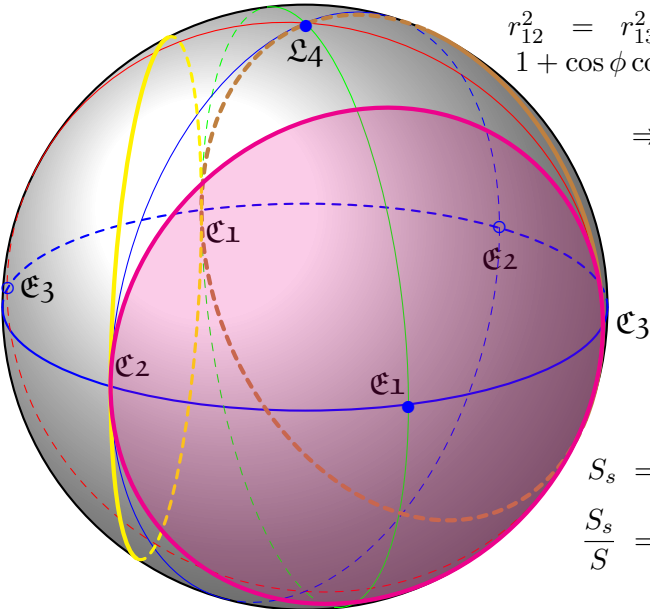
$$\begin{aligned}\xi_1 &= \rho \cos \varphi \cos \theta, & r_{12}^2 &= \frac{m_1 + m_2}{2m_1 m_2} \rho(1 + \cos \varphi \cos \theta), \\ \xi_2 &= \rho \sin \varphi \cos \theta, & r_{13}^2 &= \frac{m_1 + m_3}{2m_1 m_3} \rho(1 - \cos(\varphi - \varphi_{13}) \cos \theta), \\ \xi_3 &= \rho \sin \theta, & r_{23}^2 &= \frac{m_2 + m_3}{2m_2 m_3} \rho(1 - \cos(\varphi - \varphi_{23}) \cos \theta).\end{aligned}$$

$$V(\rho, \theta, \varphi) = \frac{1}{\sqrt{\rho}} \left(\frac{\nu_{12}}{\sqrt{1 + \cos \theta \cos \varphi}} + \frac{\nu_{13}}{\sqrt{1 - \cos \theta \cos(\varphi - \varphi_{13})}} + \frac{\nu_{23}}{\sqrt{1 - \cos \theta \cos(\varphi - \varphi_{23})}} \right) = \frac{1}{\sqrt{\rho}} D(\theta, \varphi),$$

Invariant configuration ($\theta = \text{const}, \varphi = \text{const}$):

$$\begin{aligned}\frac{\partial D(\varphi, \theta)}{\partial \theta} &= 0, \\ \frac{\partial D(\varphi, \theta)}{\partial \varphi} &= 0\end{aligned}$$

Lewis Carroll problem



$$\begin{aligned}
 r_{12}^2 &= r_{13}^2 + r_{23}^2 \Rightarrow \\
 1 + \cos \phi \cos \theta &= 1 - \cos\left(\phi - \frac{\pi}{3}\right) \cos \theta \\
 &\quad + 1 - \cos\left(\phi + \frac{\pi}{3}\right) \cos \theta \\
 \Rightarrow \cos \phi \cos \theta &= \frac{1}{2}
 \end{aligned}$$

$$S_s = 2\pi\left(1 - \cos \frac{\pi}{3}\right) = \pi$$

$$\frac{S_s}{S} = \frac{3\pi}{4\pi} = \frac{3}{4}$$

Energy integral

$$\begin{aligned}T &= \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}}, \\J &= \mu_1 \mathbf{Q}_1 \times \dot{\mathbf{Q}}_1 + \mu_2 \mathbf{Q}_2 \times \dot{\mathbf{Q}}_2, \\&= \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} \frac{d\lambda}{dt} + \frac{\xi_2 \frac{d\xi_3}{dt} - \xi_3 \frac{d\xi_2}{dt}}{2(\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} + \xi_1)}\end{aligned}$$

Clear, the angular momentum J is conjugate to the angle λ .

