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Nonlinear Formation Flying Control of Small Spacecraft in the Vicinity of Asteroid 4179 Toutatis

Ekaterina Eroshenko and Mahdi Reza Akhloumadi

Abstract. This study addresses the problem of long-term bounded relative motion maintenance for a spacecraft swarm operating in the vicinity of an asteroid. Two nonlinear control strategies are comparatively investigated: a State-Dependent Riccati Equation (SDRE)-based controller and a Lyapunov-based control approach. The performance of both methods is evaluated in terms of stability, robustness, and control efficiency for spacecraft formation-keeping in the perturbed and uncertain dynamical environment surrounding a small celestial body.

Introduction

Asteroid exploration is challenging due to the small size, irregular geometry, and uncertain gravitational fields of these bodies. Distributed spacecraft formations enable improved spatial coverage, simultaneous observations, and accurate characterization of the asteroid environment, supporting gravity-field reconstruction, autonomous navigation, and future resource utilization[1, 2] .

This study investigates a spacecraft formation near asteroid (4179) Toutatis. Its slow rotation enables coordinated observations and gradual refinement of the gravitational model. A chief spacecraft in a quasi-frozen orbit provides a stable reference trajectory for long-term operations.

To counteract secular drift caused by gravitational perturbations and model uncertainties, two nonlinear control strategies are compared for maintaining bounded relative motion: the State-Dependent Riccati Equation (SDRE) method and a Lyapunov-based control approach [4].

Equations of motion

The relative motion between a chief and a deputy spacecraft is considered. The chief spacecraft defines the reference trajectory by following a prescribed nominal

orbit, which may be either passive or actively controlled. The deputy spacecraft is required to maintain a bounded relative trajectory with respect to the chief.

Due to the highly perturbed and uncertain gravitational environment of the asteroid, secular drift occurs in the relative motion, requiring active formation-keeping control. Both control strategies are designed using an approximate relative dynamics model formulated in terms of relative orbital elements [5]. The averaged equations describing the motion near the chief spacecraft's quasi-frozen orbit are derived from the Lagrange planetary equations and given as follows:

$$\begin{aligned}
\dot{\bar{a}} &= 0, \\
\dot{\bar{e}} &= 0, \\
\dot{\bar{i}} &= \frac{3\bar{n}}{(\bar{p}/R_0)^2} C_{22} \sin \bar{i} \sin 2\bar{\Omega}_R, \\
\dot{\bar{\Omega}}_R &= \frac{3\bar{n} \cos \bar{i}}{2(\bar{p}/R_0)^2} (C_{20} + 2C_{22} \cos 2\bar{\Omega}_R) - n_T, \\
\dot{\bar{\omega}} &= -\frac{3\bar{n}}{8(\bar{p}/R_0)^2} [C_{20} (3 + 5 \cos 2\bar{i}) + 2C_{22} \cos 2\bar{\Omega}_R (-1 + 5 \cos 2\bar{i})], \\
\dot{\bar{M}} &= \bar{n} - \frac{3\bar{n} \sqrt{1 - \bar{e}^2}}{8(\bar{p}/R_0)^2} [C_{20} (1 + 3 \cos 2\bar{i}) + 6C_{22} \cos 2\bar{\Omega}_R (\cos 2\bar{i} - 1)].
\end{aligned} \tag{1}$$

The notation of these elements (where the overbar denotes the averaged value of the corresponding quantity) is as follows: $\bar{a}$ is semi-major axis, $\bar{e}$ stands for eccentricity, $\bar{i}$ is orbital inclination, $\bar{\Omega}$ the longitude of the ascending node, $\bar{\omega}$ denotes argument of periaapsis, $\bar{M}$ is mean anomaly, C_{20} is zonal harmonic coefficient, and C_{22} is sectoral harmonic coefficient, and defining orbital element vector $\boldsymbol{\alpha}$ is:

$$\boldsymbol{\alpha} = [\bar{a}, \bar{e}, \bar{i}, \bar{\Omega}, \bar{\omega}, \bar{M}]^T.$$

Control of motion

the dynamical system with control is given:

$$\dot{\boldsymbol{\alpha}}_D = \mathbf{f}(\boldsymbol{\alpha}_D) + \mathbf{B}(\boldsymbol{\alpha}_D) \mathbf{u}_D + \boldsymbol{\delta} \tag{2}$$

where the system $\mathbf{f}(\boldsymbol{\alpha}_D)$ is defined by the following equation, $\boldsymbol{\delta}$ represents the dominant disturbances, and $\mathbf{B}(\boldsymbol{\alpha}_D)$ is the control input matrix:

$$\mathbf{B}(\alpha_D) = \begin{bmatrix} \frac{2e \sin \nu}{n\sqrt{1-e^2}} & \frac{2(1+e \cos \nu)}{n\sqrt{1-e^2}} & 0 \\ \frac{\sqrt{1-e^2} \sin \nu}{na} & \frac{\sqrt{1-e^2}(e+2 \cos \nu + e \cos^2 \nu)}{na(1+e \cos \nu)} & 0 \\ 0 & 0 & \frac{\sqrt{1-e^2} \cos(\omega + \nu)}{na(1+e \cos \nu)} \\ 0 & 0 & \frac{\sqrt{1-e^2} \sin(\omega + \nu)}{na \sin i(1+e \cos \nu)} \end{bmatrix} \quad (3)$$

where ν is the true anomaly. The SDRE approach [3] synthesizes a feedback controller by transforming the nonlinear system into a state-dependent linear form and solving a Riccati equation online at each control step. This provides a near-optimal control solution with respect to the selected performance criterion but requires significant computational resources.

In contrast, the Lyapunov-based approach derives the control law analytically by selecting a positive-definite Lyapunov function and ensuring a negative-definite time derivative, which guarantees closed-loop stability [4]. This method offers a computationally efficient and robust solution, although it is generally not optimal with respect to a predefined performance index.

The proposed nonlinear control architecture combines a nominal stabilization controller with an adaptive compensation mechanism. The nominal controller, based on either the SDRE or Lyapunov approach, ensures bounded relative motion of the deputy spacecraft, while the adaptive component compensates for parametric uncertainties and unmodeled disturbances, enhancing robustness against dynamical model imperfections.

Conclusion

This study addresses the problem of long-term bounded relative motion maintenance for a spacecraft swarm operating in the vicinity of an asteroid. Numerical simulations compare the performance of the two control strategies in terms of stability, robustness, and control effort. The results provide practical insights into the suitability of each approach for maintaining distributed spacecraft formations in the highly uncertain dynamical environment surrounding small celestial bodies. Furthermore, the proposed methodology can be readily extended to centralized control architectures for large swarms of small spacecraft operating around asteroids.

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On the resonant multi-planet system TOI-1408

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Abstract. We perform a self-consistent analysis of Doppler and photometric transit observations of TOI-1408, known to host a two-planet system in 2:1 mean-motion resonance. In general we confirm its orbital configuration, however we found that the likelihood function contains multiple local maxima spread over a wider parametric domain, thus making some uncertainties significantly larger than deemed before. In particular, the planet b radius still appears poorly constrained (from $\sim 1.2 R_{\text{Jup}}$ to $\sim 3.5 R_{\text{Jup}}$) because of the near-tangential transit geometry. Simultaneously, the task allows for a purely dynamical estimation of the star mass which becomes $1.321 \left(\begin{smallmatrix} +0.049 \\ -0.042 \end{smallmatrix} \right) M_{\odot}$.

Introduction

TOI-1408 b is a TESS planet candidate confirmed in [5] based on Doppler measurements acquired in 2021-2023 by the FFOREST spectrograph mounted at the SAO RAS BTA-6 telescope [10, 11]. The planet mass was estimated as $1.69 \pm 0.20 M_{\text{Jup}}$, but the planet radius was ill-fitted because of the tangential transit, with a low limit of about $1 R_{\text{Jup}}$. The Doppler curve implied a significant non-sinusoidal shape consistent with the orbital eccentricity of $e_b \simeq 0.25$, possibly indicating the presence of other unseen companions in the system. Later on, in [7] a massive set of new Doppler observations for TOI-1408 obtained at multiple sites was presented and analysed together with TESS and ground-based photometry. They revealed the second small planet c with $r_c = 2.22 \pm 0.06 R_{\oplus}$, moving on an internal orbit in the 2:1 mean motion resonance with planet b . The second planet is transiting as well, although its transits cannot be detected individually. The planets are gravitationally perturbing each other, causing observationally detectable transit timing variations (TTVs). Contrary to [5], in [7] a robust estimate for $r_b = 2.2 \pm 0.4 R_{\text{Jup}}$ was obtained. They also noticed a nonlinear Doppler trend indicating a yet another, long-period, body. The goal of the present work is to perform a more detailed analysis treating several unresolved issues, and also taking into account new TESS observations of TOI-1408 that appeared since [7].

1. Data

We used the same 12 TESS Sectors as in [7], but also added newer observations for six more Sectors 77, 78, 79, 84, 85, and 86, downloadable from the Mikulski Archive for Space Telescopes. Each sector is about a month long. In [7] a set of 18 ground-based lightcurves of planet b transits was also released. We removed four of these lightcurves from our final analysis because they revealed small but statistically significant long-term or quasiperiodic variations.

First high-precision Doppler data for TOI-1408 were published in [5] and contained 19 RV measurements. Further on, [7] released a large set of additional Doppler measurements obtained at multiple other sites. In total we had 12 independent Doppler datasets corresponding to either different telescopes or, in some cases, to different detectors on the same telescope, or to different spectrum calibration methods.

2. Models

TESS data were treated using individual per-transit model that involved a cubic polynomial and a classic transit model [8, 1] with quadratic limb-darkening. The limb-darkening law was tuned to better fit the near-limb ring of the stellar disc, because of the grazing geometry of planet b transits. We omit these details here.

In [7] it was noticed that Doppler data may be affected by stellar rotation which they modelled by “a sinusoidal stellar rotational signal”. However, such variations do not usually manifest themselves as a literal sinusoid. A standard way to treat such variations is the Gaussian Process (GP) method relying on some parametric model for the correlation function [2, 3, 9], which we adopt here. The Doppler noise was fitted with an individual model per each dataset.

Regarding the planetary system itself, we assumed a three-planet model, in which the two inner planets (b and c) were gravitationally interacting, while the third body (d) was gravitationally detached, moving along a distant unperturbed circular orbit. The system was assumed to be coplanar. We also considered newer estimations for the star mass and radius from [6].

3. Results

All the fitting was performed using the PLANETPACK software [4], which implements the maximum-likelihood approach involving GP noise models. In general we can confirm the orbital model given in [7], in particular, the planetary TTVs coming from the primary secular period of the mean motion resonance, see Fig 1.

However we found that the likelihood function has numerous local maxima, demonstrating a “rippled” shape. In Fig 2 we show 1D slices of the likelihood along the axis of the planet radius r_b (the most affected parameter). The log of the likelihood ratio statistic, $\Delta \log L$, is shown as a function of the given parameter, relative to the global optimum (the maxima are inverted into minima here).

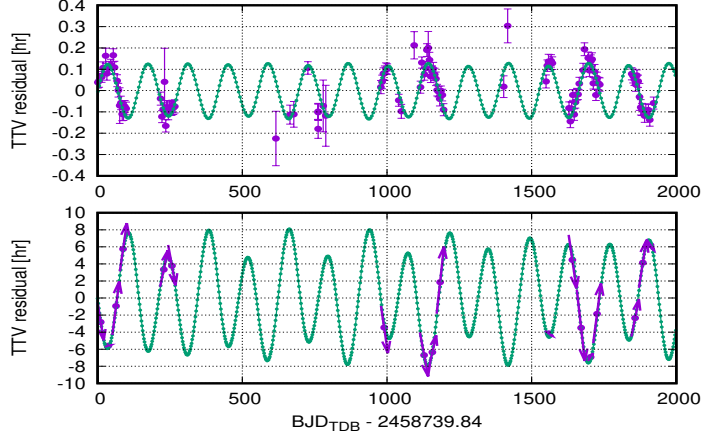


FIGURE 1. TTVs of TOI-1408 *b* (top) and *c* (bottom) against their dynamical models. In the second case we show per-sector TTV trends as vectors, because individual transits are not detectable; vector uncertainties are omitted for clarity.

As one can also see, the effect of the astrophysical constraints on the star mass and radius is important, for example it can shift different local maxima vertically, and may affect their total number. Unfortunately crucial information, like the correlation between $M_\star$ and $R_\star$, is not available to assess this effect reliably.

Conclusion

In this work we performed a detailed reassessment of the TOI-1408 planetary system, including additional new TESS data as well as involving a more detailed analysis of the limb darkening, photometric and Doppler noises, and the structure of the likelihood function. We generally confirm the three-body system interpretation from [7]. However we derived considerably larger parametric uncertainties, in particular the planet *b* radius still remains ill-determined.

We found that the astrophysical estimates for the star mass and radius has to be characterized very detailedly to be useful with TOI-1408. For example, the model depends significantly on the correlation between $M_\star$ and $R_\star$. Otherwise we can only place conservative ranges on fit parameters, for example $[1.2-3.5] R_{\text{Jup}}$ for r_b . Our purely dynamical estimation for $M_\star$ is $1.321 \left(\begin{smallmatrix} +0.049 \\ -0.042 \end{smallmatrix} \right) M_\odot$. The parameters of the long-period body TOI-1408 *d* are poorly constrained, but our conservative lower limit is $m_d \gtrsim 50 M_{\text{Jup}}$. Probably its planetary nature can be already ruled out under an assumption that there is just a single body contributing to the trend.

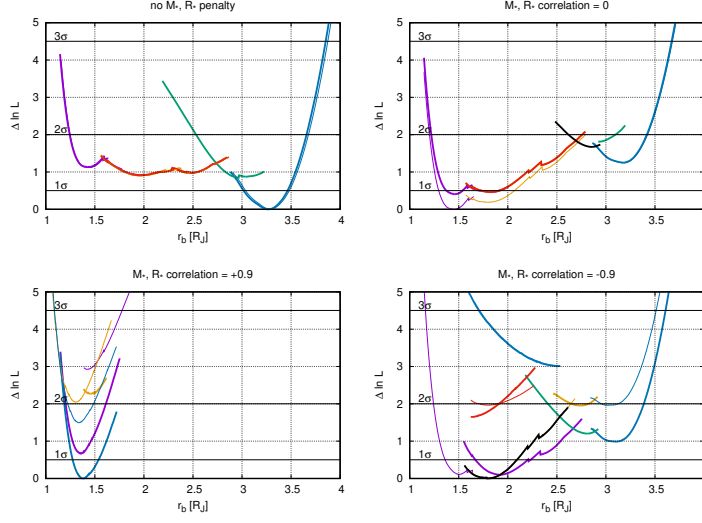


FIGURE 2. The log-likelihood for TOI-1408 b radius, assuming different correlation between the star mass and radius.

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Families of asymmetric periodic solutions to the Hill 3-body problem

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Abstract. We study families of asymmetric periodic solutions in the planar Hill 3-body problem (Hill3BP), a parameter-free limiting form of the circular R2BP with a Klein four-group of discrete symmetries. This structure allows periodic-orbit families to be organized by symmetry type and limiting generating solutions. We combine the symbolic theory of second-species generating solutions with a bifurcation-based continuation method applied to critical singly symmetric periodic orbits. We identify ten previously unknown families of asymmetric planar periodic solutions, belonging to two main classes: bridge families linking distinct symmetric families, and families that, in the second-species limit, continue to asymmetric generating sequences. We also examine the spatial stability of these planar families. Our results extend the known network of periodic-orbit families in the Hill3BP and provide a systematic path from highly symmetric backbone families to much less symmetric dynamics.

1. Motivation and description of the Hill3BP

Periodic solutions form a natural organizing framework for the phase space of a non-integrable Hamiltonian system. In the classical Poincaré viewpoint, they constitute a skeleton around which invariant manifolds of different dimensions are arranged. When the equations of motion possess additional discrete symmetries, the periodic solutions are divided into classes with distinct global properties. Families having the highest degree of symmetry frequently act as a backbone of a more complicated network: less symmetric families emerge from them through bifurcations, connect them to one another, or terminate at singular limiting solutions. The purpose of this work is to study how asymmetric families arise in this network for the planar Hill3BP and to develop a procedure by which such families can be found systematically.

The circular Hill3BP was originally introduced by G. W. Hill in lunar theory [1]. It is obtained as a limiting case of the circular restricted three-body problem by translating the origin to the smaller primary and blowing up the local coordinates. In this limit, the second primary is displaced infinitely far away, while its action remains as a velocity-independent tidal perturbation of the rotating Kepler problem. The model therefore gives an adequate local description of motion inside and near the Hill sphere of the smaller primary. Its principal advantages are the absence of a mass parameter, the appearance of an additional discrete symmetry, and the possibility of continuing its solutions to the circular restricted problem and, further, to the general three-body problem.

In canonical variables $\mathbf{z} = (x_1, x_2, y_1, y_2) \in \mathbb{R}^4$, the planar problem is governed by the Hamiltonian

$$H(\mathbf{z}) = \frac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - x_1^2 + \frac{1}{2}x_2^2 - \frac{1}{r}, \quad r = \sqrt{x_1^2 + x_2^2}. \quad (1)$$

The Jacobi integral can be written as

$$\mathcal{J} \equiv 3x_1^2 + \frac{2}{r} - \dot{x}_1^2 - \dot{x}_2^2 = \Gamma = -2H. \quad (2)$$

The equations are invariant under four transformations

$$\rho_{(\alpha, \beta)} : (t, x_1, x_2, y_1, y_2) \mapsto (\alpha t, \beta x_1, \alpha \beta x_2, \alpha \beta y_1, \beta y_2), \quad \alpha, \beta = \pm 1, \quad (3)$$

which form the Klein group $V_4 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Depending on its isotropy subgroup, a periodic orbit may be doubly symmetric, singly symmetric, or asymmetric. The symmetry type is a global invariant along a regular one-parameter family, although it may change when families meet at a bifurcating critical orbit.

The limiting variants of the Hill3BP can be used to obtain integrable approximations. One truncation is the Kepler Problem in the rotating frame Hamiltonian

$$\hat{H}_1 = \frac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - \frac{1}{r}, \quad (4)$$

whereas another is the integrable Hénon problem

$$\hat{H}_2 = \frac{1}{2}(y_1^2 + y_2^2) + x_2 y_1 - x_1 y_2 - x_1^2 + \frac{1}{2}x_2^2. \quad (5)$$

Power Geometry provides a systematic derivation of the limiting variant by associating exponent vectors with the monomials of the Hamiltonian and selecting truncations from the faces of its Newton polygon [3]. The resulting integrable limits provide the asymptotic building blocks used in the theory of generating solutions.

2. Theory of generating solutions [4]

As $\Gamma \rightarrow -\infty$, a family of periodic orbits that approaches a second-species solution can be encoded by a finite symbolic sequence called a generating solution (GS). The sequence is composed of two kinds of elementary arcs. Type-I arcs are denoted by $\{\pm j\}$, $j = 1, 2, 3, \dots$, and type-II arcs are denoted by i and e . Two identical arcs

$\{ii\}$ or $\{ee\}$ are not allowed consecutively. According to this restriction, each family that converges to a second-species limit is represented by a generating sequence.

A GS contains considerably more information than a label. It determines the limiting symmetry type of the family, its global multiplicity, the asymptotic form of the initial conditions, and the leading stability index. A sequence is defined only up to cyclic permutation; for example, $\{+1, +2\} \equiv \{+2, +1\}$. The generated solutions can be divided into asymmetric, singly symmetric, and doubly symmetric classes. The latter two are associated with the symmetry sets usually denoted by Σ_1 , Σ_2 , and their composition. Sequences containing only type-II arcs have a particularly restricted structure, while mixed sequences allow a much richer set of symmetry types.