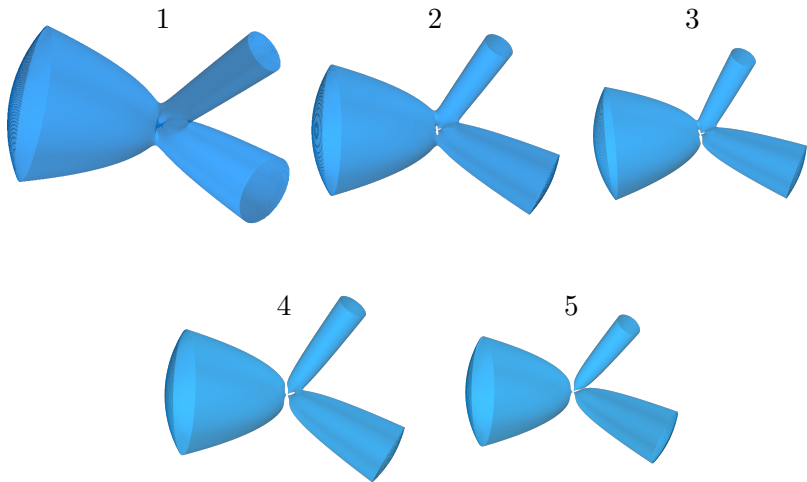
If we know ξ_1, ξ_2, ξ_3 , we can obtain this angle λ from the quadrature:

$$\lambda(t) = \int_0^t \frac{\partial R}{\partial J} d\tau = \int_0^t \frac{J - \frac{\xi_2 \dot{\xi}_3 - \xi_3 \dot{\xi}_2}{2\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} + \xi_1}}{\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} d\tau.$$

The system is conservative, therefore we have an energy integral:

$$T - U = \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h$$

Zero velocity surfaces (general three body problem)



Final motion, hierarchial systems

Periodic orbits

Finite symmetries. For the planar three-body problem, there are only 10 finite symmetry groups. Only two of them are considered here to illustrate trajectories in shape space: **dihedral group** (D_{12} , simple choreography), и **Line** symmetry.

Periodic orbits can be found as minimizers of the action functional

$$\mathcal{A} = \int_{t_1}^{t_2} L(\mathbf{q}_i, \dot{\mathbf{q}}_i, t).$$

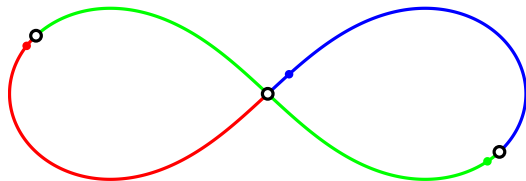
in the form

$$\begin{aligned}x_j(t) &= C_x^0 + \sum_{i=1} C_{x_i}^j \cos it + S_{x_i}^j \sin it \\y_j(t) &= C_y^0 + \sum_{i=1} C_{y_i}^j \cos it + S_{y_i}^j \sin it,\end{aligned}$$

Figure-eight

$$\begin{aligned}
 x(t) &= 1.0959 \sin t & - & 0.0253 \sin 5t & - & 0.0058 \sin 7t \\
 &+ 0.0004 \sin 11t & + & 0.0001 \sin 13t \\
 y(t) &= 0.3373 \sin 2t & + & 0.0557 \sin 4t & - & 0.0030 \sin 8t \\
 &- 0.0008 \sin 10t & + & 0.0001 \sin 14t
 \end{aligned}$$

Found numerically by Cr. Moore
in "Braids in classical dynamics",
Phys. Rev. Lett. 1993, **70**.



$$(b_1 b_2^{-1})^3$$

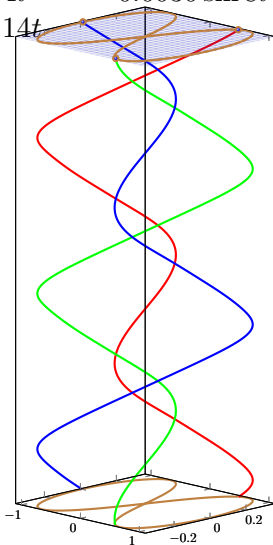
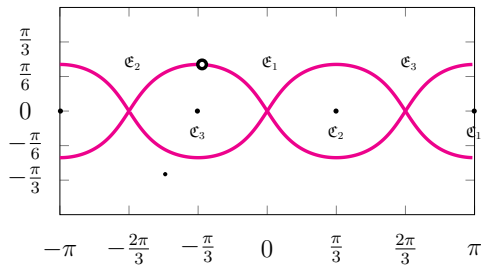
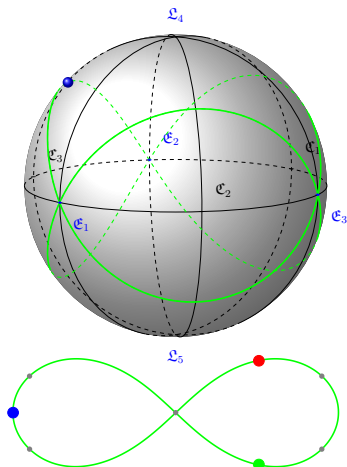


Figure-eight on the shape sphere

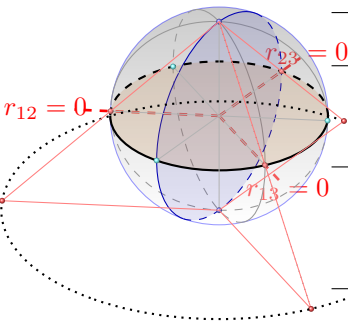
$$\begin{aligned}\xi_1 &= 1.7446017 \cos(2t) + 0.2542500 \cos(4t) - 0.0137548 \cos(8t) \\ &\quad - 0.0034641 \cos(10t) + 0.0002985 \cos(14t) + 0.0000878 \cos(16t) \\ &\quad - 0.0000089 \cos(20t) - 0.0000029 \cos(22t) \\ \xi_2 &= 1.7446017 \sin(2t) - 0.2542500 \sin(4t) - 0.0137548 \sin(8t) \\ &\quad + 0.0034641 \sin(10t) + 0.0002985 \sin(14t) - 0.0000878 \sin(16t) \\ &\quad - 0.0000089 \sin(20t) + 0.0000029 \sin(22t) \\ \xi_3 &= -1.2672068 \sin(3t) + 0.0137536 \sin(9t) - 0.0003240 \sin(15t) \\ &\quad + 0.0000102 \sin(21t) \\ \xi &= 1.9777440 + 0.0043046 \cos(6t) - 0.0000416 \cos(12t) + 0.0000007 \cos \\ &\quad - 0.0000005 \cos(24t)\end{aligned}$$

Figure-eight on the shape sphere



Global Lemaître regularization

$$\zeta = z \frac{\sqrt{8} + z^3}{1 - \sqrt{8}z^3},$$



Суд	расш. компл. плоскость z	ζ
C_{12}	$\frac{\sqrt{2}-\sqrt{6}}{2}$ $\frac{\sqrt{2}+\sqrt{6}}{2}$	-1
C_{13}	$-\frac{\sqrt{2}}{2} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) + \frac{\sqrt{6}}{2} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right)$ $\frac{\sqrt{2}}{2} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} \right) - \frac{\sqrt{6}}{2} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} \right)$	$\frac{1}{2} - \frac{\sqrt{3}}{2}$
C_{23}	$-\frac{\sqrt{2}}{2} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right) - \frac{\sqrt{6}}{2} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} \right)$ $\frac{\sqrt{2}}{2} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} \right) + \frac{\sqrt{6}}{2} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} \right)$	$\frac{1}{2} + \frac{\sqrt{3}}{2}$