A periodic orbit of an autonomous 2DOF Hamiltonian system belongs to a one-parameter family. Its period, integral value, geometry, and stability vary smoothly along a regular branch, whereas the symmetry type is preserved. Hence, symmetry-breaking critical solutions of symmetric families provide natural entry points to asymmetric branches.

3. Systematic search for asymmetric solutions and results

Two complementary search strategies were implemented. The first starts from asymmetric generating solutions and attempts to continue them from their second-species limits. The second starts from critical members of known singly symmetric families and detects branches created by symmetry-breaking bifurcations. The latter proved to be more robust and produced the main new results reported here.

Let $\mathcal{M}$ be the monodromy matrix of a periodic orbit. It is obtained from the fundamental matrix $Z(t)$ of the variational system

$$\dot{Z} = J \text{Hess } H(\mathbf{z}(t))Z, \quad Z(0) = E^4, \quad Z(t+T) = Z(t)\mathcal{M}. \quad (6)$$

Because $\mathcal{M}$ is symplectic, its multipliers occur in reciprocal and complex-conjugate pairs. In an autonomous 2DOF Hamiltonian system, $\mathcal{M}$ has the eigenvector $\mathbf{v}_1 = J\nabla H(\mathbf{z}_0)$ associated with the double multiplier 1. Its characteristic polynomial has the form

$$\chi(\lambda) = (\lambda - 1)^2(\lambda^2 - 2S\lambda + 1), \quad (7)$$

where S is the stability index. A periodic orbit is called critical when $S = \pm 1$. At $S = +1$, two qualitatively different cases occur. If S has an extremum as the family parameter passes through the critical orbit, the event is normally a saddle-node bifurcation. If S crosses the value $+1$ monotonically, a pitchfork-type loss of symmetry is possible. Hénon used such critical symmetric orbits to discover four asymmetric families [2].

For a periodic solution of period T , small corrections $\delta\mathbf{z}$ and δT satisfy the linearized continuation equation

$$(\mathcal{M} - E)\delta\mathbf{z}(T) + J\nabla H(\mathbf{z}_0)\delta T = 0. \quad (8)$$

Direct work with this singular equation is inconvenient near a critical solution. Therefore, we employ an orthogonal-symplectic transformation A of the monodromy matrix. A generic four-dimensional form is

$$\mathcal{A} = \begin{pmatrix} a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & -a_1 \\ -a_1 & -a_2 & a_3 & -a_4 \\ -a_2 & a_1 & a_4 & a_3 \end{pmatrix}, \quad \mathcal{A}^\top \mathcal{A} = E, \quad \mathcal{A}^\top J \mathcal{A} = J, \quad (9)$$

with the third column chosen parallel to the normalized gradient $\nabla H(\mathbf{z}_0)$. This construction is a special case of the parametric orthogonal-symplectic representation developed in [5]. In the transformed coordinates, the monodromy matrix can be reduced to a sparse form, and the solution of (8) becomes a two-parameter family. At a singly symmetric critical solution undergoing a pitchfork bifurcation, the reduction separates two directions: one continues the original symmetric family, whereas the second generates a new family with the same period but lower symmetry. Starting the differential correction along this second direction gives a stable numerical mechanism for entering the asymmetric branch.

More than 30 families of singly symmetric periodic solutions were processed using this procedure. Ten new families of planar asymmetric solutions were found. They can be divided into two groups.

- The first group consists of bridge families joining distinct symmetric families. A typical connection starts from a multiply covered doubly symmetric family, from which two pairs of singly symmetric branches emerge. Each singly symmetric branch contains a critical orbit with a pitchfork bifurcation. The resulting asymmetric family successively connects the singly symmetric branches and changes the symmetry set associated with its limiting endpoints. Thus, asymmetric solutions are not isolated curiosities; they provide missing links in the global network of symmetric families.

- The second group consists of families that can be continued toward asymmetric second-species generating solutions. These branches connect the local bifurcation picture with symbolic asymptotic theory. They demonstrate that the two search approaches – continuation from a critical symmetric orbit and continuation from an asymmetric GS – describe different parts of the same global family. In practical computations, the critical-orbit approach provides accurate initial conditions in a moderately unstable region, while the generating sequence identifies the limiting structure and global multiplicity of the branch.

All newly found planar asymmetric families were also tested for spatial stability, since their critical members may generate three-dimensional branches. For the families of periodic symmetric solutions, a kind of spatial bridge was studied in [6]. The results show that symmetry-breaking bifurcations create asymmetric bridges between backbone families, whereas second-species limits organize their remote endpoints. Symbolic asymptotics and adapted symplectic continuation therefore provide complementary parts of a systematic search method.

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Bifurcations of Periodic Solutions to Hamiltonian System with Symmetries

Alexander Bruno and Alexander Batkhin

Abstract. For an autonomous, real Hamiltonian system with two degrees of freedom in a generic setting, normal form (NF) theory and power-geometry techniques are employed to analyze bifurcations of periodic solutions associated with various groups of discrete linear automorphisms of the phase space. A Hamiltonian normal form is constructed in a neighborhood of a given periodic solution and, by means of an additional canonical transformation, is further reduced to an autonomous Hamiltonian system with a single degree of freedom and a single small parameter (the reduced normal form). The equilibrium points of this reduced normal form correspond to families of periodic orbits of the original system, and the resulting bifurcation scenarios depend crucially on the symmetry properties of the initial Hamiltonian system.

1. Introduction

The Poincaré approach for studying a non-integrable system of ODEs is based on finding a certain “**skeleton**” in the phase space around which the hierarchical structure of invariant manifolds of various dimensions is formed. Many problems in Celestial Mechanics can be reduced to Hamiltonian systems whose canonical equations admit a discrete group of symplectomorphisms. Each additional symmetry gives rise to new properties of the phase flow, which are closely related to bifurcations of periodic solutions of the equations of motion. This work is a continuation of [1], where considered only bifurcation scenarios of periodic solution to a generic time-independent Hamiltonian system with 2 degrees of freedom (DOF).

Consider a real Hamiltonian system with 2DOF and an analytic Hamiltonian $\gamma(\xi_1, \xi_2, \eta_1, \eta_2)$ defined in a domain $\mathcal{C}$. Suppose that the system has a periodic solution $\mathcal{M} \subset \mathcal{C}$ with period $T = T(\mathcal{M})$, and consider a sufficiently small neighborhood $\mathcal{U} \subset \mathcal{C}$ of $\mathcal{M}$. Then there exist analytic functions ξ , η , r , and ψ in $\mathcal{U}$ such that

- 1) the trajectory $\mathcal{M}$ is defined by the equalities $\xi = 0$, $\eta = 0$, $r = 0$;

- 2) the function ψ is cyclic (angular) modulo 2π ;
- 3) the neighborhood $\mathcal{U}$ is a skew product of a 3-dimensional ball and the cycle $\psi \in [0, 2\pi]$;
- 4) the coordinates ξ, η and ψ, r are canonically conjugate.

The Hamiltonian γ is 2π -periodic in ψ in the neighborhood $\mathcal{U}$ and can be expanded into the convergent Taylor series

$$\gamma = \sum \gamma_{pq\ell}(\psi) \xi^p \eta^q r^\ell,$$

where $p, q, \ell \in \mathbb{N}_0$. In these coordinates, γ defines the Hamiltonian system

$$\dot{\xi} = \frac{\partial \gamma}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial \gamma}{\partial \xi}, \quad \dot{r} = \frac{\partial \gamma}{\partial \psi}, \quad \dot{\psi} = -\frac{\partial \gamma}{\partial r}. \quad (1)$$

By applying standard complex and nonlinear transformations [2], the Hamiltonian γ is reduced to the complex normal form

$$h(\tilde{u}, \tilde{v}, \tilde{r}, \tilde{\varphi}) = \tilde{h}_{20} + \tilde{h}_{01} + \sum \tilde{h}_{pq\ell m} \tilde{u}^p \tilde{v}^q \tilde{r}^\ell e^{im\tilde{\varphi}}, \quad (2)$$

where the main diagonal of the matrix JG contains its eigenvalues ν and $-\nu$, with $\nu = \exp(2\pi\lambda)$. The Hamiltonian (2) is called a *complex normal form* (NF) if:

- 1) its quadratic part $\tilde{h}_{20} + \tilde{h}_{01}$ is in NF;
- 2) the expansion (2) contains only resonant terms satisfying so called *resonant equation*

$$(p - q)\lambda + im = 0. \quad (3)$$

Theorem 1 ([3]). *The canonical transformation*

$$\tilde{u} = u \exp(-i \operatorname{Im} \lambda \varphi), \quad \tilde{v} = v \exp(i \operatorname{Im} \lambda \varphi), \quad \tilde{r} = \rho - i \operatorname{Im} \lambda uv, \quad \tilde{\varphi} = \varphi, \quad (4)$$

reduces the NF of the Hamiltonian (2) to a constant power series

$$h(u, v, \rho) = \sum h_{pq\ell m} u^p v^q \rho^\ell, \quad (p - q) \operatorname{Re} \lambda = 0, \quad (p - q) \operatorname{Im} \lambda = -m, \quad (5)$$

where p, q, ℓ, m are integers, $p, q, \ell \geq 0$. Here $h_{pq\ell m} = \tilde{h}_{pq\ell m}$ if $p + q \geq 3$, the quadratic terms have the form $h_{20} + h_{01} = \frac{2\pi}{T} \left(\frac{1}{2} \langle \mathbf{w}, G\mathbf{w} \rangle - \rho \right)$, where the matrix $JG = JG - i \operatorname{Im} \Lambda$ with the diagonal matrix $\Lambda = \operatorname{diag}(\boldsymbol{\nu}, -\boldsymbol{\nu})$ and there are no linear terms in $\mathbf{w} = (u, v)$.

Thus, we obtain an autonomous 1DOF Hamiltonian system $\dot{u} = h_v, \dot{v} = -h_u$ with an additional equation $\dot{\varphi} = -h_\rho$ and an additional small parameter $\rho = \text{const}$. We call (5) the *reduced normal form* (RNF). It allows us to study the bifurcations of families of periodic solutions of the system (1) in a neighborhood of the resonant periodic solution, because each periodic solution $\mathcal{M}$ to system (1) corresponds to a stationary point (SP) of the RNF Hamilton system.

For that to each germ of family of periodic solutions of the truncated system we make transformations:

- reverse to (4) and
- reverse to the standard transformation for given λ [2, S. 1.4].

Hypothesis 1. *If result of these transformations is a real germ, then corresponding family of periodic solutions is real.*

2. Classes of Symmetries

A periodic solution $\mathcal{M}$ has a monodromy matrix $\mathcal{N}$ with characteristic polynomial $\chi(\nu) = (\nu - 1)^2 (\nu^2 - 2S\nu + 1)$, where S is the stability index. The solution is linearly stable if $|S| \leq 1$, with $S = \operatorname{Re} \exp(2\pi\lambda)$, $\nu_{3,4} = \exp(\pm 2\pi\lambda)$. If the argument of the non-trivial multipliers $\nu_{3,4}$ is commensurable with π , then, according to Poincaré [4, Ch. XXX], additional periodic solutions may bifurcate near the generating solution. Their period T' is a multiple of the original period T .

In this case, the resonant equation (3) becomes

$$(p - q)\mathfrak{r} + \mathfrak{s}m = 0, \quad \lambda = \frac{\mathfrak{r}}{\mathfrak{s}}, \quad |\mathfrak{r}| < \mathfrak{s}, \quad \mathfrak{r} \in \mathbb{Z}, \quad \mathfrak{s} \in \mathbb{N}. \quad (6)$$

Let $F_k^{(\mathfrak{s})}(u, v)$ denote a homogeneous resonant form of degree k in the canonical variables (u, v) :

$$F_k^{(\mathfrak{s})}(u, v) \stackrel{\text{def}}{=} \sum_{p+q=k} a_{pq} u^p v^q, \quad a_{pq} = \text{const}, \quad (7)$$

and p, q satisfy (6). Accordingly, the RNF can be represented as

$$h(u, v, \rho) = \sum_{\ell=0}^{\infty} \rho^{\ell} \sum_{k=1}^{\infty} F_k^{(\mathfrak{s})}(u, v), \quad (8)$$

where triple indices $a_{pq\ell}$ are used to distinguish the coefficients corresponding to different powers of ρ .

We consider finite groups of linear canonical automorphisms of the two-dimensional phase space that preserve the homogeneity of the forms (7) and the integral character of their monomial exponents. Such transformations change the signs of u and v , interchange the variables, or combine these operations. The full group of these transformations is the dihedral group D_4 of order 8,

$$D_4 = \langle r, s \mid r^4 = s^2 = e, \, srs = r^{-1} \rangle,$$

with generators $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

A linear map A is called a Hamiltonian symmetry if $H \circ A = H$ and A are symplectic. A linear map $\varkappa$ is called a reversing symmetry, or reversor, if $H \circ \varkappa = H$ and $\varkappa$ is anti-symplectic. In the latter case, $\varkappa \circ \Phi_t = \Phi_{-t} \circ \varkappa$. For a homogeneous form, invariance under a subgroup $G \subset D_4$ always means literal invariance $F_k^{(\mathfrak{s})} \circ A = F_k^{(\mathfrak{s})}$ for all $A \in G$ including anti-symplectic elements.

The subgroups of D_4 relevant to classification are the trivial subgroup $\{e\}$; five cyclic subgroups of order 2: $\langle r^2 \rangle$, $\langle s \rangle$, $\langle rs \rangle$, $\langle r^2 s \rangle$, $\langle r^3 s \rangle$; the cyclic subgroup of order 4: $C_4 = \langle r \rangle$; the two Klein four subgroups $V_4^{\text{ax}} = \langle r^2, s \rangle$, $V_4^{\text{diag}} = \langle r^2, rs \rangle$; and the full group D_4 . The elements of C_4 are symplectic, whereas the four reflections are anti-symplectic.

Definition 1. Let $\text{Sym}(z) \subset D_4$ be the isotropy subgroup of a solution z . The solution is called **asymmetric** if $\text{Sym}(z) = \{e\}$, **singly symmetric** if $\text{Sym}(z)$ is cyclic, and **doubly symmetric** if $\text{Sym}(z)$ requires two generators.

Thus, singly symmetric solutions correspond to the cyclic subgroups of orders 2 or 4, while doubly symmetric solutions correspond to V_4^{ax} , V_4^{diag} , or D_4 . The symmetry subgroup imposes algebraic restrictions on the coefficients $a_{pq\ell}$; consequently, some coefficients may vanish and the admissible RNF terms, and hence the possible bifurcation scenarios of periodic solutions, become symmetry-dependent. For $\rho = 0$, the degrees of $F_k^{(s)}$ begin with $k = 3$ for the prime elementary divisors of the matrix M and with $k = 2$ otherwise.

3. Results

Here we give some results of the computations for singly and doubly symmetric cases for $\mathfrak{s} = 1, 2$, whereas a generic case with a trivial group of symplectomorphisms was described in [1]. The scheme of computation presented in [1] is slightly modified here and is mainly based on the power geometry technique applied rather to Hamiltonian than to its canonical system. Briefly, the main steps of the computations are the following: getting the support $\mathbf{S}$ of the RNF H , calculating its convex hull Γ , selecting the facets $\Gamma_j^{(2)}$ with negative normal vectors N_j and obtaining the corresponding truncated Hamiltonians $\hat{H}_j$. Solutions to the truncated system $\hat{\mathcal{S}}_j \stackrel{\text{def}}{=} \left\{ \partial \hat{H}_j / \partial u = \partial \hat{H}_j / \partial v = 0 \right\}$ provide the germs of periodic solution. After that only those germs should be selected that give real periodic solutions.

In the following, we consider only the Klein four subgroup V^{ax} with axial symmetries s , r^2s and central symmetry r^2 . The case of axial symmetry with generator r^2s is the same as the case with generator s . For cases $\mathfrak{s} = 1, 2$ we should distinguish sub cases for the presence and absence of the Jordan block (JB).

For $\mathfrak{s} = 1$ with JB we consider single axial symmetry (AS), central symmetry (CS) and double symmetry (DS).

Symmetry AS. There is the only normal vector $N_1 = -(2, 3, 4)$, corresponding truncated Hamiltonian is $\hat{H}_1 = \frac{1}{2}\sigma v^2 + a_{300}u^3 + a_{101}\rho u$, and truncated system $\hat{\mathcal{S}}_1 = \{\sigma v = 0, -3u^2a_{300} - \rho a_{101} = 0\}$. The family of SP solutions takes the form $\{u = \pm\sqrt{\beta\rho}, v = 0\}$, where β depends on the NF coefficients. Here we have a *saddle-node* bifurcation.

Symmetries CS and DS give us the only normal vector $N_1 = -(1, 2, 2)$, truncated Hamiltonian $\hat{H}_1 = \frac{1}{2}\sigma v^2 + a_{201}u^2\rho + a_{400}u^4$ and truncated system $\hat{\mathcal{S}}_1 = \{\sigma v = 0, -4a_{400}u^3 - 2a_{201}\rho u = 0\}$. Here we have the same asymptotic form of SP solutions and the same saddle-node bifurcation.

For $\mathfrak{s} = 1$ without JB we get the following.

Symmetry AS gives two normal vectors $N_1 = -(1, 1, 2)$ and $N_2 = -(2, 1, 2)$. The corresponding Hamiltonians are the following: $\hat{H}_1 = a_{101}u\rho + a_{120}uv^2 + a_{300}u^3$ and $\hat{H}_2 = a_{101}u\rho + a_{021}v^2\rho + a_{120}uv^2 + a_{040}v^4$. The first system $\hat{\mathcal{S}}_1 = \{2uva_{120} = 0, -3u^2a_{300} - v^2a_{120} - \rho a_{101} = 0\}$ has only symmetrical solution $\{u = \pm\sqrt{\beta_1\rho}, v = 0\}$, whereas the second system $\hat{\mathcal{S}}_2 = \{4v^3a_{040} + 2\rho va_{021} + 2uva_{120} = 0, -v^2a_{120} - \rho a_{101} = 0\}$ has the solutions of the form $\{u = \beta_1\rho, v = \pm\sqrt{\beta_2\rho}\}$, which are asymmetrical. So here we have a *pitch-fork* or *symmetric break* bifurcation. Along the first solution the Hamiltonian value changes monotonically and along the second one the Hamiltonian has an extrema.

Conclusion. In the case of JB the symmetry of periodic solution does not change while crossing the bifurcation point. The birth of asymmetrical periodic solution appears in the case of JB absence and monotonical changes of the Hamiltonian value.

For $\mathfrak{s} = 2$ with JB we consider only the case with single axial symmetry, because the monodromy matrix $\mathcal{N}$ in the cases of central and double symmetries is the full square matrix, hence it cannot have a JB with negative eigenvalue. So the truncated Hamiltonian corresponding to the normal vector $N_1 = -(1, 1, 2)$ is $\hat{H}_1 = \frac{1}{2}\sigma v^2 + a_{201}u^2\rho + a_{400}u^4$, and we get 2 solutions: a trivial one $\{u = v = 0\}$ corresponding to the original periodic solution and a symmetrical one: $\{u = \pm\sqrt{\beta\rho}, v = 0\}$. This solution gives us a family of solutions with double period, i.e. here *period doubling* bifurcation takes place.

For $\mathfrak{s} = 2$ without JB we consider a doubly symmetrical situation. Convex hull gives us the only normal vector $N_1 = -(1, 1, 2)$ and to it the truncated Hamiltonian $\hat{H}_1 = (a_{021}v^2 + a_{201}u^2)\rho + a_{040}v^4 + a_{220}u^2v^2 + a_{400}u^4$ is corresponding. The truncated system

$$\hat{\mathcal{S}}_1 = \{2a_{220}u^2v + 4a_{040}v^3 + 2\rho va_{021} = 0, -4a_{400}u^3 - 2a_{220}uv^2 - 2\rho ua_{201} = 0\}$$

has 4 series of solutions. The first solution $Sol_1 : \{u = v = 0\}$ corresponds to the original periodic solution. The second $Sol_2 : \{u = \pm\sqrt{\beta_2\rho}, v = 0\}$ and the third $Sol_3 : \{u = 0, v = \pm\sqrt{\beta_3\rho}\}$ correspond to new families of doubly periodic solutions with single symmetry each. Along each of the new families the Hamiltonian $\hat{H}_1$ reaches its extremum. There are possible 2 different scenarios depending on whether the extrema have the same types (both minima or maxima) or different types (one is minimum and another is maximum or vice versa). In scenario 1 one new family consists of stable solutions and the other consists of unstable ones. In scenario 2 both two new families consist of unstable solutions. Finally, the fourth solution of the form $Sol_4 : \{u = \pm\sqrt{\beta_4\rho}, v = \pm\sqrt{\beta_5\rho}\}$ does not correspond to real periodic solutions.