Preimages of init circle

$$|\zeta|^2 = \zeta\bar{\zeta} = 1.$$

Direct calculation leads to equations

$$\begin{aligned} & \left((x + \sqrt{2})^2 + y^2 - 3 \right) \\ & \left((x - \sqrt{2}/2)^2 + (y - \sqrt{6}/2)^2 - 3 \right) \\ & \left((x - \sqrt{2}/2)^2 + (y + \sqrt{6}/2)^2 - 3 \right) = 0 \end{aligned}$$

Наклон плоскостей
образов

The inclination
of the preimage planes
 $i = \arccos 1/\sqrt{3} \approx 54.7356^\circ$.

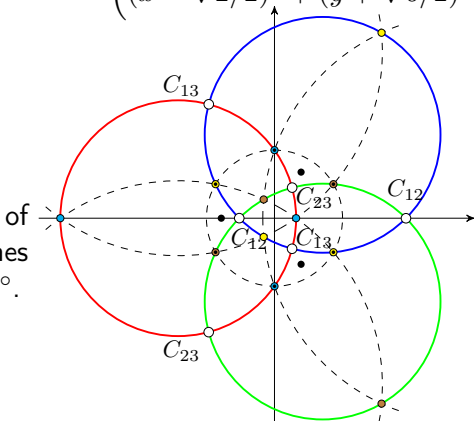


Figure-eight curve on the Riemann sphere

Loop $\zeta(t)$:

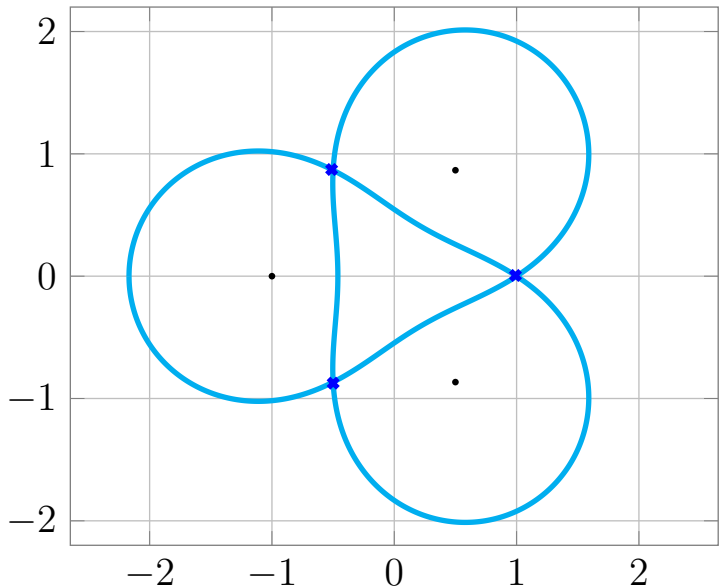


Figure-eight in the extended complex plane

$$\begin{aligned}
 \zeta_x &= -0.3600461 \cos(t) & -1.1308580 \cos(2t) & +0.0101961 \cos(4t) \\
 & -0.4151381 \cos(5t) & -0.0010348 \cos(7t) & -0.1633463 \cos(8t) \\
 & +0.0001264 \cos(10t) & -0.0650747 \cos(11t) & -0.0000210 \cos(13t) \\
 & -0.0260053 \cos(14t) & +0.0000033 \cos(16t) & -0.0104057 \cos(17t) \\
 & -0.0000007 \cos(19t) & -0.0041656 \cos(20t) & +0.0000001 \cos(22t) \\
 & -0.0016679 \cos(23t) & +0.0000001 \cos(25t) & -0.0006678 \cos(26t) \\
 & -0.0002673 \cos(29t) & -0.0001070 \cos(32t) & -0.0000428 \cos(35t) \\
 & -0.0000171 \cos(38t) & -0.0000069 \cos(41t) & -0.0000027 \cos(44t) \\
 & -0.0000011 \cos(47t) & -0.0000004 \cos(50t) & -0.0000002 \cos(53t) \\
 \zeta_y &= -0.3600461 \sin(t) & +1.1308580 \sin(2t) & +0.0101961 \sin(4t) \\
 & +0.4151381 \sin(5t) & -0.0010348 \sin(7t) & +0.1633463 \sin(8t) \\
 & +0.0001264 \sin(10t) & +0.0650747 \sin(11t) & -0.0000210 \sin(13t) \\
 & +0.0260053 \sin(14t) & +0.0000033 \sin(16t) & +0.0104057 \sin(17t) \\
 & -0.0000007 \sin(19t) & +0.0041656 \sin(20t) & +0.0000001 \sin(22t) \\
 & +0.0016679 \sin(23t) & +0.0000001 \sin(25t) & +0.0006678 \sin(26t) \\
 & +0.0002673 \sin(29t) & +0.0001070 \sin(32t) & +0.0000428 \sin(35t) \\
 & +0.0000171 \sin(38t) & +0.0000069 \sin(41t) & +0.0000027 \sin(44t) \\
 & +0.0000011 \sin(47t) & +0.0000004 \sin(50t) & +0.0000002 \sin(53t)
 \end{aligned}$$

Figure-eight in the regularized extended complex plane

We need to solve the equation

$$\zeta = z \frac{\sqrt{8} + z^3}{1 - \sqrt{8}z^3},$$

with respect to z .

The roots $z_k(t)$ must be a continuous function of t (in fact, $z_k(t) \in \mathbb{C}^2$), possibly except at points where the root becomes multiple (bifurcation points).

For any $t \in [0, 2\pi)$, the point $\zeta(t)$ in the extended plane $\mathbb{C}$ yields 4 roots of this equation.

This is a 4-sheeted mapping. The curve $\zeta(t) \in \mathbb{C}^2$ is mapped to $z(t)$.

The simplest method. Continuation

- 1 For $t = 0$, we find all 4 roots of $z_k(0)$.
As a rule, these four different values identify the leaf.
- 2 For each subsequence t_{n+1} :
The polynomial gives 4 new roots.
- 3 We compare them with the roots of the previous step, minimizing the distance:

$$\min_{\sigma \in S_4} \sum_{k=1}^4 |z_{\sigma(k)}(t_{n+1}) - z_k(t_n)|$$

Figure-eight in the regularized extended complex plane

Если $\zeta(t)$ это аналитическая функция, мы можем вывести дифференциальное уравнение для корней. Продифференцируем уравнение по t :

If $\zeta(t)$ is a known function, we can derive a differential equation for the roots. Differentiate the equation with respect to t :

$$4z^3\dot{z} + \sqrt{8}\dot{\zeta}z^3 + 3\sqrt{8}\zeta z^2\dot{z} + \sqrt{8}\dot{z} - \dot{\zeta} = 0$$

Therefore:

$$\frac{dz}{dt} = -\dot{\zeta}(t) \cdot \frac{\sqrt{8}z(t)^3 - 1}{4z(t)^3 + 3\sqrt{8}\zeta(t)z(t)^2 + \sqrt{8}},$$

unless the denominator is equal to zero (no multiple roots).

This is ODE for each root $z_k(t)$. By specifying the initial conditions $z_k(0)$, one can integrate along the trajectory $\zeta(t)$.