All the scenarios of periodic solution bifurcations described above were studied numerically in the Hill 3-body problem by M. Hénon [5, 6] and the second author [7, 8].

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Assessing Chirikov's Diffusion Theory on the Arnold Model

Pablo M. Cincotta and Claudia M. Giordano

Abstract. This work focuses on an Arnold diffusion-like process along the homoclinic tangle of a single resonance in a multidimensional Hamiltonian system. Following Chirikov's theory [2], we consider the two-parameter Arnold model [1], consisting of a pendulum (as a resonance model) and a free rotator coupled through a perturbation. Both the low- and high-frequency regimes of the free rotator are analyzed. We derive a reliable diffusion coefficient (or diffusion timescale) using a relatively novel formulation based on Shannon entropy [3]. We compare the results with Chirikov's estimates and find agreement only in the high-frequency regime. Evidence supporting both the agreement and the discrepancies is presented and discussed using a variety of approaches, ranging from theoretical analysis to numerical experiments. In addition, we compare two characteristic timescales of chaotic dynamics—the Lyapunov time and the diffusion time—showing that the latter provides the relevant timescale for predictable dynamics, yielding another example of stable chaos.

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On the solution of the special BVP related to a full set of geodetic and gravimetric data

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Abstract. The work is devoted to the problem of determining the anomalous gravitational potential on the earth's surface in conditions of the simultaneous availability of data on the spatial position of points, their height above sea level and the measured force of gravity. A solution to the boundary value problem is proposed based on gravity anomalies of a special type. The necessary mathematical expressions have been obtained. Computational experiments were performed on model data.

Introduction

There are two main types of gravity reductions: “free-air” reductions and topographic reductions. The former are used to solve boundary value problems in physical geodesy, while the latter are used to improve the accuracy of calculations and facilitate interpolation using a gravimetric map.

For a known boundary surface, to determine the anomalous potential, L.P. Pellinen [1] proposed to introduce anomalies of a special type that combine the advantages of known “free air” anomalies (gravity anomalies – used with a known potential difference, gravity disturbance – with a known boundary surface) and take into account the influence of topographic masses on the calculation point. He formulated a boundary condition in the spherical approximation and proposed a solution based on the method of expansion in powers of a small parameter of M.S. Molodensky [3]. These anomalies are of particular interest because it is now possible to simultaneously obtain geodetic, leveling and gravimetric determinations for one point, leading to different in gravity reductions.

1. Analytical solution

First, a solution to the boundary value problem was obtained in integral form for a spherical reference surface. In addition, algorithms for taking into account the

influence of the “zero-zone” have been developed for the numerical implementation of the obtained formula, taking into account the singularity at the evaluation point.

To ensure the possibility of increasing the accuracy of calculations, an integral equation is obtained based on the density of a simple layer, taking into account the flattened of the reference surface. A necessary boundary condition is proposed. A connection was obtained between gravity anomalies in special reduction and other types of anomalies.

The properties of the obtained solutions are described.

2. Computational experiments

To test the obtained solutions, Earth models were created: sources of normal and anomalous fields were selected, the physical surface was defined, reference values of the determined quantities and measurement results were generated. Two types of calculations were performed:

- in the first case, the boundary surface and the values of gravity anomalies are considered known, and the anomalous potential is subject to determination;
- in the second case, the boundary surface is also known, the values of the gravity anomaly are given, and it is necessary to calculate the anomalous potential.

Since it is recommended to move from gravity anomalies to gravity disturbance ones to refine the solution of the boundary value problem [2], in both experiments the results of calculations based on anomalies in a special reduction and gravity disturbance were compared.

Conclusion

The spherical solution has a number of interesting properties: the integration kernel does not contain transcendental functions, the first-degree harmonic is preserved, the considered type of anomalies in this form is endowed with the meaning of the density of a simple layer distributed over the surface under study. The algorithms for taking into account the influence of the “zero-zone” seem to be quite simple.

The solution in the form of an integral equation over the surface has a simpler structure than a similar representation for other types of anomalies. With numerical integration, the rate of convergence of its solution is several times higher than for gravity disturbance. There is a tendency towards a uniform distribution of calculation errors over the entire integration surface, in contrast to gravity disturbance, which strongly correlate with the magnitude of the anomalous potential itself.

A solution based on special type anomalies will be convenient for calculating the gravimetric height anomaly for satellite leveling, since the calculation results will contain only the methodological error in performing the numerical integration, which can be studied separately.

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On Refining the Applicability Conditions of the Power Series in the Circular Restricted Three-Body Problem

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Abstract. The secular evolution of an asteroid orbit linked with the circular orbit of Jupiter is studied in the inner restricted three-body problem. It is shown that the power series expansion of the doubly averaged perturbing function has the radius of convergence $\rho(e) = 1/(1 + e)$ and is therefore inapplicable in the region $a(1 + e) > r_J$, where spurious equilibria appear on the phase portraits. This region is wider than the one indicated by von Zeipel.

Keywords. restricted three-body problem, secular perturbations, averaging, power series, radius of convergence.

We consider the classical problem of analyzing the secular perturbations of the Keplerian orbital elements of an asteroid in the inner restricted three-body problem, when the asteroid's orbit, with its focus at the Sun, is linked with the circular orbit of the perturbing body (Jupiter). As is well known, the linking condition reads

$$\frac{a(1 - e^2)}{1 + e \cos \omega} < r_J < \frac{a(1 - e^2)}{1 - e \cos \omega}, \quad (1)$$

where a , e , ω are the semi-major axis, the eccentricity, and the argument of pericenter of the asteroid's orbit, and r_J is the radius of Jupiter's orbit. Based on condition (1), von Zeipel [1] introduces the parameter region A , where both nodes of the asteroid's orbit lie inside Jupiter's orbit; the orbit-linking regions B and B' , where one of the nodes of the asteroid's orbit lies outside Jupiter's orbit; and the region C , where both nodes of the asteroid's orbit lie outside Jupiter's orbit.

Averaging over the fast variables (λ_J , the longitude of Jupiter, and l , the mean longitude of the asteroid) yields the following expression for the perturbing function in region A :

$$R^{**} = \frac{f m_J}{r_J} \sum_{n=1}^{\infty} D_n \mu^{2n}, \quad (2)$$

where

$$\begin{aligned} \mu &= \frac{a}{r_J}, \quad D_n = P_{2n}(0) \left(P_{2n}(0) P_{2n}(\cos i) \Phi(0, n) + \right. \\ & 2 \sum_{m=1}^n (-1)^m \binom{2(n+m)+1}{2m} \Phi(m, n) \left(\frac{e}{2} \right)^{2m} P_{2n}^{(-2m)}(0) P_{2n}^{(2m)}(\cos i) \cos 2m\omega \Big), \\ & \Phi(m, n) = {}_2F_1\left(m-n, m-n-\frac{1}{2}; 2m+1; e^2\right). \end{aligned}$$

Here i is the inclination of the asteroid's orbit, m_J is the mass of Jupiter, f is the gravitational constant, P_{2n} and $P_{2n}^{(2m)}$ are the Legendre polynomials, and ${}_2F_1$ is the Gauss hypergeometric function.

In the case of the doubly averaged perturbing function, the system of Lagrange equations in the orbital elements possesses a complete set of first integrals:

$$a = c_0, \quad (1 - e^2) \cos^2 i = c_1, \quad R^{**} = c_2,$$

and admits reduction to a system with one degree of freedom in (e, ω) . Analysis of the phase portraits in (e, ω) has shown that the phase curves in the region $A' : a(1+e) > r_J$ are unreliable: spurious equilibria appear, and their number grows as the number of retained terms of the series increases.

This fact has been confirmed analytically: it was established that the power series (2) converges absolutely in the interval

$$0 < \mu < \frac{1}{1+e},$$

so that the radius of convergence is

$$\rho(e) = \frac{1}{1+e}.$$

This result was also verified numerically: relying on the asymptoticity estimate from [2], the following quantity was computed for each point of the parameter space (e, ω, μ, c_1) :

$$n_{\max} = \sup \left\{ n \in \mathbb{N} \mid |R^{(i)} - R_n^{**}| < O(\mu^{2n+2}) \right\},$$

where $R^{(i)}$ is the numerically computed value of the doubly averaged perturbing function and R_n^{**} is the truncation of the series R^{**} to n terms. Maps of $n_{\max}$ in the (e, ω) plane were constructed for various values of c_1 and μ , confirming the non-asymptotic behavior of the series R_n^{**} in the region A' . It should be noted that the region A' is wider than the region indicated in [1], for which von Zeipel proposes using the power series, since it includes both the regions B and B' and a part of the region A .

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Figure-eight orbit with unequal masses in the planar three-body problem

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Abstract. The generalization of the figure-eight orbit to the unequal-mass case in the planar three-body problem is studied within the space-of-forms framework. A high-precision shooting method yields solutions that pass through Eulerian configurations exactly four times for small deviations of the third mass from unity. These trajectories are shown to lose periodicity and choreographic property; for 10% mass deviation, no such solutions are found. Topological constraints on periodic solutions are also derived: the absence of 2π -periodic-in- φ orbits at zero angular momentum is proved, and an integral condition necessary for 4π -periodic solutions is obtained. The results confirm the space of forms as a useful tool for analysing the existence and properties of periodic regimes in the three-body problem.

Introduction

The figure-eight orbit [1] of the planar three-body problem exists for equal masses and zero angular momentum. Whether similar orbits exist for unequal masses is an open question. This question is investigated using the space-of-forms formalism [2, 5], which factors out rotations and provides a geometric description of the dynamics.

1. Space of forms

Following [5], the space of forms is obtained by factoring the Jacobi coordinates over rotations. The shape coordinates are

$$\xi_1 = \frac{1}{2}(Q_1^2 - Q_2^2), \quad \xi_2 = Q_1 \cdot Q_2, \quad \xi_3 = Q_1 \times Q_2,$$

with $\xi_1^2 + \xi_2^2 + \xi_3^2 = I^2$. Generalised spherical coordinates are

$$\xi_1 = \rho^2 \cos \varphi \cos \theta, \quad \xi_2 = \rho^2 \sin \varphi \cos \theta, \quad \xi_3 = \rho^2 \sin \theta,$$

where ρ is the scale factor, φ is the equatorial angle, and θ is the latitude. Collinear configurations correspond to $\theta = 0$. The Eulerian collinear points are the minima of the triangular potential

$$D(\varphi, \theta) = \sum_{i < j} \frac{(m_i m_j)^{3/2}}{\sqrt{m_i + m_j}} \frac{1}{\sqrt{1 + \cos(\varphi - \varphi_{ij}) \cos \theta}}. \quad (1)$$

The Hamiltonian system (for zero angular momentum) is

$$\begin{cases} \dot{\rho} = p_\rho, \\ \dot{\varphi} = \frac{4p_\varphi}{\rho^2 \cos^2 \theta}, \\ \dot{\theta} = \frac{4p_\theta}{\rho^2}, \\ \dot{p}_\rho = \frac{4}{\rho^3} \left(\frac{p_\varphi^2}{\cos^2 \theta} + p_\theta^2 \right) - \frac{D}{\rho^2}, \\ \dot{p}_\varphi = -\frac{D_\varphi}{\rho}, \\ \dot{p}_\theta = -\frac{D_\theta}{\rho} - \frac{4p_\varphi^2 \sin \theta}{\rho^2 \cos^3 \theta}. \end{cases} \quad (2)$$

The equal-mass figure-eight has initial conditions

$$\begin{cases} \rho(0) = \sqrt{2}, \quad \varphi(0) = 0, \quad \theta(0) = 0, \\ p_\rho(0) = 0, \quad p_\varphi(0) = 0.6012240891875504, \quad p_\theta(0) = 0.922706672027483. \end{cases} \quad (3)$$

2. Numerical search

Two approaches to defining a (pseudo) figure-eight for unequal masses are considered. The first preserves the maximal number of passages through Eulerian configurations. The second minimises a cost function measuring deviation from such passages. Since the shooting equations (first approach) are locally non-degenerate, any trajectory with many exact crossings coincides with the first approach's solution. Thus the first approach is used.

For given m_3 ($m_1 = m_2 = 1$), the initial point is fixed at an Eulerian configuration: $\varphi = 0, \theta = 0, \rho = \sqrt{2}$. The remaining unknowns are found by requiring $\theta = 0$ at the next three Eulerian angles $\varphi_1, \varphi_2, \varphi_3$ (minima of $D(\varphi, 0)$):

$$F(p_\rho, p_\varphi, p_\theta; \Delta m_3) = \begin{pmatrix} \theta(\varphi_1) \\ \theta(\varphi_2) \\ \theta(\varphi_3) \end{pmatrix} = 0. \quad (4)$$

The system is solved by Newton's method; equation 2 is integrated with a 3-stage collocation Runge–Kutta method using 100-digit arithmetic.

Table 1 lists the initial data for several m_3 . ϵ is the precision.

The corresponding trajectories are shown in figure 1. Each solution passes through $\theta = 0$ four times with residual $\sim 10^{-23}$. The trajectories lose

TABLE 1. Initial conditions for (pseudo) figure-eight solutions.

m_3	p_ρ	p_φ	p_θ	ϵ
1.001	5.01722e-3	6.00834e-1	-9.23871e-1	3e-6
1.01	4.68088e-2	6.00971e-1	-9.34137e-1	3e-6
0.99	-4.94253e-2	6.00853e-1	-9.11090e-1	3.5e-6
0.999	-4.81961e-3	6.01225e-1	-9.21555e-1	1e-20

periodicity and choreographic property. For $m_3 = 1.1$ and 0.9 no solution was found.

Since the obtained solutions are not periodic, the question of whether periodic (pseudo) figure-eights exist remains. This motivates the topological analysis below, which provides necessary conditions for periodicity.

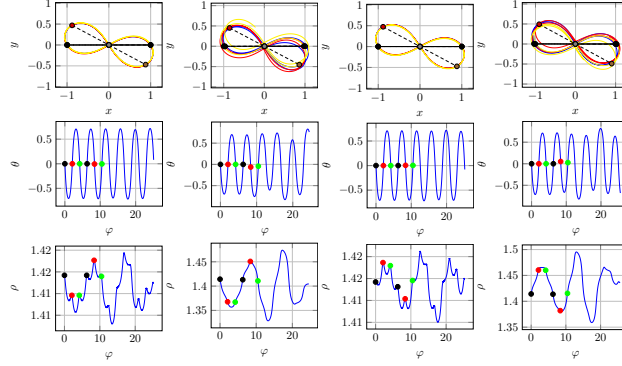


FIGURE 1. Trajectories for $m_3 = 0.999, 0.99, 1.001, 1.01$ (left to right). Eulerian configurations marked with dots.

3. Topological constraints on periodic orbits

For a periodic solution, let J be the angular momentum. The identity

$$\frac{J}{\rho^2} = \dot{\lambda} + \frac{\dot{\theta} \sin \varphi + \dot{\varphi} \sin \theta}{2(\cos \varphi \cos \theta + 1)} - \frac{1}{2} \dot{\varphi} \sin \theta \quad (5)$$

holds, where λ is the rotation angle. For a periodic orbit of period T , $\lambda(T) - \lambda(0) = 2\pi k$. Integrating and using the closedness of

$$\omega = \frac{d\theta \sin \varphi + d\varphi \sin \theta}{2(\cos \varphi \cos \theta + 1)} \quad (6)$$

on $\Pi = \mathbb{R}/2\pi\mathbb{Z} \times (-\pi/2, \pi/2) \setminus \{(\pi, 0)\}$ gives:

Lemma 1. *For any closed curve $C \subset \Pi$:*

1. $\oint_C \omega = \pi l$, $l \in \mathbb{Z}$, if C is transversal to φ ;
2. $\oint_C \omega = 2\pi l$, $l \in \mathbb{Z}$, otherwise.

Hence any periodic solution satisfies

$$\int_0^T \left(\frac{1}{2} \dot{\varphi} \sin \theta + \frac{J}{\rho^2} \right) dt = \begin{cases} \pi l, & \text{if transversal in } \varphi, \\ 2\pi l, & \text{otherwise.} \end{cases} \quad (7)$$

Two corollaries follow for $J = 0$:

1. No 2π -periodic-in- φ solution exists, since $\left| \int_0^{2\pi} \frac{1}{2} \dot{\varphi} \sin \theta dt \right| < \pi$.
2. Any 4π -periodic-in- φ solution must satisfy

$$\int_0^T \dot{\varphi} \sin \theta dt = 0, \quad (8)$$

which implies crossing $\theta = 0$ at least once.

Thus, any periodic generalization of the figure-eight must be at least 4π -periodic-in- φ and must cross the collinear plane.

Conclusion

The numerical search found (pseudo) figure-eight solutions for masses close to equal that pass through Eulerian configurations four times but are not periodic. Topological analysis shows that any periodic generalization must be 4π -periodic in φ and cross $\theta = 0$. The space of forms proves useful for both numerical and analytical studies of the three-body problem.

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Exoplanet WASP-76 b secondary eclipse timing deviation possibly indicating the existence of additional unseen components

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Abstract. The work involves the homogeneous processing of observations of 10 exoplanets, each with at least four secondary eclipses, using data from the Spitzer/IRAC space telescope in the $3.6\ \mu\text{m}$ band. The aim is to search for infrared brightness profile asymmetry on the surface of these exoplanets. To achieve this, the model parameters were estimated using the maximum likelihood method, incorporating correlated noise modeled as a stationary Gaussian process. The effect of intrapixel sensitivity variations is modeled using a 2D Fourier series. For WASP-76 b, a statistically significant shift of secondary eclipse center of -2.42 ± 0.57 min was detected. This shift possibly indicates the presence of additional invisible components in the planetary system.

Introduction

Observations in the infrared spectrum allow us to detect not only transits, but also secondary eclipses (when the planet sets behind the star). These observations enable us to determine the orbit of the planet and its average temperature, but in this study, we were interested in more refined effects, specifically, irregularities in the shape of the light curve at the beginning and end of the eclipse, which could be caused by variations in brightness across the planetary surface.

The most suitable candidates for observing the effect in question are hot jupiters. These planets have the highest temperatures and radii, and as a consequence, the deepest secondary eclipses. Furthermore, these types of planets should be tidally locked to their host star, allowing for the observation of stationary current patterns from the heated dayside to the nightside. At the same time, the hottest region should shift in relation to the substellar point, making it easier to detect the uneven surface brightness effect.

In this study, we analyzed the observational data from secondary eclipses of ten exoplanets located close to their host stars (hot jupiters, hot neptun, and super-earth), which were obtained using the IRAC instrument on the Spitzer Space Telescope. This work builds on previous research [1].

Several articles by other researchers have addressed a similar research topic. Specifically, in [2, 3, 4], images of the surface of HD 189733b were obtained. The unique aspect of this study is that data from ten exoplanets was analyzed simultaneously in order to identify the characteristic features of their light curves. Additionally, a different IRAC bandpass was used, $3.6\ \mu\text{m}$ instead of $8\ \mu\text{m}$.

The choice of this spectral range was due to its previous use in the aforementioned study [1]. Although the eclipse depth in this range is less than that observed at $8\ \mu\text{m}$, it does not allow for direct recording of surface brightness inhomogeneities. Transitioning to the $8\ \mu\text{m}$ range would necessitate revising the noise model, which is noticeably distinct from those employed in previous studies, according to [5].

The aim of this study is to analyze several exoplanets simultaneously in order to identify the distinctive features of their light curves.

1. Methodology

During the course of the work, observations were analyzed for 10 exoplanets: HD 209458 b, HD 189733 b, GJ 3470 b, WASP-12 b, WASP-43 b, WASP-76 b, HAT-P-2 b, 55 CANCRI e, WASP-80 b, and WASP-52 b.

The complete photometry algorithm has been previously presented [1]. A distinctive aspect of this study is a more thorough consideration of outliers in the data, and the fact that for several objects, frames containing only one or two outliers were not excluded from analysis, as otherwise a significant portion of the observational data would have been disregarded.

To account for strong correlated noise, a maximum likelihood method was employed, with the correlated noise being represented as a stationary Gaussian process.

In this study, a uniformly bright disk model was employed as a representation of the signal from the target object (the most basic light curve from the article [6]). As the aim of the research was not to simulate the heterogeneity of surface brightness, but rather to identify discrepancies with the simplest model, this model included two primary free parameters that did not account for noise: the relative brightness of the planet compared to the star and a time offset for the center of secondary eclipse relative to the model-predicted orbital position. For most targets, the assumed orbit was circular, with the notable exception of exoplanet HAT-P-2b, which exhibits a significant level of eccentricity.

The modeling of the noise component involved a modification of the approach proposed in [1]. Specifically, the part responsible for uneven pixel sensitivity included a variable number of coefficients rather than the fixed 20 previously used.

The number of coefficients was selected individually for each pixel in each object and ranged from 4 to 68. This decision was based on the statistical significance of model improvement, with the choice of number of coefficients determined by when the significance reached a level of 2 standard deviations.

Conclusion

As a result, the following eclipse parameters were obtained (see Table 1). It should be noted that for all planets, the data are presented as deviations from a circular orbit, except for HAT-P-2 b, where the result is compared to a highly elongated orbit ($e = 0.50833, \omega = 186.96^\circ$). The offset in the eclipse timing for each exoplanet can be explained by two factors: the heterogeneity in surface brightness and the non-zero orbital eccentricity.

Object	Center offset (min.)	(rad. of the planet)
HD 209458 b	-0.38 ± 0.64	-0.036 ± 0.057
HD 189733 b	0.18 ± 0.39	0.021 ± 0.044
GJ 3470 b	33.22 ± 8.4	9.32 ± 2.35
WASP-12 b	0.33 ± 0.98	0.033 ± 0.098
WASP-43 b	-0.37 ± 0.45	-0.063 ± 0.076
WASP-76 b	-2.42 ± 0.57	-0.217 ± 0.051
HAT-P-2 b	0.30 ± 1.47	0.039 ± 0.193
55 CANCRI e	3.00 ± 1.53	3.428 ± 1.755
WASP-80 b	-1.18 ± 1.72	-0.126 ± 0.184
WASP-52 b	0.02 ± 1.77	0.003 ± 0.198
weighted average score	-0.10 ± 0.25	-0.010 ± 0.029
Object	$e \cos(\omega) (\times 10^{-3})$	Depth of eclipse (ppm)
HD 209458 b	-0.13 ± 0.20	1205 ± 43
HD 189733 b	0.09 ± 0.19	1470 ± 38
GJ 3470 b	10.86 ± 2.74	48 ± 50
WASP-12 b	0.33 ± 0.98	3461 ± 194
WASP-43 b	-0.50 ± 0.60	3211 ± 108
WASP-76 b	-1.46 ± 0.34	2755 ± 92
HAT-P-2 b	—	916 ± 97
55 CANCRI e	4.44 ± 2.27	101 ± 69
WASP-80 b	-0.42 ± 0.61	959 ± 123
WASP-52 b	0.01 ± 1.11	2010 ± 171

TABLE 1. Model parameters

It is worth noting the data obtained for exoplanets GJ 3470 b and 55 CANCRI e, the eclipses of which are recorded with a significant error. In particular, GJ 3470 b. It is also significant that the offset of the eclipse centre varies greatly

depending on the model used, which could indicate that there is no eclipse as such in the observational data and the recorded effect is a measurement error.

For most objects, there is no statistically significant offset in the center of secondary eclipses. However, there are two exceptions: 55 Cancri e, which is part of a multi-planet system, and WASP-76 b. In the case of 55 Cancri e, the planet may have a non-zero eccentricity, while in the case of WASP-76 b, it looks unlikely. This object also show an earlier occurrence of the eclipse than would be expected from a non-uniform brightness, despite the absence of any other exoplanets in the system. It is worth noting that the second star in this system is located at a distance of 53.0 ± 8.8 AU from the main star, and the expected period of the system is approximately 300 years. This should not significantly affect the timing of eclipses observed over the two-year period covered by the Spitzer Space Telescope.

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Stability of Periodic Oscillations of a Two-body Gravity Gradient System in the Plane of an Elliptic Orbit

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Abstract. The dynamics of a two-body system connected by a spherical hinge moving within a gravitational field in an elliptical orbit is investigated. The dynamics of the system is described by a second-order system of differential equations with periodic coefficients. By applying perturbation techniques and symbolic computation, a periodic solution to the equations of motion is constructed as a power series in a small parameter. The conditions under which such periodic motion occurs were determined and its stability in the linear approximation was investigated. The stability analysis involves the computation of the Hamiltonian of the system, which determines the linearized differential equations of the perturbed motion with periodic coefficients, and the evaluation of the monodromy matrix eigenvalues.

Introduction

In this work, we study the oscillations of a system of two bodies connected by a spherical hinge that moves within a gravitational field in the plane of an elliptical orbit. The investigation of the motion of a system of two connected bodies in an elliptical orbit is of practical interest for the design of composite schemes for a passive gravitational attitude control systems. While the dynamics of various gravitational orientation schemes for satellite-stabilizer configurations has been widely discussed in the literature, some reviews can be found in [1, 2, 3].

The oscillations of a satellite (one rigid body) in an elliptical orbit were investigated in detail in [4]. Research on the planar oscillations of two connected bodies has largely been restricted to cases where their centers of mass coincide [5, 6, 7]. In [8], the eigen-oscillations of a system of two bodies were studied and optimal parameters for transition to equilibrium were found.

The present work extends this research by considering the planar oscillations of a two-body system in an elliptical orbit, specifically where the spherical hinge is located at the intersection of the principal central axes of inertia of the first and second bodies. Expanding upon our previous study [9], which was limited to a first-order linear approximation, this paper calculates the cubic approximation and provides a solution accurate to the third order. By employing perturbation techniques and symbolic computation with the help of Wolfram Mathematica [10], we construct the periodic solution in the form of a power series in a small parameter and investigate its stability in the linear approximation.

Equations of Motion

The equations of motion of a system of two bodies connected by a spherical hinge in the plane of an elliptical orbit relative to the center of mass in the orbital coordinate system: $OXYZ$ (where the OZ axis is directed along the radius vector connecting the center of mass of the Earth and the center of mass of the two-body system O ; the axis OX is directed along the vector of the linear velocity of the center of mass, and the axis OY coincides with the normal to the plane of the orbit) have the following form:

$$\begin{aligned}
& - (1 + e \cos \vartheta) \alpha_2'' + 2e \alpha_2' \sin \vartheta + \frac{B_1 + Ma_1^2}{Ma_1 a_2} ((1 + e \cos \vartheta) \alpha_1'' - 2e \alpha_1' \sin \vartheta) \\
& - e(1 + e \cos \vartheta)(\alpha_2' + 1)^2 + 2e \sin \vartheta \left(1 - \frac{B_1 + Ma_1^2}{Ma_1 a_2}\right) + 4e \\
& + \frac{3e(A_1 - C_1 - Ma_1^2)}{Ma_1 a_2} = 0, \\
& - (1 + e \cos \vartheta) \alpha_1'' + 2e \alpha_1' \sin \vartheta + \frac{B_2 + Ma_2^2}{Ma_1 a_2} ((1 + e \cos \vartheta) \alpha_2'' - 2e \alpha_2' \sin \vartheta) \\
& + e(1 + e \cos \vartheta)(\alpha_1' + 1)^2 + 2e \sin \vartheta \left(1 - \frac{B_2 + Ma_2^2}{Ma_1 a_2}\right) + 2e \\
& + \frac{3e(A_2 - C_2 - Ma_2^2)}{Ma_1 a_2} = 0,
\end{aligned} \tag{1}$$

The axes of the coordinate systems $O_1 x_1 y_1 z_1$ and $O_2 x_2 y_2 z_2$, are directed along the principal central axes of inertia of the first and the second body, respectively. The orientation of the coordinate system $O_i x_i y_i z_i$ with respect to the orbital coordinate system is determined by the aircraft angles α_i (pitch), β_i (yaw), and γ_i (roll) (see [3]). The index $i = 1$ ($i = 2$) refers to body 1 (body 2).

In equations (1) α_1, α_2 are the aircraft pitch angles of orientation of the axis of inertia of the first and second bodies relative to the orbital coordinate system, ϑ is the true anomaly and e is the orbital eccentricity. Here A_i, B_i, C_i are principal central moments of inertia and $M = M_1 M_2 / (M_1 + M_2)$ is the reduced mass, where M_i is the mass of the i th body. The case where the hinge is located at the

intersection of the principal central axes of inertia of the satellite and stabilizer is considered. Then the coordinates of the spherical hinge P in the coordinate system associated with each body are $(a_1, 0, 0)$, $(a_2, 0, 0)$. The prime in (1) denotes differentiation with respect to ϑ . System (1) admits the zero stationary solution

$$\alpha_1 = \alpha_2 = 0. \quad (2)$$

In an elliptical orbit, a zero equilibrium orientation (2) of the system (1) does not exist due to the non-uniform rotation of the radius vector connecting the centers of mass of the Earth and the two-body system. The equilibrium orientation that exists in a circular orbit transforms into periodic eccentric oscillations in the orbital plane. These eccentric oscillations represent a particular periodic solution of the system of equations (1).

The objective of this work is to construct a periodic solution of the system of equations of motion (1) in the form of power series in a small eccentricity parameter e ($e \ll 1$) in the neighborhood of the zero stationary solution (2) (which we denote by the dynamic equilibrium position) using procedures for symbolic calculations of power series, and to analyze the stability of the zero dynamic equilibrium position in the linear approximation.

Periodic solutions

It can be verified that a general solution of the nonlinear system (1) cannot be found in analytic form. To find solutions to system (1), we use methods of perturbation theory and algorithms for symbolic calculations of power series proposed in [11, 12, 13]. The periodic solution determining the eccentric oscillations of the system of two connected bodies was obtained as an expansion in a series in powers of a small parameter e up to a cubic term:

$$\alpha_i(\vartheta) = e\alpha_i^{(1)}(\vartheta) + e^2\alpha_i^{(2)}(\vartheta) + e^3\alpha_i^{(3)}(\vartheta). \quad (3)$$

The computation of the unknown functions $\alpha_i^{(k)}(\vartheta)$ in (3) is done in accordance with the techniques proposed in [13] and involves complex symbolic computations.

In this study, all symbolic computations were performed using the Wolfram Mathematica environment, specifically using functions such as: `Expand`, `TrigExpand`, `Series`, `Map`, `Simplify`, `Replace`, `DSolve`.

In the present work, we have derived the expressions for the periodic motion of the system in linear, quadratic and cubic approximations.

Stability Analysis

The existence of a periodic solution to the equations of motion (1) means that for a given value of the parameter e one can choose initial conditions such that the system remains in a state of dynamical equilibrium, with the bodies oscillating near the zero stationary position (2).

The stability analysis of the dynamic equilibrium position was conducted according to the following scheme. Using expressions for the kinetic energy and force function, and performing the Legendre transformation, the Hamiltonian H was determined. To study the stability of periodic solutions, new canonical variables q_1, q_2, p_1, p_2 were defined.

Rewriting the Hamiltonian H in the new canonical variables and expanding it into a power series in terms of q_i, p_i up to the second order inclusive yields $H = H_0 + H_1 + H_2$, where H_k is the k th order homogeneous polynomial with respect to the canonical variables q_i, p_i which are considered as small perturbations of the periodic solution. The quadratic part H_2 of the Hamiltonian determines the linearized differential equations of the perturbed motion with periodic coefficients. The perturbed motion of the system is described by a linear system of four differential equations with periodic coefficients.

The behavior of the solutions to this system of linearized equations is governed by their characteristic multipliers, which are the eigenvalues of the monodromy matrix. By calculating the monodromy matrix and analyzing its eigenvalues, we obtain, according to Lyapunov's theory, the stability conditions for periodic solutions in the linear approximation. Symbolic-numerical simulation indicate that this periodic motion is stable in the linear approximation.

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Stable periodic orbits of the equal-mass zero angular momentum three-body problem

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In 2009, Martynova, Orlov and Rubinov investigated the stability of a special 2D initial conditions' domain of the equal-mass zero angular momentum planar three-body problem [1]. Their motivation for considering this particular domain was that the well-known stable periodic orbits consisting of the Figure-8 orbit, the Schubart orbit and the Broucke orbit, belong to it. Based on the results from [1], one may conjecture that the rather large stability regions found in [1] are generated by a large number of stable periodic orbits. Indeed, during the period of 2013-2018, hundreds of new collisionless periodic orbits were discovered in that same initial conditions' domain, many of them shown to be linearly stable [2, 3, 4].

In this presentation, we report the results from our recently conducted high-precision numerical study, which is intended to confirm and expand the knowledge about the 2D domain of initial conditions in [1]. The main results are as follows: four stability regions are established in the domain and 971 verified initial conditions for linearly stable periodic collisionless orbits are found. Many of the identified initial conditions are new ones. The periodic orbits within each stability region exhibit characteristic patterns in their syzygy sequences. Additional computations suggest that these orbits are KAM-stable orbit candidates. The presentation is based on our recent work [5].

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A theory of secular orbital evolution of a binary black hole in a non-spherical stellar cluster

Pavel Ivanov and Alexandre Polnarev

Abstract. In this study we propose a new solution to the so-called "central parsec problem" of the orbital evolution of a supermassive binary black hole at scales of one parsec or less. At these scales, the standard mechanism of orbital decay driven by dynamical friction is inefficient. We discuss the orbital evolution through the formation of highly elongated orbits due to the action of a non-spherical part of gravitational potential of the central stellar cluster followed by their circularisation through the emission of gravitational waves. The non-spherical part of gravitation potential is assumed to be small and mainly determined by distribution of the stars at distances much larger than the orbital semi-major axis. In this approximation it has two non-trivial parts with azimuthal numbers $m = 0$ and 2 . While the former part leads to the well studied orbital dynamics of von Zeipel-Lidov-Kozai type, the latter causes a non-trivial secular evolution, which is studied analytically and numerically.

Introduction

The formation of gravitationally bound supermassive binary black holes (SBBH) is a natural consequence of the process of merging galaxies containing supermassive black holes in their centres, and the subsequent orbital evolution of the supermassive black holes to the centre of the newly formed galaxy due to dynamic friction. However, when the distance between these black holes becomes sufficiently small, order of $0.01 - 1$ pc, dynamic friction becomes inefficient, since the stars efficiently interacting with the SBBH are scattered off by the binary to hyperbolic orbits. On the other hand, at these distances the loss of orbital energy due to emission of gravitational waves (GW) is typically too small to lead to the merging of the system in Hubble time. The problem of the formation of sources of GW emission due to the SBBH merger was called "the central parsec problem." We propose a new possible solution to this problem determined by a potential non-sphericity of the central star cluster (CSC), which can influence the orbital evolution of the SBBH.

In our analysis we assume that SBBH of unequal mass ratio can reach the distances smaller than the radius of influence of the more massive black hole, where it becomes gravitationally bound. At such distances it has an approximately Keplerian orbit that slowly evolves due to the influence of external perturbations caused by a non-spherical part of the gravitational potential the central star cluster (CSC) and other perturbing factors. It is shown that the orbit can acquire large eccentricities as a result of such evolution, corresponding to periastra order of a few gravitational radii of the more massive black hole. In this case, for certain parameters of the the problem, bursts of GW emission during the periastron passages are efficient enough to cause the loss of the orbital energy sufficient for the SBBH merger in Hubble time.

1. Basic assumptions and the main results

As we've mentioned above it is assumed that the mass ratio of the black holes is small. It is also implied that the distribution of the stellar mass inside the radius of influence of the more massive black hole obeys the standard Young's law. In this case, it is expected that the characteristic size of the orbit (say, its semi-major axis) will quickly decrease due to the influence of dynamic friction down to the radius within which the initial mass of the stars is a quantity of the order of the mass of the lighter black hole. Then, it is also expected that, during several dynamic times, the stars close to the orbit will be ejected from the central cluster by the action of the non-stationary gravitational field of SBBH, and the orbital evolution due to dynamic friction will be stalled. A non-sphericity of the part of CSC external with respect to the orbit of the SBBH is considered in the framework of the quadrupole approximation. In this approximation, generally speaking, the non-spherical part of gravitational potential has two significant independent components determined by spherical harmonics with the azimuthal numbers $m = 0$ and 2 . The first component leads to the usual evolution of von Zeipel-Lidov-Kozai type, where a change of angular momentum of the orbit and, accordingly, its eccentricity is constrained by the requirement that the projection of the angular momentum onto the symmetry plane of the orbit is conserved during the secular evolution. In our work we mainly consider the secular orbital evolution due to the presence of the second component of the perturbing potential having the azimuthal number $m = 2$. In the course of such evolution all three components of the angular momentum are not conserved and the orbit can, in principle, become so elongated, that the effects associated with the presence of the GW emission are significant. Since the secular equations, describing the orbital evolution of such a type, do not have any integrals of motions apart from the Hamiltonian, for their analytical analysis we propose a new method of "quasi-conserved integrals of motion." It is based on the observation that one can find exact solutions to the secular equations corresponding to the singular initial value of the orbital inclination, i_0 , tending to zero at the initial time $t = 0$, $i_0 \rightarrow 0^+$. We also assume, for simplicity, that the initial eccentricity e_0 is small

but finite. These 'singular' solutions have the property that the eccentricity e becomes equal to one at some particular time. They can be continued to the later times using the requirement that the nodal and apsidal angles are continuous at the time corresponding to $e = 1$, while the inclination angle i is discontinuous; it experiences a jump from 0^+ to π^- at this time. The evolution at $i \rightarrow \pi^-$ can again be described analytically. It proceeds until another moment of time corresponding to $e = 1$ is reached, where we again assume that the nodal and apsidal angles are continuous, while i jumps from π^- to 0^+ . These singular solutions have a cyclic character. We then treat non-singular solutions having a small, but finite value of i_0 as perturbations over the singular ones. They have a small, but finite, deviation of the maximal eccentricity from one $\propto i_0$. These non-singular solutions can be described in terms of a slow evolution of the initial values of the dynamical variables describing the singular system, which are, by definition, its integrals of motion, and i_0 . In the same way we can also describe the influence of other perturbing factors, most importantly, the influence of Einstein's precession of the apsidal angle. In general, similar to the singular case, the orbital evolution starting with some small values of eccentricity e_0 and i_0 can be divided on four subsequent stages. At first stage i stays sufficiently close to i_0 , while eccentricity grows to values sufficiently close to 1. We then have a sharp transition of i from the small value to a value close to π . When the inclination is equal to $\pi/2$ during this sharp transition the eccentricity passes through a local maximum. Then the eccentricity goes again to a small value close to e_0 , while the inclination stays close to $\pi/2$. Then the eccentricity starts to grow again, until the next sharp transition of inclination from $\sim \pi/2$ to ~ 0 happens. Then the eccentricity decreases again to a value close to the initial one, while the inclination stays close to the small value. Then the cycle repeats. A distinctive feature of these non-singular solutions is that their parameters, such as e_0 , i_0 , the maximal values of eccentricity are slowly changing from one cycle to another, and the cycles are only quasi-periodic. The numerically obtained evolution of inclination and eccentricity during one quasi-periodic cycle is shown in Fig. 1figure.1. We find some mapping relations, describing the small changes of the parameters over one cycle of quasi-periodic evolution. However, a typical duration time of these cycles is always of the order of the von Zeipel-Lidov-Kozai time period. We have checked that this behaviour is in an excellent quantitative agreement with the numerical results. Moreover, it was numerically demonstrated that when the evolution starts with values of e_0 and i_0 order of one, it has some transient periods when the evolution is qualitatively similar to the one described above. When the Einstein precession is significant enough, it may be shown using the arguments based on the conservation of the Hamiltonian of the dynamical system that the deviation of a maximal value of eccentricity from one over many cycles of the evolution is proportional to its typical rate. This result allows us to estimate a typical value of the periastron distance and the role of GW emission on the orbital evolution and find the parameters of SBBH and the star cluster corresponding to SBBH merger over one Hubble time. A distinctive feature of GW sources formed as a result of the action of our effect is that the eccentricity

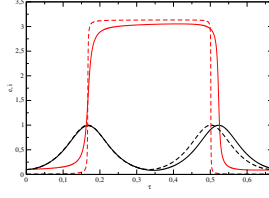


FIGURE 1. The eccentricities e (black lines) and inclinations i (red lines) shown as functions of the 'slow' time τ , which is chosen in such a way, that one quasi-period has always a value of τ order of one. Solid and dashed lines have $i_0 = 0.1$ and 0.01 , respectively.

may remain significant in the course of all stages of the evolution leading to the merger, see [1] for more detail.