Figure-eight in the regularized extended complex plane

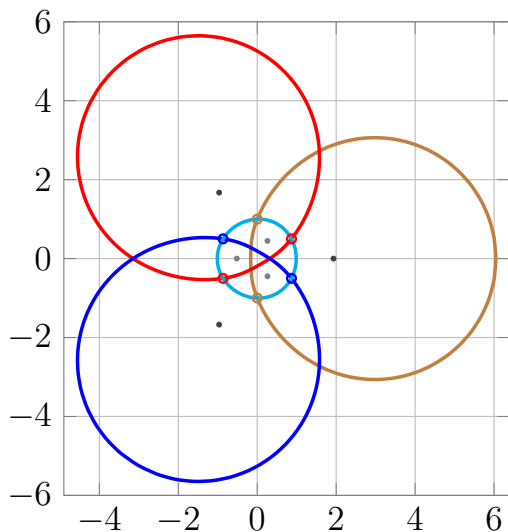
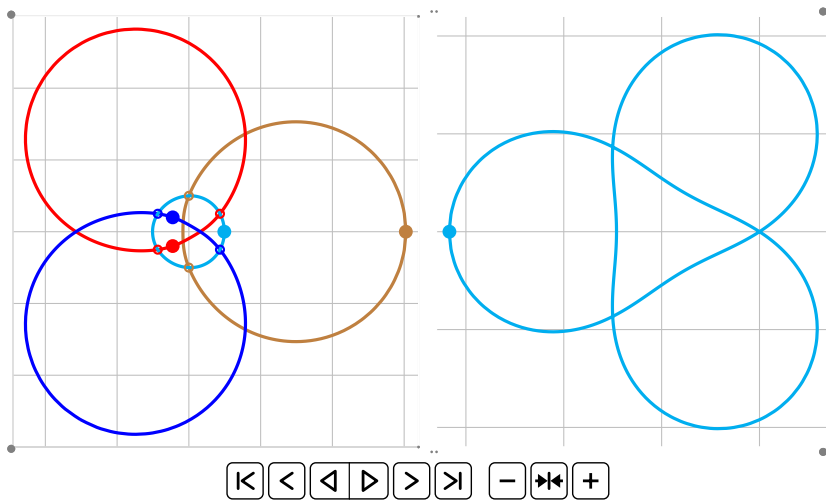
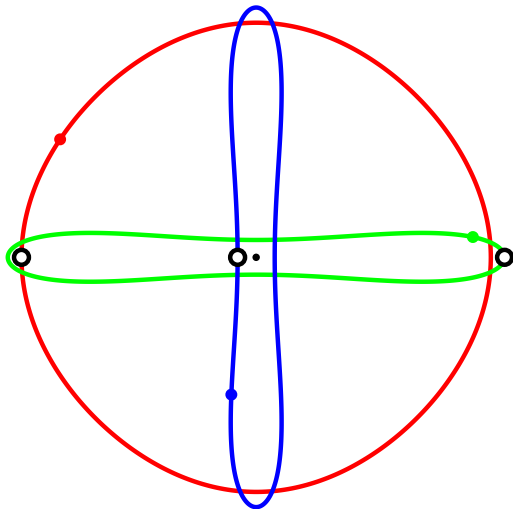


Figure-eight in the regularized extended complex plane



Another orbits

$$m_1 = 0.99, m_2 = 1.01, m_3 = 1.0; \quad \omega = \frac{1}{2}, \quad J = 0.92132$$



$$m_1 = 0.99, \quad m_2 = 1.01, \quad m_3 = 1.0$$

$$x_1 = -0.000186$$

$$+1.584708 \cos(t) + 0.000001 \cos(2t) + 0.017772 \cos(3t) - 0.000038 \cos(4t) \\ + 0.001112 \cos(5t) - 0.000006 \cos(6t) + 0.000100 \cos(7t) - 0.000001 \cos(8t) \\ + 0.000010 \cos(9t) + 0.000001 \cos(11t),$$

$$y_1 = -1.570546 \sin(t) - 0.000020 \sin(2t) - 0.017455 \sin(3t) + 0.000037 \sin(4t) \\ - 0.001095 \sin(5t) + 0.000006 \sin(6t) - 0.000098 \sin(7t) + 0.000001 \sin(8t) \\ - 0.000010 \sin(9t) - 0.000001 \sin(11t);$$

$$x_2 = 0.954909$$

$$-0.780250 \cos(t) - 0.044762 \cos(2t) - 0.008675 \cos(3t) - 0.001877 \cos(4t) \\ - 0.000538 \cos(5t) - 0.000149 \cos(6t) - 0.000048 \cos(7t) - 0.000015 \cos(8t) \\ - 0.000005 \cos(9t) - 0.000002 \cos(10t) + 0.000001 \cos(11t);$$

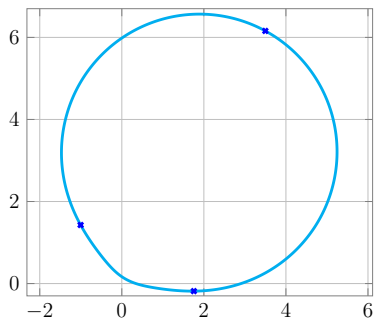
$$y_2 = 0.773255 \sin(t) + 0.050203 \sin(2t) + 0.008514 \sin(3t) + 0.002046 \sin(4t) \\ + 0.000529 \sin(5t) + 0.000158 \sin(6t) + 0.000047 \sin(7t) + 0.000015 \sin(8t) \\ + 0.000005 \sin(9t) + 0.000002 \sin(10t) + 0.000001 \sin(11t);$$

$$x_3 = -\frac{m_1 x_1 + m_2 x_2}{m_3}$$

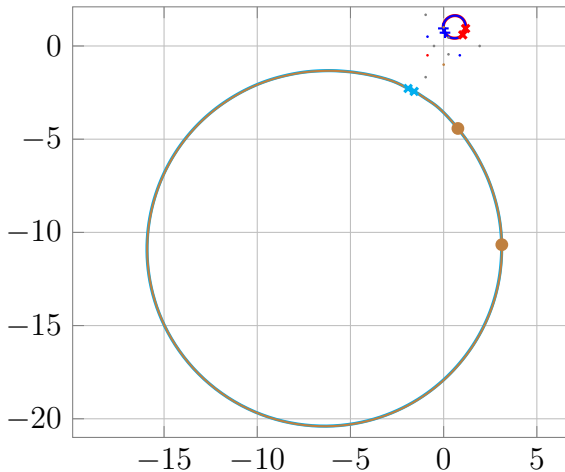
$$y_3 = -\frac{m_1 y_1 + m_2 y_2}{m_3}$$