Conclusions

We have proposed a new mechanism leading to the solution of the central parsec problem based on a potential non-sphericity of the central star cluster. The main attention was paid to the perturbing gravitational potential with $m = 2$ leading to a non-standard secular evolution. This evolution was treated analytically and numerically. It was shown that the evolution has a quasi-cyclic character and the Einstein precession is, for typical parameters of SBBH and the cluster, the main effect limiting a value of eccentricity over many (say, ~ 100) cycles. We estimated the parameters of SBBH and the cluster corresponding to SBBH merger in one Hubble time.

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The Euler-Lambert problem of determining the orbit of the body based on its two positions

Alexey Ivanyukhin and Vyacheslav Ivashkin

Abstract. The Euler–Lambert problem is a classical problem in celestial mechanics concerned with determining the orbits of celestial bodies. A wide range of solution methods have been developed. Most of these methods rely on the fact that the trajectories connecting two points in space form a one-parameter family of orbits that link the specified points. This paper presents a concise overview of approaches to solving the problem, spanning from the 18th century — when the problem was first systematically studied — to the present day. Particular emphasis is placed on the description of the Okhotsimsky–Egorov method, which was developed by the authors on the basis of Okhotsimsky’s formula for solving the Euler–Lambert problem as an inverse dynamics problem. A comparative analysis of the computational efficiency of several widely used methods is also provided.

Introduction

The Euler–Lambert problem is a classical problem in celestial mechanics concerned with determining the orbits of celestial bodies. Traditionally, this problem is formulated as a two-point boundary value problem within the two-body model, in which the position vectors at two distinct time instants are known.

In practice, solving the Lambert problem does not require solving the boundary value problem directly, since a complete set of first integrals for this problem is known. These integrals enable the reduction of the solution to a single nonlinear equation. The form of this equation depends on the formulation of Kepler’s equation relating the orbital position and time.

Initially, the objective of solving this problem was to determine the orbits of solar system bodies from two observations. In practical astronautics, the problem has acquired new significance and is often used as a basis for designing spacecraft transfer trajectories. In this context, it is assumed that the orbit connecting two given positions represents the transfer trajectory.

Currently, the problem is frequently employed at the initial stage of mission design to identify suitable launch dates and transfer durations. Moreover, it serves as a foundation for developing more sophisticated methods for solving trajectory problems involving n -impulse maneuvers, multiple gravity-assist fly-bys, and visits to several targets. Consequently, developing an efficient solution algorithm with an explicit determination of the velocity along the transfer orbit remains a relevant task.

1. Review of solution methods

Over the long history of the problem, several dozen solution methods have been proposed. The majority of these methods are based on the fact that trajectories between two points in space form a one-parameter family of orbits connecting the given points.

The earliest algebraic-geometric methods date back to the works of Newton, Euler, Lambert, Lagrange, and Gauss [?, ?]. They are distinguished by their clarity and transparency of approach. However, they exhibit high computational complexity and significant numerical error. Arguably, the first computationally efficient method of its time was Gauss's method, as it spawned an entire direction of development that remains active to this day.

Modern methods for solving the Lambert problem can be classified according to the type of nonlinear equation employed, the techniques used for its solution, and the rule for selecting the initial guess. Competition among approaches now occurs primarily at the level of expression compactness, computational complexity, and ease of use in applied problems.

Broadly, the main groups of methods can be divided into the following categories:

- Methods based on the classical form of Kepler's equation, employing iterations in one of the Keplerian elements (typically semi-major axis, eccentricity, focal parameter, or anomaly) [?, ?, ?, ?];
- Methods based on the universal form of Kepler's equation (with a universal variable) [?, ?];
- Methods using regularized variables in the Kustaanheimo–Stiefel (K–S) formulation [?];
- Methods based on the geometry of the eccentricity vector [?];
- Methods based on the inverse ballistic problem [?, ?, ?, ?].

2. Okhotsimsky–Egorov Method

In particular, the one-parameter family of orbits connecting two given points is generated by varying the flight path angle (the angle between the velocity vector and the local horizontal) at the initial point. This ballistic formulation without constraints on transfer time was thoroughly examined by D.E. Okhotsimsky [?].

For each specified value of the initial flight path angle, the velocity—and consequently all orbital parameters—is determined from Okhotsimsky's equation [?]. Thus, the problem of determining the transfer orbit for a given time reduces to determining the initial flight path angle. This scheme by Yegorov for finding a solution based on Okhotsimsky's equation defines the general idea of the Okhotsimsky–Egorov method.

The authors have considered the solution of the Euler–Lambert problem based on the ballistic approach of Okhotsimsky–Egorov as an inverse ballistic problem [?]. An efficient algorithm for solving the problem based on Lagrange's time equation [?] has been proposed. Convergence rate estimates demonstrate that when using Halley's method for determining the solution, the typical number of iterations is approximately 5–7 for all solution types, which represents a very favorable performance for Euler–Lambert problem algorithms.

The advantages of this approach include the bounded and well-structured solution domain, the overall transparency of the method due to the use of physically intuitive trajectory parameters, and the explicit dependence of the solution on the initial velocity vector. These features enable effective integration of this solution scheme into the optimization of impulsive trajectories and interplanetary mission design.

Conclusion

This work continues the development of the Okhotsimsky–Egorov method. An attempt is made to determine the place of the solution technique based on the inverse ballistic problem within the general structure of methods for solving the Euler–Lambert problem.

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The authors congratulate Elena Nikolaevna Palyakhova on her anniversary and wish her good health and continued creative success.

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Low-Thrust Inclination Control on Elliptical Orbits with Constant Orbital Period

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Abstract. This study addresses the problem of low-thrust control aimed at modifying the orbital geometry while maintaining the orbital period. More specifically, the control strategy focuses on altering the orbital eccentricity and inclination while keeping the semi-major axis constant. This problem formulation is relevant for combined insertion schemes into geostationary and high elliptical orbits. The problem is analogous to Edelbaum's problem, but it differs significantly due to a more complex strategy for changing the orbital geometry. New analytical feedback control laws have been derived for the considered problem. The efficiency of the obtained control laws has been evaluated.

Introduction

Many spacecraft employ low-thrust propulsion systems (electric thrusters) for interorbital transfers, orbit maintenance, and correction in near-Earth space. Combined launch strategies utilizing both high-thrust and low-thrust propulsion for delivering spacecraft to target orbits are of significant interest, as such transfer schemes can substantially increase the spacecraft mass delivered to the target orbit while maintaining acceptable transfer durations. In practical implementations of combined transfer schemes, cases with a constant orbital period are of particular interest.

A key feature of spacecraft control with low-thrust propulsion systems is the significant dominance of gravitational acceleration over thrust acceleration, typically differing by several orders of magnitude. This low acceleration level results in long transfer durations and necessitates determination of the control at each instant.

The analysis of interorbital transfers and combined launch strategies requires the development of efficient and simple control laws for executing typical orbital

maneuvers with low-thrust propulsion. A number of special cases admitting analytical solutions are known, including acceleration with transverse or radial thrust [?]. Of significant practical importance is the solution obtained by Edelbaum for transfers between non-coplanar circular orbits [?].

1. Optimal Control Problem

For qualitative analysis of orbital evolution under low-thrust acceleration, it is convenient to employ the planetary Lagrange equations [?]. In this study, the averaged equations of motion in Gauss form are considered, and an analysis of secular variations of orbital elements under low-thrust acceleration is conducted in both orbital and Cartesian coordinate systems.

Following the ideas of Spitzer and Pollard [?, ?], optimal control problems for the yaw angle are formulated for transfers between elliptical orbits while maintaining a constant value of the semi-major axis, with the objectives of minimizing transfer time and maximizing final spacecraft mass. In contrast to the known works [?, ?] and similar studies, continuous yaw angle control and a long active arc per revolution are considered.

The solution of the optimal control problem is reduced to an isoperimetric problem, analogous to the Edelbaum problem [?]. The formulations considered involve a more complex strategy for modifying the orbital geometry. For the time-optimal case, a closed-form solution in terms of Legendre elliptic integrals has been obtained, while for the problem of maximizing final mass, a numerical-analytical solution requiring neither numerical integration nor optimization has been derived.

Conclusion

New analytical feedback control laws have been derived for the problems under consideration. Efficiency assessments of the obtained control laws demonstrate a reduction in the required ΔV by 10 % compared to analogous, previously known control laws.

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The system of Ephemeris Research in Astronomy ERA, Ephemeris of Planets and the Moon EPM: current status and future plans

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Abstract. This paper presents the current status of work on the new version of the EPM2026 solar system ephemerides and plans for modernizing the astronomical ephemeris calculation system ERA. The main differences between the EPM2021 and EPM2026 ephemerides are outlined, including the set of adjusted dynamic model parameters and the database of radio, optical, and light-ranging observations used. A programming language for the planned ERA system update is selected, and a plan for automating this work is outlined.

Introduction

In December 1985, the first experimental versions of the ERA system (Ephemeris Research in Astronomy) and the object-oriented SLON language were presented on the BESM-6 computer [1, 2]. The first versions of the ERA (ERA-5) system and the SLON language, which were ready for use in practical scientific work, appeared only 10 years later [3]. In 2015, the ERA system was redesigned [4]. During the redesign, the source code was rewritten from Pascal to Racket (for running programs written in the SLON language, data management, and much of the computing) and C (libraries for computing).

EPM ephemeris (Ephemeris of Planets and Moon) has been produced for more than 20 years and contains information about the position of the planets of the Solar system, the Sun, the Moon, five asteroids (Ceres, Pallas, Vesta, Iris, Bamberga) and five TNO (Pluto, Eris, Makemake, Haumea, Sedna). As well as TT-TDB and Euler angles for lunar physical libration. The ERA system and the database of observations of Solar system bodies are used to create

EPM ephemerides. To date, several versions of EPM ephemerides have been released: EPM2004, EPM2008, EPM2011/m, EPM2015, EPM2017 and EPM2021. The EPM ephemerides cover a time interval of more than 400 years (1787-2214).

Current status of EPM ephemerides

In recent years, a number of works have been published on the development of the ERA system and the construction of new versions of EPM ephemerides [5, 6]. Based on these works and the latest version of the ERA system, the next version of the EPM ephemeris, EPM2026, is planned to be released this year.

Some time ago, due to the geopolitical situation, we had limited access to high-precision and most informative radio observations of spacecraft for building ephemeris of planets. For this reason, the main addition to the measurement database was the re-processed observations of planets and satellites made at the Pulkovo Observatory from 1930 to 2018 [7]. To reduce the correlations of the specified parameters, the EPM2026 ephemeris model will be assigned to the reference Julian date 2453371.5, instead of the date 2446000.5 used in the EPM2021 ephemeris.

Access to the Lunar Laser Ranging (LLR) observations remains available for the time being. To compile the lunar section of the ephemeris, we processed new LLR measurements up to 2025. In addition, the first LLR observations of the NGLR-1 retroreflector, obtained by the Apache Point Observatory Lunar Laser-ranging Operation (APOLLO) and Wettzell stations, have been analysed. One of the results of the analysis was the refinement of the coordinates of the NGLR-1 retroreflector, which are in good agreement with the results obtained by our Chinese colleagues [8], calculated using the American and French ephemerides DE440 and INPOP21a, respectively. The rates of change in the coordinates of LLR stations were also added to the list of parameters to be refined, which has improved the representation of LLR observations in the EPM ephemerides.

Modernization of the ERA system

When developing the latest, eighth, version of the ERA system, a number of decisions were taken which, under our circumstances, prevent this software product from being developed to its full potential and, as a result, hinder further scientific research. In addition to the difficulties involved in maintaining the source code written in Racket without the involvement of qualified programmers, a whole range of further problems arose. First and foremost, these include the relatively slow execution speed of calculations using Racket compared with the more widely used programming languages C/C++ and Fortran, as well as the removal of support for 80-bit variables in newer versions of Racket, which is critical when performing high-precision astronomical calculations. Consequently, when work began on modernising the ERA system, a choice had to be made: either to continue using

one of the latest versions of Racket that supported 80-bit variables, with the associated risks, or to switch to other programming languages that would support the required computational precision in their current versions for the foreseeable future. Taking into account the experience gained and the length of time the ERA system and other software products in the field of ephemeris astronomy have been in use, as well as the testing carried out, the programming languages C/C++ and Fortran were selected [9].

Currently, work has begun on porting the main computing modules of the ERA system to the C/C++ programming language. As well as maintaining the required accuracy of ephemeris calculations, the use of this language will significantly reduce computation time and lay the groundwork for improving their accuracy. There are plans to speed up the porting of computational modules from Racket to C/C++ by using AI. It is also envisaged that AI will be used to create a suite of unit tests, which should reduce risks and accelerate the forthcoming modernisation of the ERA system.

Conclusion

By the end of 2026, a new version of the EPM ephemeris, EPM2026, is being prepared for release based on an updated database of optical and lunar laser ranging observations of Solar System bodies, as well as an expanded set of dynamic model parameters. For the planned ERA system update, the C/C++ programming language has been selected, and a plan has been outlined to automate the conversion of the system's main computing modules from Racket to C/C++.

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Cosmology on the extended spacetime

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Abstract. We consider a multidimensional cosmological model with spacetime $M^4 \times SU(2)$, where the extra dimensions form the Lie group $SU(2)$ equipped with a bi-invariant metric. The Riemannian curvature of $SU(2)$ contributes to the scalar curvature of the total manifold, naturally generating the cosmological constant $\Lambda = 3/b_0^2$, where b_0 is the stabilized scale factor of the internal space. The generalized Friedmann equations are reduced to a form suitable for numerical integration and solved.

We also propose the concept of a *quasiperiodic time manifold*. It is generally accepted that if the time is a coordinate on a compact manifold then it is cyclic and there is the problem of closed timelike paths. But the time coordinate can also be defined as a coordinate on the *universal covering space of a compact manifold* G . This eliminates closed timelike curves without changing the metric. Hence the spacetime of the general relativity theory becomes $G \times \mathbb{R}^3$ with the covering space $\mathbb{R}_1^4$.

Keywords: Kaluza–Klein cosmology, Supernova Ia, Dark energy, Cosmological constant.

1. Introduction

The accelerated expansion of the Universe is a central problem in modern cosmology. Within the standard Λ CDM model, this acceleration is attributed to a cosmological constant Λ , which is treated as a free parameter. Multidimensional Kaluza–Klein theories offer a geometric explanation for Λ arising from the curvature of extra dimensions [1]. In this work, we consider the spacetime $M^4 \times SU(2)$, where M^4 is the Friedmann–Lemaître–Robertson–Walker (FLRW) space and the extra dimensions form the compact Lie group $SU(2)$ (the 3-sphere) with a bi-invariant metric. It was previously shown [2] that the curvature of $SU(2)$ yields a nonzero contribution to the scalar curvature, exactly corresponding to Λ .

A known issue in higher-dimensional cosmologies is the collapse of extra dimensions as in the work of D. Sahdev [4], where the internal scale factor $b(t)$ inevitably tends to zero. In our model, the intrinsic geometry of $SU(2)$ provides a

natural stabilization mechanism, allowing $b(t)$ to asymptotically approach a finite value b_0 , thereby generating the effective cosmological constant.

The objectives of this work are: (i) to reduce the generalized Friedmann equations to a form suitable for numerical integration; (ii) to study the dynamical stabilization of the internal dimensions; (iii) to estimate cosmological parameters from Type Ia supernova data using a nonlinear regression method.

2. Generalized Metric and Friedmann Equations

The FLRW metric is

$$ds_{\text{FRW}}^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right), \quad (1)$$

and the bi-invariant metric on $SU(2)$ is

$$ds_G^2 = b(t)^2 (d\xi^2 + \sin^2 \xi d\psi^2 + \sin^2 \xi \sin^2 \psi d\zeta^2), \quad (2)$$

where $a(t)$ and $b(t)$ are the scale factors of the four-dimensional space and the internal space, respectively. The total metric is $ds^2 = ds_{\text{FRW}}^2 + ds_G^2$.

The stress-energy tensor is assumed to be

$$T_\nu^\mu = \text{Diag}(\rho, -P, -P, -P, \sigma, \sigma, \sigma), \quad (3)$$

where ρ is the mass density, P is the pressure, and σ is the density associated with the extra dimensions. From the Einstein equations, we obtain the generalized Friedmann equations [2]:

$$\frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{c^2 a^2} + \frac{9\dot{a}\dot{b}}{c^2 ab} + \frac{3\dot{b}^2}{c^2 b^2} - \frac{3}{b^2} = \varkappa \rho, \quad (4)$$

$$\frac{\kappa}{a^2} + \frac{\dot{a}^2}{c^2 a^2} + \frac{2\ddot{a}}{c^2 a} + \frac{6\dot{a}\dot{b}}{c^2 ab} + \frac{3\ddot{b}}{c^2 b} + \frac{3\dot{b}^2}{c^2 b^2} - \frac{3}{b^2} = -\varkappa P, \quad (5)$$

$$\frac{3\kappa}{a^2} + \frac{3\dot{a}^2}{c^2 a^2} + \frac{3\ddot{a}}{c^2 a} + \frac{6\dot{a}\dot{b}}{c^2 ab} + \frac{2\ddot{b}}{c^2 b} + \frac{\dot{b}^2}{c^2 b^2} - \frac{1}{b^2} = \varkappa \sigma. \quad (6)$$

The consistency condition from the Bianchi identity yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) + 3\frac{\dot{b}}{b}(\rho - \sigma) = 0. \quad (7)$$

Assuming $P = (\gamma - 1)\rho$ and $\sigma = (1 - \gamma_1)\rho$, we obtain

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0} \right)^{-3\gamma} \left(\frac{b}{b_0} \right)^{-3\gamma_1}. \quad (8)$$

In what follows, we take $\gamma = 1$ (dust-like matter).