$m_1 = 0.99$, $m_2 = 1.01$, $m_3 = 1.0$
 extended complex plane

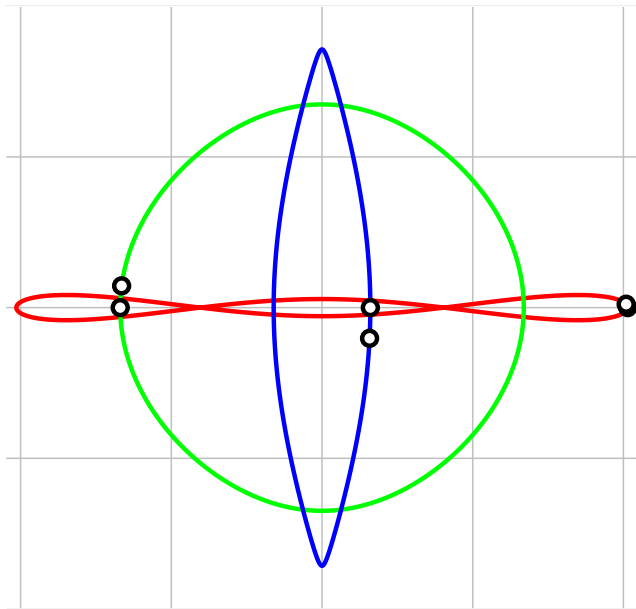
Extended complex plane



Regularized complex plane



$$m_1 = 0.7, \quad m_2 = 1.3, \quad m_3 = 1.0$$



$$m_1 = 0.7, \quad m_2 = 1.3, \quad m_3 = 1.0$$

$$\begin{aligned} x_1 = & 1.054023 + 1.027929 \cos(t) - 0.064567 \cos(2t) + 0.013572 \cos(3t) \\ & - 0.003322 \cos(4t) + 0.001028 \cos(5t) - 0.000324 \cos(6t) + 0.000112 \cos(7t) \\ & - 0.000039 \cos(8t) + 0.000014 \cos(9t) - 0.000005 \cos(10t) + 0.000002 \cos(11t); \\ y_1 = & -1.020632 \sin(t) + 0.072456 \sin(2t) - 0.013662 \sin(3t) + 0.003607 \sin(4t) \\ & - 0.001046 \sin(5t) + 0.000342 \sin(6t) - 0.000115 \sin(7t) + 0.000041 \sin(8t) \\ & - 0.000015 \sin(9t) + 0.000006 \sin(10t) - 0.000002 \sin(11t); \end{aligned}$$

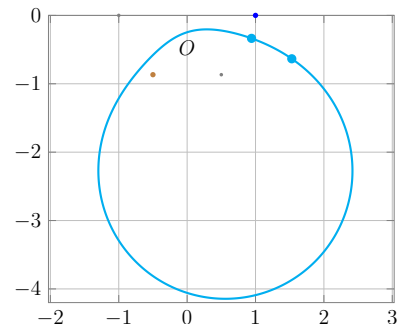
$$\begin{aligned} x_2 = & 0.003912 - 1.330058 \cos(t) - 0.000246 \cos(2t) - 0.012806 \cos(3t) \\ & + 0.000927 \cos(4t) - 0.000779 \cos(5t) + 0.000133 \cos(6t) - 0.000074 \cos(7t) \\ & + 0.000019 \cos(8t) - 0.000009 \cos(9t) + 0.000003 \cos(10t) - 0.000001 \cos(11t); \\ y_2 = & 1.320138 \sin(t) + 0.000681 \sin(2t) + 0.012582 \sin(3t) - 0.000896 \sin(4t) \\ & + 0.000766 \sin(5t) - 0.000130 \sin(6t) + 0.000073 \sin(7t) - 0.000018 \sin(8t) \\ & + 0.000009 \sin(9t) - 0.000003 \sin(10t) + 0.000001 \sin(11t); \end{aligned}$$

$$x_3 = -\frac{m_1 x_1 + m_2 x_2}{m_3};$$

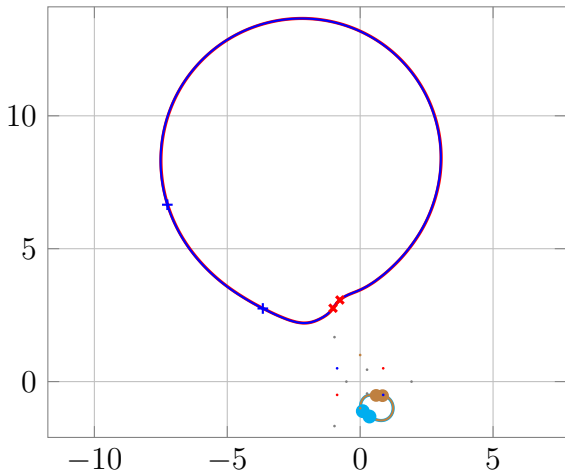
$$y_3 = -\frac{m_1 y_1 + m_2 y_2}{m_3};$$

$m_1 = 0.7, m_2 = 1.3, m_3 = 1.0$
 extended complex plane

Extended complex plane



Regularized complex plane



ToDo

- Increase the accuracy
- Consider more orbits (both with zero angular momentum and with non-zero)
- Consider projections of these orbits onto the shape sphere (both ordinary and Lemaitre-regularised).
- Consider “Figure-eights” in the case of unequal masses
- Vanishing angular momentum in shape space conjecture.

THANKS!!!