3. Special Solution: $b = \text{const}$ and the Cosmological Constant

If $b \equiv b_0 = \text{const}$, equation (4) reduces to

$$\frac{\dot{a}^2}{c^2 a^2} = \frac{\varkappa}{3} \rho + \frac{1}{b_0^2} - \frac{\kappa}{a^2}. \quad (9)$$

Comparing with the standard Λ CDM Friedmann equation,

$$\frac{\dot{a}^2}{c^2 a^2} = \frac{\varkappa}{3} \rho + \frac{\Lambda}{3} - \frac{\kappa}{a^2},$$

we obtain

$$\Lambda = \frac{3}{b_0^2}, \quad b_0 = \sqrt{\frac{3}{\Lambda}}. \quad (10)$$

Thus, the cosmological constant is naturally determined by the stabilized scale factor of the extra dimensions. The dark energy density parameter is

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} = \frac{c^2}{H_0^2 b_0^2} = 1 - \Omega_m - \Omega_\kappa. \quad (11)$$

4. Numerical Solutions and Stabilization of Extra Dimensions

To solve the system numerically, we rewrite equations (4) and (5) in terms of the dimensionless Hubble parameter. From (4):

$$\frac{\dot{a}}{a} = \sqrt{\frac{\varkappa \rho c^2}{3} - \frac{\kappa c^2}{a^2} + \frac{c^2}{b^2} + \frac{5\dot{b}^2}{4b^2} - \frac{3}{2} \frac{\dot{b}}{b}}. \quad (12)$$

The second derivative of b is

$$\ddot{b} = \frac{1}{5} \left(-\frac{2H_0^2 \Omega_\kappa b^2 \dot{a}}{a^3 \dot{b}} + \frac{3b(6\dot{a}^2 - 7H_0^2 \Omega_\kappa)}{a^2} + \frac{3b(2\gamma_1 - 7)H_0^2 \Omega_m}{a^3} \left(\frac{b}{b_0} \right)^{-3\gamma_1} - \frac{8(c^2 - \dot{b}^2)}{b} + \frac{39a\dot{b}}{a} \right). \quad (13)$$

The initial conditions are set at the recombination epoch: $t_{\text{start}} = 0.00038$ billion years, $a_{\text{start}} = 1/(1 + 1089) = 0.000917$. The initial value b_{start} is chosen such that $b(t_0) = b_0$ at the present epoch.

We distinguish three cases based on the spatial curvature Ω_κ :

1. **Positive curvature** ($\Omega_\kappa > 0$): The function $b(t)$ decreases monotonically and asymptotically approaches b_0 . The derivative $\dot{b}$ reaches zero at finite time t_1 , where the solution must be glued to the special solution $b \equiv b_0$. This corresponds to the Λ CDM phase with accelerated expansion.
2. **Flat Universe** ($\Omega_\kappa = 0$): The function $b(t)$ first decreases, then increases. The derivative changes sign without singularities, but the expansion of $a(t)$ is not accelerated. This case is not cosmologically interesting.

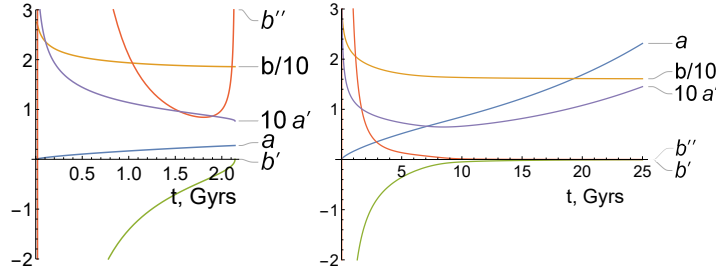


FIGURE 1. Scale factors $a(t)$, $b(t)$ and their derivatives for $\Omega_\kappa = 0.1$ (left) and $\Omega_\kappa = -0.1$ (right).

3. **Negative curvature** ($\Omega_\kappa < 0$): The function $b(t)$ decreases monotonically and asymptotically approaches b_0 as $t \rightarrow \infty$. The expansion is accelerated in the current epoch.

For $\Omega_\kappa < 0$, the asymptotic behavior is exponential:

$$b(t) - b_0 \sim e^{-ct/b_0}, \quad a(t) \sim e^{ct/b_0}.$$

Parameter Estimation for this 3-dimensional parameter space can be done based on the Type Ia Supernova Data [5, 6].

We can also propose the concept of a *quasiperiodic time manifold*. It is generally accepted that if the time is a coordinate on a compact manifold then it is cyclic and there is the problem of closed timelike paths. But the time coordinate can also be defined as a coordinate on the *universal covering space of a compact manifold* G . This eliminates closed timelike curves without changing the metric. Hence the spacetime of the general relativity theory becomes $G \times \mathbb{R}^3$ with the covering space $\mathbb{R}_1^4$ (with the classical Lorentz signature) [3].

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Total Effect of the Earth’s Gravitational Field on the Tide-Generating Potential

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Extended abstract

The Earth tide-generating potential (TGP) is one of the most important potentials of celestial mechanics. It is used, e.g., in theories of Earth nutation, developments of solid and ocean tides, analysis of geodetic and gravimetric observations, etc.

In the current presentation exact formulae are given for calculating a new part of the Earth TGP caused by a joint action of external gravitating bodies (the Moon, the Sun, major planets, ...) and an *arbitrary* spherical harmonic of the geopotential expansion. This extends the results of the previous studies [1, 2, 3], where only the effect of the second-degree geopotential coefficients was considered. The formulae explicitly separate some time variables (named the “TGP coefficients”), which are functions of the geopotential harmonics, masses and instantaneous positions of the external bodies, from the geographical coordinates of an observation point on (or above) the Earth’s surface, where the TGP is evaluated.

By using these formulae we calculate the TGP coefficients over a time interval of 30,000+ years (-13,200–17,191) with a small sampling step of one day and then expand them in accurate harmonic series. The modern values for the masses and instantaneous positions of the Moon/the Sun/major planets provided by the recent long-term numerical ephemeris DE441 [4] are used as the source. An author’s modification of the spectral analysis method [5, 6] is applied as the expansion tool. It allows one to represent an arbitrary tabulated function of planetary/lunar coordinates by a generalized harmonic series, where both amplitudes and arguments of the series’ terms are high-degree polynomials of time.

As a result, new terms in the Earth TGP development are found. They are due to the geopotential coefficients of the third degree. The minimum amplitude of terms in the obtained harmonic series was chosen equal to $1 \times 10^{-9} \text{ m}^2/\text{s}^2$. It is by an order of magnitude less than the similar threshold employed by the latest TGP catalogues HW95 [7] and KSM03 [5]. The use of the thus decreased threshold and the modern planetary/lunar ephemeris allowed us to also improve the already

known terms in the Earth's TGP harmonic expansion caused by the geopotential coefficients of the second degree and increase their total number by a factor of two.

The new and updated terms are available on the web-server of SAI MSU. They are presented in two formats: KSM25 (http://lnfm1.sai.msu.ru/neb/ksm/tgp/k25gp_format-ksm25.dat, see the format description on http://lnfm1.sai.msu.ru/neb/ksm/tgp/readme_format-ksm25.pdf) and a generalized HW95 (http://lnfm1.sai.msu.ru/neb/ksm/tgp/k25gp_format-hw95gen.dat). The former format allows one to represent the TGP coefficients over long intervals of time (thousands and tens of thousands of years), and the latter is currently used in some popular software packages for processing tidal measurements, e.g., ETERNA [8].

A new type of the “permanent tides” is revealed (the permanent tides on the Earth's surface are caused by the constant terms in the harmonic developments of some TGP coefficients). It is originated from the tesseral geopotential coefficients of order 1 (mostly, c_{31} , s_{31}), and the corresponding permanent tides depend on not only the geographical latitude of the observation site (like any other permanent tide found before), but also on the site longitude.

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Selecting Orbits for Safe Storage of Space Debris in the GNSS Region

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Abstract. The problem of selecting disposal orbits for global navigation system satellites is considered. Averaged analytical models were used to interpret the results of numerical simulation. The dynamic evolution of objects in the studied region is determined by the competition between two resonances. Luni-solar resonances, dependent on the inclination, lead to an increase in orbital eccentricity. The secondary secular apsidal-nodal resonance, caused by the influence of solar radiation pressure, in the vicinity of the stationary point limits the variation of eccentricity. It is recommended to choose disposal orbits from 100 to 500 km higher on the semi-major axis with respect to corresponding navigation constellation orbits. By suitable positioning of the orbital plane with respect to the direction to the Sun, it is possible to achieve a minimal variation in eccentricity over a long-time interval of up to 200 years for area-to-mass ratios less than $1 \text{ m}^2 \text{ kg}^{-1}$.

Introduction

Typically considered two possible disposal strategies for the GNSS (Global Navigation Satellite System) region involving either stable or unstable orbits from the eccentricity growth point of view [1]. Unstable scenario seems favorite from point of view the possibility to dilute the collision risk and to aim at the re-entry in the atmosphere. Stable scenarios seem to minimize the interactions (crossings) with the operational constellations and, therefore, might be preferred for operational reasons. The aim of the study is to justify the choice of disposal orbits within the framework of the Stable scenario. Requirement for disposal orbits is that objects in disposal orbits should not enter the areas of motion of GNSS and IGSO (Inclined Geosynchronous Orbits) satellites over tens or hundreds of years (up to 200 years). During this time, effective systems for the active debris removal (ADR) from disposal orbits must be developed.

1. Methodology

Our research is based on interactions between analytical methods and numerical simulations. We use numerical simulation to accurately predict the motion of objects in the medium Earth orbit (MEO) region. Analytical methods are used to analyze and interpret the results of numerical modeling, justify the choice of disposal orbits and their stability.

The theoretical part of our research is based on the following fundamental results. Firstly, the theory of the secular luni-solar resonances dependent on inclination [2, 3]. Secondly, the theory of the secondary secular apsidal-nodal resonances due to solar radiation pressure [4, 5]. Thirdly, consideration of orbital evolution under the influence of solar radiation pressure in the vicinity of a stationary point on the "eccentricity – longitude of perigee" plane [6].

The dynamic evolution of space debris in the vicinity of the area of motion of satellites of global navigation systems is studied based on numerical simulation. The orbital evolution of space objects is modeled in "Numerical Model of the Motion of an Artificial Satellites" [7] developed at the Tomsk State University. The model of perturbing forces takes into account the major perturbing factors:

- the gravitational field of the Earth (EGM96 model, harmonics up to the 29th order and degree, inclusive, for the correct description of the movement of objects in eccentric orbits and before re-entering),
- the gravitation of the Moon and the Sun,
- the tides of the Earth,
- the direct solar radiation pressure including the eclipses due to the Earth's shadow,
- the Poynting-Robertson effect,
- the atmospheric drag.

The equations of motion are integrated by Everhart's method of the 19th order. The coordinates of the Moon and the Sun during the numerical integration of the equations of motion are extracted from the JPL Planetary and Lunar Ephemerides DE405/LE405.

We used several sets of initial data to study the dynamic evolution of objects in the vicinity of the GNSS region. We chose initial conditions for the space objects in MEO near the GNSS regions (GLONASS, GPS, BeiDou, and Galileo). We varied the initial values of the semi-major axis and the longitude of the ascending node. The initial value of the eccentricity corresponds to a near-circular orbit with $e_0 = 0.0001$. The initial value of the argument of perigee is 270° . The area-to-mass ratio A/m is equal to 0.001, 0.01, 0.1, and $1 \text{ m}^2 \text{ kg}^{-1}$ from values corresponding to satellites to values corresponding to space debris. The orbital evolution covers 200 years.

2. Results

One of the two resonances, either the secular luni-solar resonance or the secondary secular apsidal-nodal resonance, dominates and determines the evolution of the orbital eccentricity for a given initial value of the longitude of the ascending node. Luni-solar resonances, dependent on inclination, operate in all regions of the GNSS. These resonances lead to increased orbital eccentricity. For this reason, regions dominated by luni-solar resonances are excluded from the disposal orbit analysis for the Stable scenario from the eccentricity growth point of view.

The secondary secular apsidal-nodal resonance due to solar radiation pressure ensures evolution in the vicinity of a stationary point, which limits the growth of the orbital eccentricity. The areas of dominance of secondary secular apsidal-nodal resonances are suitable for the selection of disposal orbits within the Stable scenario. Space debris, including objects with high ratios A/m , could potentially accumulate in the vicinity of this region.

We propose placing the end-of-life GNSS satellites in orbits from 100 to 500 km higher on the semi-major axis with respect to corresponding operational orbits. By suitable positioning of the orbital plane with respect to the direction to the Sun, it is possible to achieve a minimal variation in eccentricity over a long-time interval of up to 200 years for area-to-mass ratios $A/m < 1 \text{ m}^2 \text{ kg}^{-1}$.

Conclusion

Interaction between analytical methods and numerical simulation made it possible to interpret the results of numerical modeling and determine the initial values of the semi-major axes and longitudes of the ascending nodes of the disposal orbits for each of the global navigation satellite systems. The evolution of objects in MEO is controlled by luni-solar resonances and secondary secular apsidal-nodal resonance caused by solar radiation pressure. In combination with the evolution in the vicinity of the stationary point, this resonance provides a limited variation of eccentricity, which is necessary to ensure the Stable disposal scenario. It is recommended to select disposal orbits in areas where secondary secular apsidal-nodal resonances caused by solar radiation pressure dominate.

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Birkhoff conjecture and finite energy foliations in Hill's lunar problem

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Abstract. We prove Birkhoff's retrograde orbit conjecture in Hill's lunar problem by showing that the retrograde orbit bounds a disk-like global surface of section for every energy below the critical value. We also obtain a global description of the dynamics through the critical energy level by constructing finite energy foliations for energies slightly above it. The binding of these foliations consists of the retrograde orbit together with the Lyapunov orbits near the critical points. As a consequence, there exist infinitely many periodic orbits and infinitely many trajectories asymptotic to the Lyapunov orbits. The proof combines pseudo-holomorphic curve techniques with a new convexity theorem for Hill's lunar problem. More precisely, we construct an explicit global symplectic change of coordinates under which the bounded regularized component becomes strictly convex up to the critical energy level. This convexity implies strong dynamical consequences, including lower bounds for the Conley-Zehnder indices of periodic orbits, and allows the application of the Hofer-Wysocki-Zehnder theory of finite energy foliations. As a result, we obtain disk-like global surfaces of section below the critical level and $2-3-2$ foliations for energies slightly above it.

Introduction

The restricted three-body problem has a central place in celestial mechanics since the works of Euler, Lagrange, Poincaré, and Birkhoff. In its planar circular version, one studies the motion of a massless body under the gravitational influence of two primaries moving on circular orbits around their center of mass. Despite its simple formulation, the problem has a wide variety of dynamical phenomena, including periodic orbits, homoclinic intersections, chaotic scattering, and transport through the neck regions near the Lagrange equilibrium points.

One of the most influential models arising from the restricted three-body problem is Hill's lunar problem. Introduced by George W. Hill (1877,1878), the

model was developed as part of Hill's program to obtain a precise analytical description of the motion of the Moon under the perturbative influence of the Sun. Rather than studying the full three-body problem directly, Hill derived a limiting system describing the dynamics near the Earth after a suitable rescaling procedure. The resulting equations contain the principal nonlinear effects of the solar perturbation while presenting a considerably simpler structure.

Hill's work represents one of the major achievements of nineteenth-century celestial mechanics. Using Fourier series and numerical calculations, he obtained highly accurate approximations of lunar periodic motions and set the foundations of a new lunar theory. His methods influenced generations of mathematicians and astronomers, and his lunar theory remained the standard reference for many years. See Brown(1896) and Wintner(1941) for historical accounts and further developments. A modern discussion of Hill's original lunar theory, including the convergence of the associated series expansions, was given by Ligon(2025).

For much of the twentieth century, progress on Hill's lunar problem was driven primarily by perturbative methods, normal form techniques, and numerical investigations. Important advances were obtained through the study of invariant manifolds, periodic trajectories, and chaotic dynamics near the equilibrium points. In particular, the numerical investigations of Simó and Stuchi(2000) revealed a rich global structure involving Lyapunov orbits, stable and unstable manifolds, homoclinic and heteroclinic trajectories, and complicated transport phenomena near the critical energy level. The problem is known to be non-integrable Meletlidou, Ichtiaroglou, and Winterberg(2001), Morales-Ruiz, Simó, and Simon(2005) and transverse intersections of invariant manifolds produce chaotic recurrent dynamics in the sense of Poincaré. See Conley(1968), McGehee(1969), Simó and Stuchi(2000) and references therein.

From the perspective of Hamiltonian and symplectic dynamics, Albers, Frauenfelder, Paternain and van Koert(2012) proved that the regularized energy surface below the critical value is fiberwise star-shaped. In particular, the regularized dynamics naturally defines a Reeb flow on a contact three-manifold. Later, Lee(2017) proved fiberwise convexity of the regularized energy surface below the critical value, relating the dynamics to a Finsler geodesic flow on the two-sphere. Quantitative aspects of the dynamics for very negative energies were studied in Grotta-Ragazzo, L. and Salomão(2025) using Birkhoff normal forms and symplectic techniques. Also, recent works Aydin(2023,2024) combined numerical investigations with techniques from symplectic dynamics to study bifurcating families of periodic solutions in the spatial Hill problem.

The development of pseudo-holomorphic curve techniques by Hofer(1993) and Hofer, Wysocki and Zehnder(1995-2023) introduced powerful new methods for studying global properties of three-dimensional Hamiltonian dynamics. In particular, finite energy foliations organize the dynamics of Reeb flows through families of pseudo-holomorphic curves in symplectizations. These methods led to the construction of global surfaces of section and finite energy foliations in several situations, with strong dynamical consequences.

One of the central open questions in Hill's lunar problem concerns the existence of global surfaces of section. Birkhoff(1915) conjectured that the retrograde orbit bounds a disk-like global surface of section on the regularized energy surface below the critical value. In particular, it provides a complete reduction of the dynamics on the energy surface to a two-dimensional conservative system and implies the existence of a "direct" periodic orbit as a fixed point of the return map.

Conclusion

In this presentation, we will resolve Birkhoff's conjecture and gives the first global geometric description of Hill's lunar problem through the critical energy level. Our first main result proves that the retrograde orbit bounds a disk-like global surface of section for every energy below the critical value. Our second main result concerns energies slightly above the critical value. In this case, we construct $2 - 3 - 2$ finite energy foliations whose binding consists of the retrograde orbit together with the Lyapunov orbits near the critical points. These foliations provide the first global description of the dynamics in a neighborhood of the critical energy level. In particular, they organize the dynamics by families of planes and cylinders transverse to the flow and imply the existence of infinitely many periodic orbits and infinitely many trajectories asymptotic to the Lyapunov orbits.

The results in this paper complete a program initiated by Birkhoff more than a century ago and further demonstrate the effectiveness of the Hofer-Wysocki-Zehnder theory of pseudo-holomorphic curves and finite energy foliations in the study of classical problems from celestial mechanics.

0.1. Ingredients of the proofs

The key ingredient in the proofs is a convexity theorem for the bounded regularized component of the energy surface. More precisely, we show that after a suitable symplectic change of coordinates, the bounded component becomes strictly convex up to the critical energy level. This convexity implies lower bounds for the Conley-Zehnder indices of periodic orbits and provides the dynamical conditions required for the theory of pseudo-holomorphic curves. The resulting finite energy curves give rise to open book decompositions by disk-like global surfaces of section below the critical level, and $2 - 3 - 2$ foliations for energies slightly above it, see Figure 1. To obtain such a foliation for the energy surfaces, we apply the a theorem from [1], which is a general criterion for the existence of finite energy foliations with prescribed binding orbits.

The convexity properties established in this paper appear to have applications beyond Hill's lunar problem. In particular, the symplectic coordinates constructed here naturally extend to the regularized planar restricted three-body problem for sufficiently small mass ratio μ . This suggests a possible approach to constructing finite energy foliations for energy levels ranging from the first to slightly above the second Lagrange value. Such foliations would provide a global description of the dynamics across the neck regions and will be investigated in future work.

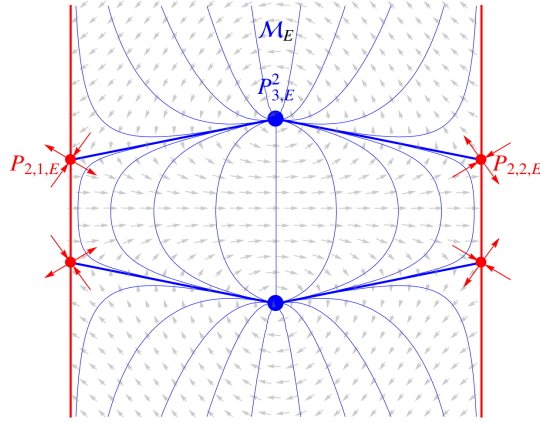


FIGURE 1. The 2 – 3 – 2 foliation in the Hill's lunar problem

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On the analysis of the operation of a double electrostatic layer in the ion sail mode

A.S. Lozovoi, K.A. Orekhov, K.G. Petrov,
A.A. Tikhonov and A.B. Yakovlev

Abstract. The paper presents analytical estimates and numerical calculations of the pressure on a double electrostatic layer in the ion sail regime in low-Earth orbits and in interplanetary space. An estimate of the leakage current density for the system under consideration is also given.

Introduction

A logical step in the plans for further space exploration was the development of fundamentally new propulsion systems capable of operating with minimal consumption of on-board energy sources. Such systems, in particular, include various sailing structures: film [1, 2], electric [3-6] and magnetic [7-9]. The thrust of a magnetic sail arises as a result of the magnetostatic interaction between the solar wind and an artificial magnetic field, which is generated on board the spacecraft by an electric current flowing through an electrical circuit of a given shape. In this paper, another version of an electric sail is considered – the so-called dipole drive (double electrostatic layer), proposed in [10], which structurally consists of two parallel conductive surfaces made in the form of grids of conductive filaments. The conductive surfaces are separated by an insulating gap, the design of which is determined by the necessary electrical and mechanical strength and the minimum possible effect on the electric field between the conductive surfaces. To maintain a constant potential difference between the grids, a device is connected to a negatively charged surface that emits electrons into the surrounding space.

1. Calculation method and main results

The double electrostatic layer is capable of operating in the so-called ion sail mode. When the flow of ions and electrons comes from a negatively charged surface, the

positively charged ions, entering the space between the charged surfaces, receive an inhibitory effect from the electric field between them. With a sufficiently high accelerating potential difference, these ions are pushed back out. The effect of the dipole motor on each ion is similar to an elastic collision. Thus, the dipole engine rejects the incoming ion flow without changing their kinetic energy, acting as an ion sail (Fig. 1). In Figure 1, $\mathbf{E}$ is the electric field strength in the space between

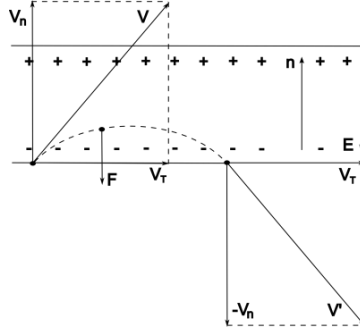


FIGURE 1. Device diagram

the grids, $\mathbf{V}$ is the velocity at the entrance to the gap, $\mathbf{V}'$ is the velocity at the exit from it, $\mathbf{V}_n$ is the normal component of the velocity. In the analytical evaluation of the ion sail, it is assumed that there is no field from the outside of the double electrostatic layer, but inside it is constant. Then the condition for the reflection of an ion with charge Z and mass m_i has the form $ZeU \geq \frac{m_i V_n^2}{2}$.

In this case, the change in the impulse of one ion will be determined by the expression $\Delta \mathbf{p} = -2m_i V_n \mathbf{n}$.

By defining the distribution of particle velocities by velocities and directions in the spherical coordinate system $dn(V, \varphi, \psi)$, it is possible to obtain a change in the impulse of the flow reflected during dt by a section dS of the conductive surface $d\mathbf{P} = -2m_i dn(V, \varphi, \psi) V_n^2 dS dt \mathbf{n}$.

The total pressure on the ion sail is obtained by integrating this value over all possible speeds and their directions.

The described approach does not take into account two important features of a real double electrostatic layer moving in a cosmic plasma: 1) charge distribution on a grid rather than on a solid plane, which leads to the penetration of an electric field into the outer space, and 2) shielding of the electric field by cosmic plasma.

Figure 2 shows the results of calculations of ion trajectories for a near-Earth plasma temperature of 10^3 K, the angle of incidence on the plane of the surface grid $\pi/6$ and a potential difference of 0.24 V. In conditions of solar wind (substorm) with a velocity of hydrogen H^+ and helium He^{++} ions of about 780 km/s, calculations have shown that it is necessary to supply a voltage of about 6700 V to the grid.

The trajectories are shown as blue solid lines, the grid is green, the flow is directed from bottom to top. According to a given ion velocity and direction distribution function, an ensemble is created to calculate the change in impulse $\Delta \mathbf{p}_k$ for the ion and the pressure on the ion sail by summing all calculated $\Delta \mathbf{p}_k$. Calculations of

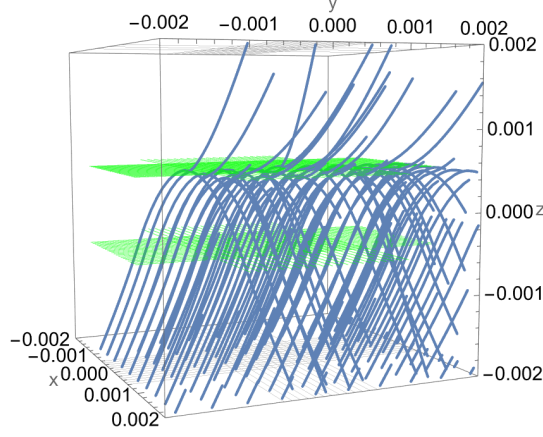


FIGURE 2. Particle trajectories

the pressure acting on the double electrostatic layer were carried out for spacecraft in low-Earth orbit and for spacecraft in near-solar orbit under the influence of the solar wind. The paper also provides an analytical estimate of the leakage current density of a double electrostatic layer in the ion sail mode $j = \frac{1}{2} n_e e \sqrt{\frac{8kT}{\pi m_e}}$ and a numerical calculation of the leakage current density was performed.

Conclusion

The efficiency of the double electrostatic layer in the ion sail mode has been evaluated. Estimates of the pressure forces on a double electrostatic layer in the ion sail regime in low-Earth orbits and in interplanetary space under the influence of the solar wind are given. Estimates of the leakage current density and the consumed electrical power of a double electrostatic layer in the ion sail mode are given. The work is supported by the Russian Science Foundation (grant No. 24-41-02031, <https://rscf.ru/project/24-41-02031/>).

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On the behavior of orbits of Vanhaecke system on integral surfaces

Mikhail Malykh, Lubov Lapshenkova and Shiwei Wang

Abstract. In the 1990s, P. Vanhecke described a Hamiltonian system with two degrees of freedom and a polynomial Hamiltonian integrable in Abelian functions of two variables. We believe that the Vanhaecke system is a good test to hone methods for studying integral manifolds of conservative dynamical systems.

1. Introduction

The idea of the motion nature of conservative dynamical systems was formed in the 1970s [1]. A dynamical system is called transitive if almost any of its orbits fills the entire phase space densely. Transitivity of the system is prevented by conservation laws that hold the orbit on some integral manifold. Therefore, the study of a conservative dynamical system splits into two parts: 1) finding all conservation laws by analytical methods; 2) studying a dynamical system whose restriction to integral manifolds is transitive by statistical methods.

An issue of not less importance is the development of algorithms for finding manifolds on which the system is transitive. In fact, the problem is to find a complete set of conservation laws. Note that these manifolds can be seen by observing the trajectory over a long period of time. It is not difficult to roughly estimate the dimension of the manifold on which the integral curve “winds” and hence determine the number of independent conservation laws. However, the accumulation of errors in numerical methods makes conclusions based on these experiments not entirely convincing.

A natural first step in verifying such judgments is to apply numerical methods to dynamic systems that have been exactly integrated. Interesting and non-trivial from this point of view are nonlinear dynamic systems integrable in Abelian functions of two variables.

2. Vanhaecke system and its integral manifolds

In this talk, we consider the Vanhaecke system [2]. This is a completely integrable Hamiltonian system with Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + (q_1^2 + q_2^2)^2 + aq_1^2 + bq_2^2 \quad (1)$$

for $a \neq b$. This system has two polynomial integrals:

$$F = (p_2q_1 - p_1q_2)^2 - (2q_1^4 + 2q_1^2q_2^2 + 2aq_1^2 + p_1^2)(a - b)$$

and

$$G = (p_2q_1 - p_1q_2)^2 + (2q_1^2q_2^2 + 2q_2^4 + 2bq_2^2 + p_2^2)(a - b)$$

in involution.

Equations

$$F(p_1, p_2, q_1, q_2) = f, \quad G(p_1, p_2, q_1, q_2) = g \quad (2)$$

define in $\mathbb{R}^4$ a surface S , the integral surface of the Vanhaecke system. We calculated its projection into $q_1q_2p_1$ using the Gröbner basis in Sage [3], it is given by the formula

$$s_2p_1^4 + s_1p_1^2 + s_0 = 0, \quad (3)$$

where s_0, s_1, s_2 are polynomials in q_1, q_2 , whose coefficients depend on the parameters a, b, f, g .

One might think that Liouville's theorem allows finding two more integrals from two polynomial integrals in involution, and therefore the system has two more integrals depending on t . After eliminating t , we should obtain a one-dimensional integral manifold, not a surface. This kind of reasoning leads to the wrong result, since it suggests carrying over local results to the global case. In fact, Liouville's theorem leads to two additional integrals of motion, which are very complicated Abelian quadratures, written explicitly in [4].

Vanhaecke's passage to the Jacobi problem is based on a guess, which we will describe as follows: we pass from the variables q_1, q_2 to two new variables x_1, x_2 , which are roots of the equation

$$x^2 + (q_1^2 + q_2^2 + a + b)x + aq_2^2 + bq_1^2 + ab = 0.$$

The problem of integrating the Vanhaecke system on the integral surface can be written as a Jacobi problem

$$\begin{cases} \int_{x_1^0}^{x_1} \frac{d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{d\xi}{\sqrt{R(\xi)}} = 0, \\ \int_{x_1^0}^{x_1} \frac{\xi d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{\xi d\xi}{\sqrt{R(\xi)}} = t, \end{cases}$$

where t is the time passed between the initial position of the system (x_1^0, x_2^0) and the final position (x_1, x_2) . Here R is a polynomial of degree 5 in ξ . All necessary calculations can be performed in the Sage system.

The fundamental theorem of the theory of Abelian functions states that every rational symmetric function s of quantities x_1, x_2 satisfying the Jacobi problem

$$\begin{cases} \int_{x_1^0}^{x_1} \frac{d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{d\xi}{\sqrt{R(\xi)}} = u, \\ \int_{x_1^0}^{x_1} \frac{\xi d\xi}{\sqrt{R(\xi)}} + \int_{x_2^0}^{x_2} \frac{\xi d\xi}{\sqrt{R(\xi)}} = v, \end{cases}$$

is a meromorphic function of u, v :

$$s = \text{Al}(u, v),$$

referred to as Abelian function. The zeros of R define the periods of the Abelian function: if e_1, e_2 are zeros of R and there are no other zeros in the interval $[e_1, e_2]$, then the numbers

$$\omega_1 = 2 \int_{e_1}^{e_2} \frac{d\xi}{\sqrt{R(\xi)}}, \quad \omega_2 = 2 \int_{e_1}^{e_2} \frac{\xi d\xi}{\sqrt{R(\xi)}}$$

form a system of common periods of the Abelian function:

$$\text{Al}(u + \omega_1, v + \omega_2) = \text{Al}(u, v).$$

Returning to our dynamical system, we see that

$$x_1 + x_2 = \text{Al}_1(0, t), \quad x_1 x_2 = \text{Al}_2(0, t).$$

Thus the squares of q_1 and q_2 are expressed linearly through these functions, which allows expressing the general solution of the Vanhaecke system in terms of two Abelian functions of two arguments.

3. Results of computer experiments and discussion

For arbitrary initial data, the orbit is not closed and fills at $t \in \mathbb{R}$ some piece of the algebraic surface of the surface everywhere densely. In other words, for almost any initial data, the restriction of the Vanhaecke system to an integral surface is transitive. This does not contradict either the complete integrability of the Vanhaecke system in the sense of Liouville or its solvability in classical transcendental functions. The very intricate course of the orbit on the surface is described analytically by means of Abelian functions of two variables and is determined by four real numbers, a set of periods of Abelian functions. The key point here is the analytical description of functions of one variable t using periodic functions of two variables.

The behavior of orbits in numerical experiments is determined not so much by the rationality or irrationality of the ratio of two periods ω_1 and ω'_1 , but by the possibility of approximating this number with a rational fraction with high accuracy. Therefore, for arbitrary initial data, we see a seemingly periodic solution with a very large period and a closed orbit that runs along the integral surface densely enough for this surface to become visible, but not densely enough to allow saying that it covers the surface everywhere densely.

Details of the experiments can be found in [3].

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On the integration of Hamiltonian systems using Liouville's theorem in CAS

Ahemed Deina Mohamed El Mokhtar and Ksaverii Malyshev

Abstract. Liouville's theorem allows one to reduce Hamiltonian systems to quadratures if there is a required number of independent integrals in involution. However, in practice, integration is usually performed using other methods. The question arises: is it possible to attempt to obtain the same results directly from Liouville's theorem? We consider several examples of systems with two degrees of freedom and polynomial integrals. For these systems, we obtain additional integrals using Liouville's theorem and analyze them using the CAS **SageMath**.

Introduction

Let us consider a Hamiltonian system with two degrees of freedom. Following Whittaker [2, p. 275], we formulate Liouville's theorem [1].

Theorem 1 (Liouville, 1855). *If a dynamical system, defined by the equations*

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \quad (r = 1, 2),$$

the integral of energy is $H(q_1, q_2, p_1, p_2) = h$, and if $V(q_1, q_2, p_1, p_2) = c$ denotes any other integral not involving the time, then the expression $p_1 dq_1 + p_2 dq_2$, where p_1 and p_2 have the values found from these integrals, is the exact differential of some function

$$\vartheta(q_1, q_2, h, c),$$

and the two remaining integrals of the system are

$$\frac{\partial \vartheta}{\partial c} = \text{const}, \quad \frac{\partial \vartheta}{\partial h} = t + \text{const}.$$

Theoretically, this statement can be regarded as a method for integrating canonical equations when an additional integral of motion is known. This becomes

particularly promising when the known integrals of motion are polynomials quadratic in the momenta [3]. On the other hand, in the polynomial case, it becomes possible to obtain additional (remaining) integrals of motion in the form of integrals of algebraic functions.

Typically, such systems are integrated using other methods, which yield results in Abelian functions [4, 5]. Integration is based on the transition to special coordinate systems [6, 7], cf. [8]. Is it possible to obtain the known results by implementing Liouville's theorem as a method for integrating a canonical system in CAS?

Examples

We considered Yoshida Hamiltonians (1984, [9, 10], [11, p. 264]) as the first examples

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^4 + q_2^4}{4} + \alpha \frac{q_1^2 q_2^2}{2},$$

which yields integrable systems only for three values of the parameter $\alpha = 0, 1, 3$.

The second example we considered is the Vanhaecke system [4, 5], with Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + (q_1^2 + q_2^2)^2 + a q_1^2 + b q_2^2,$$

which is integrable for any values of a, b and for $a = b = 0$ it is reduced to the Yoshida system with $\alpha = 1$.

The additional integrals of the indicated systems are known and have degree no higher than second in momenta. So the equations for the integrals are explicitly solved for the momenta, which allows one to write quadratures from Liouville's theorem.

In the case of $\alpha = 3$ for Yoshida system and for the Vanhaecke system with arbitrary values a, b , it is easy to obtain the hyperelliptic quadrature

$$\vartheta(q_1, q_2, h, c) = \int_{(q_1, q_2)}^{(q_1, q_2)} \sqrt{\frac{B_1 + \sqrt{D}}{A}} dq_1 + \int_{(q_1, q_2)}^{(q_1, q_2)} \sqrt{\frac{B_2 + \sqrt{D}}{A}} dq_2.$$

Here A, B_1, B_2, D are polynomials of q_1, q_2, h, c, a, b . It is necessary to monitor the choice of branches of 4-valued functions and carry out the integration path within the region of possible motion. These quadratures consist of several hundred symbols. Further, it is not difficult to construct quadrature expressions for additional integrals obtained in Liouville's theorem.

These quadratures can be simplified to elliptic integrals, for example, in the case of a particular solution $q_1 = p_1 = 0$, when q_2 is expressed in elliptic functions for Vanhaecke and Yoshida systems. Note that in the case of $\alpha = 0$, the nested quadrature disappears.

Conclusion

We have obtained expressions for additional (remained) integrals of motion, and in some special cases, the systems can indeed be integrated using known functions. Moreover, sometimes for integration there is no need to switch to special coordinate systems. The calculation results are compared with better-known results related to these systems. The interplay between the two faces of the same solution – Abelian quadratures in separated variables and hyperelliptic quadratures in Cartesian space – may well prove to be a subject of independent interest.

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A Covariant Poisson-Bracket Form of the Levi-Civita Separability Condition

Vladimir Martinusi

Abstract. We introduce a coordinate-free formulation of the classical Levi-Civita separability condition for Hamiltonian systems. By organizing candidate separation variables into an alternating canonical quartet (x_i, p_i, x_j, p_j) , a recursive sequence of nested Poisson minors fully reproduces the classical separability criteria without invoking a preferred coordinate chart. Furthermore, we demonstrate that this five-bracket factorizes under local canonical scalings and maps directly onto a quadratic form governed by iterated Poisson brackets, offering an intrinsic diagnostic tool for tracking both separable and non-separable dynamics.

Introduction

The integration of the Hamilton–Jacobi equation by separation of variables is fundamental to analyzing conservative dynamical systems. In 1904, Levi-Civita [1] established the classical necessary and sufficient structural criteria for orthogonal separability. For an arbitrary Hamiltonian $H(p, x) = h$, the phase space geometry must satisfy:

$$\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x_j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial x_i \partial p_j} - \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial x_j} + \frac{\partial H}{\partial x_i} \frac{\partial H}{\partial x_j} \frac{\partial^2 H}{\partial p_i \partial p_j} = 0. \quad (1)$$

Because classical representations rely strictly on standard partial derivatives, they cannot easily distinguish between intrinsic non-integrability and a suboptimal choice of configuration coordinates. To resolve this limitation, this note introduces a coordinate-free, recursive operator formulation of the classical Levi-Civita condition using nested Poisson minors.

1. The Functional Brackets

Let $(\mathcal{M}, \omega)$ be a symplectic phase space with the standard Poisson bracket $\{f, g\} = \omega^{-1}(df, dg)$.

Definition 1 (The Four-Bracket Minor). For four scalar functions $a, b, c, d \in C^\infty(\mathcal{M})$, the four-bracket is defined as the 2×2 Poisson minor:

$$\{a, b, c, d\}_4 = \det \begin{pmatrix} \{a, c\} & \{a, d\} \\ \{b, c\} & \{b, d\} \end{pmatrix} = \{a, c\} \{b, d\} - \{a, d\} \{b, c\}.$$

Definition 2 (The Recursive Five-Bracket). For five functions $a, b, c, d, H \in C^\infty(\mathcal{M})$, the five-bracket is built recursively via:

$$\{a, b, c, d\}_5(H) = \det \begin{pmatrix} \{H, c\} & \{H, d\} \\ \{a, b, \{H, c\}, H\}_4 & \{a, b, \{H, d\}, H\}_4 \end{pmatrix}.$$

Definition 3 (Alternating Canonical Quartet). Four functions $a, b, c, d \in C^\infty(\mathcal{M})$ form an alternating canonical quartet if $\{a, b\} = 1$, $\{c, d\} = 1$, and all other pairwise brackets vanish. Locally, these functions act as Darboux coordinate pairs (x_i, p_i, x_j, p_j) .

2. Main Results

Theorem 1 (Covariant Separability). Let $H \in C^\infty(\mathcal{M})$ be a Hamiltonian on a symplectic phase space. Suppose $a, b, c, d \in C^\infty(\mathcal{M})$ form an alternating canonical quartet. Then H satisfies the Levi-Civita separability condition for the (i, j) sector if and only if

$$\{a, b, c, d\}_5(H) = 0. \quad (2)$$

Theorem 2 (Symplectic Curvature Representation). Let (x_i, p_i, x_j, p_j) be an alternating canonical quartet. The recursive five-bracket can be expressed intrinsically as a quadratic form:

$$\{x_i, p_i, x_j, p_j\}_5(H) = -\frac{1}{2} \mathbf{v}^T \mathbf{M} \mathbf{v},$$

where $\mathbf{v} = (H_{x_i}, H_{x_j}, H_{p_i}, H_{p_j})^T$ is the structural gradient vector and $\mathbf{M}$ is a symmetric connection matrix constructed from iterated Poisson brackets [2]:

$$\mathbf{M} = \begin{pmatrix} 0 & \{\{H, x_i\}, x_j\} & 0 & -\{\{H, x_i\}, p_j\} \\ \{\{H, x_i\}, x_j\} & 0 & -\{\{H, x_j\}, p_i\} & 0 \\ 0 & -\{\{H, x_j\}, p_i\} & 0 & \{\{H, p_i\}, p_j\} \\ -\{\{H, x_i\}, p_j\} & 0 & \{\{H, p_i\}, p_j\} & 0 \end{pmatrix}.$$

3. Applications

The recursive five-bracket serves both as a diagnostic tool for mapping cross-coupling boundaries and as an algorithmic investigator for discovering adapted canonical frames.

3.1. Cartesian Systems

For a natural Cartesian Hamiltonian $H = \frac{1}{2}(X^2 + Y^2) + V(x, y)$, evaluating the five-bracket isolates mixed potential curvature:

$$\{x, X, y, Y\}_5(H) = -YX \left(\frac{\partial^2 V(x, y)}{\partial x \partial y} \right).$$

The bracket vanishes globally if and only if $V(x, y) = f_1(x) + f_2(y)$. On non-separable profiles—such as Hénon-Heiles or Toda lattice potentials—the bracket maps exact submanifolds where separability breaks [4].

3.2. Polar-Nodal (Whittaker) Systems

In polar-nodal coordinates $(r, \theta, \nu, R, \Theta, N)$ with $H = \frac{1}{2}(R^2 + \Theta^2/r^2) - \mu/r + V(\theta, \Theta, N)/r^q$ [5], the radial-angular coupling yields:

$$\{r, R, \theta, \Theta\}_5(H) = R\Theta r^{-3-q}(-2 + q) \left(\frac{\partial V}{\partial \theta} \right).$$

For $q = 2$, the bracket vanishes identically, isolating angularly separable extensions such as MICZ-Kepler systems [3]. For the uniform Stark effect ($V = \epsilon r \cos \theta$), the non-vanishing bracket formally proves the necessity of transitioning to parabolic coordinates.

Conclusion

We have reformatted the Levi-Civita orthogonal separability condition into a coordinate-free framework using nested Poisson minors. By replacing classical partial derivative strings with recursive bracket operators, integrability checks become independent of pre-selected charts and are optimized for direct implementation in computer algebra systems.

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Using orbital dynamics stability criteria in the search for exomoons

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Abstract. Within the framework of the exoplanet satellite search, we performed a comparative analysis of the applicability of various criteria for determining the size of a stable satellite orbit.

Introduction

One of the pressing tasks of modern astronomy is the search for exoplanet satellites (exomoons) [3, 4]. The long-term orbital dynamics of a satellite that actually exists in a planetary system must be stable. Numerical modeling, namely the study of motion stability allows us to assess the possibility of detecting exomoons from the analysis of observations and to obtain/refined information on the orbital and physical parameters of identified exomoon candidates. By calculating the Lyapunov characteristic exponents (LCEs; see [7] for details) in the hierarchical three-body problem (“star – planet – planetary satellite”), we analyzed the applicability of several criteria [1, 2, 8, 10] for estimating the maximum semi-major axis of a satellite’s orbit a_{crit} , corresponding to its long-term stable orbital dynamics.

Stability diagrams

To evaluate the applicability of empirical stability criteria to the study of exoplanet satellite dynamics, we calculated the Lyapunov times (the timescale of predictable dynamics) of the exomoon’s orbital dynamics over sets of initial conditions typical for this problem: (a_s, i_s) , (a_s, e_p) , and (a_s, e_s) . The orbital elements of the satellite subsystem are denoted by the subscript “s”, and those of the planetary system by the subscript “p”. Based on these sets of initial conditions, the boundaries of the stable orbital dynamics regions were determined, and the corresponding stability diagrams were constructed. Figure 1 shows examples of the constructed stability diagrams for the exoplanet satellite system. The semi-major axis of the

satellite's orbit a_s is given in units of the planet's Hill sphere radius R_H . The diagrams indicate the Lyapunov time T_L of the planet's satellite subsystem. The Lyapunov time was estimated as [9] $T_L = 1/L$, where L is the maximum LCE. The methodology for constructing the stability diagrams is described in [5, 6].

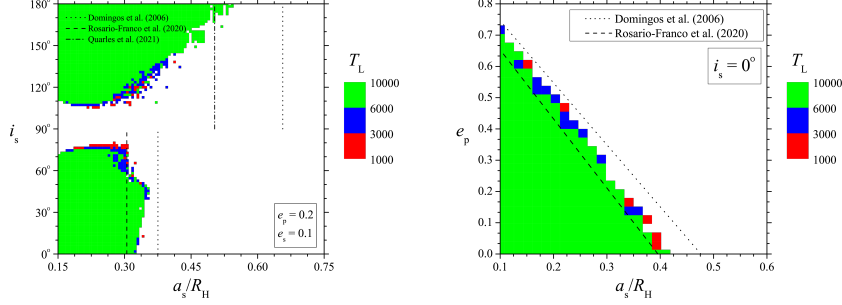


FIGURE 1. Lyapunov times (in years) of the exoplanet satellite subsystem calculated over a set of initial conditions: (a_s, i_s) for $(e_p = 0.2, e_s = 0.1)$ and (a_s, e_p) for $e_s = 0, i_s = 0$. White color corresponds to unstable exomoon orbits. The dashed, dotted, and dash-dotted lines indicate the empirical criteria considered [2, 8, 10] for the maximum value of a_s corresponding to the stable dynamics of the planetary satellite.

Conclusion

Analysis of the constructed stability diagrams showed that the empirical criterion proposed by Domingos et al. [2], frequently used to determine potentially possible exomoon orbits, significantly overestimates the theoretically possible value of a_s . In the coplanar problem for a prograde exomoon orbit (where the directions of the planet's and satellite's orbital motions coincide), the criterion proposed by Rosario-Franco et al. [8] performs better when estimating a_{crit} , while for a retrograde exomoon orbit, the criterion from [10] is more accurate. If the angle between the orbital planes of the planet and the satellite can be substantial ($i_s \geq 30^\circ$), a reliable estimate of a_{crit} can only be obtained through numerical simulation of the long-term orbital dynamics of the exomoon and by employing rigorous motion stability analysis methods (such as calculating LCEs, the MEGNO parameter, etc.).

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Search for invisible satellites of visual binary stars by studying the stability of their long-term orbital dynamics

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Abstract. We present a brief teaser of an extension of the long-term orbital stability analysis technique for application to the search for a low-mass companion in a wide binary star system. The stability analysis is performed by computing the Lyapunov characteristic exponents over a set of possible orbital parameters for a component that may exist in the system. The method was applied to the optically discovered companion in ADS 15571. The satellite is located within the ranges determined by long-term stability limits. In addition, the ranges for possible companions of 17 wide binaries with highly accurate orbital parameter values were determined.

Introduction

Multiple-star system properties partially reflect stellar formation features and trends driven by long-term dynamical evolution. To date, there is no comprehensive scenario that can provide an answer to all questions in this field of astronomy [1]. Consequently, searching for low-mass companions in wide binaries is an important aspect of building a representative multiple-star dataset.

The observational program of the Pulkovo Observatory 26-inch refractor concentrated on collecting relative positions (ρ, θ) of the wide binaries. These data, spanning several decades, make it possible to identify variations in the ρ, θ time series [3]. These quasi-periodic offsets may be caused by a set of reasons. One possibility is a genuine gravitational perturbation induced by a low-mass companion orbiting, for instance, one of the components. Historically, such low-mass companions were called invisible satellites.

Typically, detecting such low-mass stars via spectral measurements or high-resolution imaging is challenging because of their low brightness and subarcsecond angular separation from the much brighter primary star. Examining the orbital

stability of an unconfirmed satellite offers an alternative approach to assessing the reality of a potential companion. A similar methodology is routinely employed in investigations of exomoon candidates [7]. This paper provides a brief review of the approach and presents several results of its application.

1. A review of the technique of the long-term orbital stability analysis for the wide binary with a low-mass companion

The stability analysis was performed by means of numerical integration and calculation of the Lyapunov characteristic exponents (LCEs; see [8] for details). The LCEs were computed over sets of possible orbital parameters of the low-mass companion. In the representative orbital parameter planes, regions with stable dynamics and significant values of the Lyapunov time (the time of predictable system dynamics) $T_L = 1/L$ were identified, where L is the maximum LCE. It is assumed that the orbital parameters of the yet unidentified companion may reside precisely within these regions. This methodology is described in more detail in [6, 7].

2. The ADS 15571 system companion test

Numerous examples of confirmed invisible satellite detections based on Pulkovo Observatory data are known. For instance, the ADS 15571 companion had been initially detected from the Pulkovo 26-inch refractor astrometric series [2] and confirmed with BTA 6-m telescope speckle imaging [5]. Figure 1 presents an example of stability diagrams constructed for a low-mass companion in the ADS 15571 system. The integration of orbital dynamics and the calculation of LCEs were performed over a time interval of 10^5 years. As can be seen from Figure 1, the orbital parameters of the low-mass companion determined in [5] correspond to its stable dynamics with $T_L \geq 10^4$ years. This fact confirms the reliability of the observational data analysis results presented in [5].

3. Dataset forming

The dataset to examine the technique was formed from a wide binary orbit set published in [4]. This orbit set contains the results of the most accurate Pulkovo technique-based determination of orbital parameters from astrometric and radial velocity measurements.

Currently, the approach to assessing orbital stability requires data coverage over a significant orbital arc. This was possible only for binaries with a period less than 350 years. Hence, our set contains only 17 stars. Some of the dataset properties are demonstrated in Figure 2. The selected binary systems are located between 10 and ~ 60 pc from the Sun, with primary masses ranging from 1.0 to 1.5 solar masses, and component mass ratios between 0.4 and 1.0.

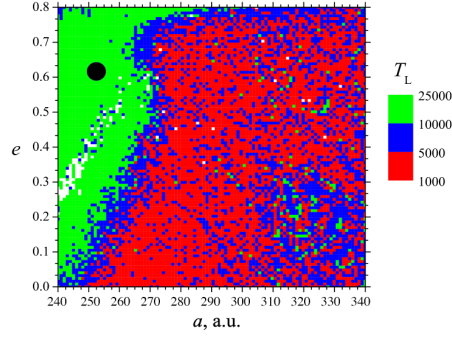


FIGURE 1. Lyapunov times (in years) for a low-mass companion in the ADS 15571 system computed on the semi-major axis–orbital eccentricity plane. White areas represent unstable orbits. The position of the low-mass component is indicated by a black dot according to [5].

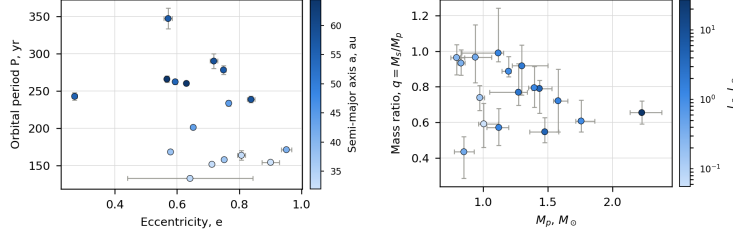


FIGURE 2. Dataset properties. Left: eccentricity – period diagram (the corresponding semi-major axis values are shown by colour). Right: primary mass – mass ratio diagram (the corresponding luminosity of primary components is shown by colour).

Long-term orbital stability analysis requires separate mass estimations for both components. We apply the `astroARIADNE`¹ [9] code to do this.

The results and conclusion

Our analysis demonstrates that long-term orbital stability investigations of wide binary stars are an effective method for establishing natural constraints on the orbital parameters of potential companions. This conclusion is supported by our results for the ADS 15571 system, where satellites detected via speckle imaging

¹<https://github.com/jvines/astroARIADNE>

were found to lie within the ranges dictated by orbital stability requirements. A subset of 17 short-period wide binaries was analyzed using the described approach. Constraints on the possible orbital parameters of low-mass companions were determined using mass estimates of the primary components, obtained from photometric spectral energy distributions (SEDs) and the astroARIADNE code.

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Analytic bounds of the distance between two elliptic orbits

D.V. Mikryukov

Abstract. In this talk we consider simple estimates of the Euclidean distance between two elliptic orbits, expressed as explicit functions of osculating elements. The primary focus is on the lower bound, as in practice the lower bound is more often required than the upper one. The practical convenience of the constructed lower bound τ is that it is nonnegative and vanishes only when the orbits intersect.

Introduction

The problem of computing the Euclidean distance function ρ between two confocal elliptic orbits $\mathcal{E}$ and $\mathcal{E}'$ arises in many areas of modern celestial mechanics. This function is now often referred to as the MOID (Minimum Orbital Intersection Distance). Unfortunately, no general analytic solution to this problem exists, and in practice it is solved numerically (see e.g. [1, 2]). As a result, processing a large number of orbital pairs can become computationally expensive.

In practice, however, the MOID usually needs to be computed only for orbits that are close to each other. Therefore, it is of great interest to construct a lower bound τ of the distance ρ that is an explicit nonnegative function of the osculating elements and vanishes only when the orbits intersect. Indeed, in practice the notion of closeness is quantified by some threshold value $\delta > 0$. If $\tau > \delta$, then $\rho > \delta$ as well, and hence computing ρ , which is much more computationally expensive than finding τ , is unnecessary for the given pair of orbits. The value of ρ is computed only when $\tau \leq \delta$. This leads to a significant acceleration of the construction of a catalog of close orbits, since sufficiently distant orbits are filtered out rather quickly.

In the present talk, we consider one of the simplest such estimates for the case of noncoplanar orbits. In practice, this estimate can always be applied, since strictly coplanar orbits do not exist. However, it should be noted that the efficiency of this estimate decreases as the orbits approach a coplanar configuration.

Mathematical setting

In the inertial Euclidean space $\mathbb{R}^3$ consider noncoplanar bounded Keplerian orbits $\mathcal{E}$ and $\mathcal{E}'$ with a common focus O . Let $\mathbf{Z}$ and $\mathbf{Z}'$ be unit vectors directed along the angular momentum vectors of the orbits $\mathcal{E}$ and $\mathcal{E}'$, respectively. The vector $\mathbf{w} = \mathbf{Z} \times \mathbf{Z}'$ defines the line of mutual nodes c , i.e., the line along which the orbital planes of $\mathcal{E}$ and $\mathcal{E}'$ intersect. Denote by e , e' , and I the eccentricities of the orbits $\mathcal{E}$, $\mathcal{E}'$ and the angle between $\mathbf{Z}$ and $\mathbf{Z}'$, respectively. By assumption, $0 \leq e, e' < 1$, $\sin I > 0$.

The point O divides the mutual nodal line into two rays. On the ray directed along the vector $\mathbf{w}$, we mark the points $M \in \mathcal{E}$ and $M' \in \mathcal{E}'$, and on the other ray, the points $N \in \mathcal{E}$ and $N' \in \mathcal{E}'$. The points M, M', N, N' always exist and are unique (although some of them may coincide).

Put

$$\sigma_1 = r - r', \quad \sigma_2 = R - R', \quad \sigma_3 = r + R', \quad \sigma_4 = R + r',$$

where $r = OM$, $r' = OM'$, $R = ON$, $R' = ON'$, and define the functions

$$\begin{aligned} \ell &= \ell(\mathcal{E}, \mathcal{E}') = \sigma_1 \sigma_2, \\ \sigma &= \sigma(\mathcal{E}, \mathcal{E}') = \min\{|\sigma_1|, |\sigma_2|, |\sigma_3|, |\sigma_4|\}. \end{aligned} \quad (1)$$

Now for the distance function $\rho = \rho(\mathcal{E}, \mathcal{E}')$ we can write the obvious upper bounds

$$\rho^2 \leq |\ell|, \quad \rho \leq \sigma \leq |\ell|^{1/2}. \quad (2)$$

Both functions ℓ and σ of the ten-dimensional argument $(\mathcal{E}, \mathcal{E}')$ are continuous, vanish if and only if the orbits intersect, and are expressed explicitly in terms of orbital elements. In the definition of σ , we take the absolute value of σ_3 and σ_4 to make formula (1) more symmetric.

The properties of the function ℓ , called the linking coefficient, are discussed in detail in [3]. That paper also raises the question of finding a positive function $C'(e, e', I)$ such that the inequality

$$\rho^2 \geq C'(e, e', I)|\ell|$$

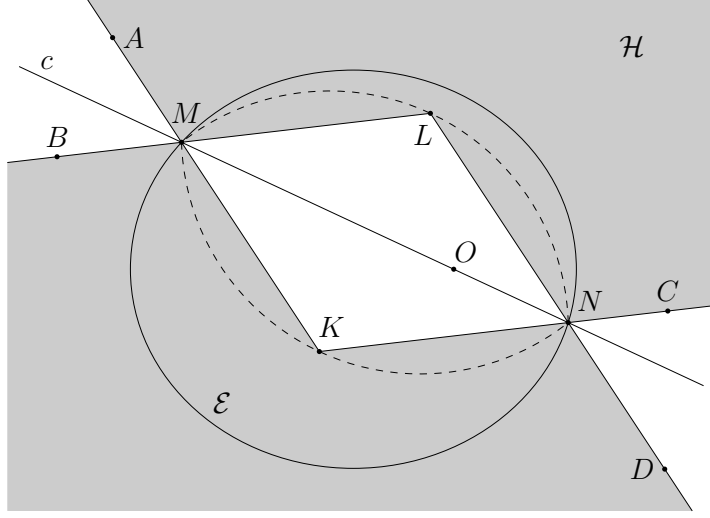
holds.

We express cautious doubts about the existence of the aforementioned function $C'(e, e', I)$. Nevertheless, one can construct a positive function $C(e, e', I)$ satisfying the lower bound

$$\rho \geq \tau = \tau(\mathcal{E}, \mathcal{E}') = C(e, e', I)\sigma. \quad (3)$$

Basic geometric constructions

The main idea behind constructing the estimate τ is that the orbits $\mathcal{E}$ and $\mathcal{E}'$ are entirely embedded into two-dimensional sets $\mathcal{H}$ and $\mathcal{H}'$ of relatively simple geometric structure, after which the distance $\rho(\mathcal{H}, \mathcal{H}')$ between them is computed and taken as τ . The sets $\mathcal{H}$ and $\mathcal{H}'$ are simple in the sense that their boundaries

FIGURE 1. The set $\mathcal{H}$ is shaded in gray.

consist only of segments and rays. As a result, the distance $\rho(\mathcal{H}, \mathcal{H}')$ turns out to be equal to the distance between two skew lines. In this way, we avoid dealing with the difficult problem of computing the distance between two second-order curves.

Let us outline the construction of the set $\mathcal{H} \supset \mathcal{E}$. The set $\mathcal{H}' \supset \mathcal{E}'$ is constructed similarly.

Draw the tangent m to the orbit $\mathcal{E}$ at the point M . If we hold e fixed, the quantity $\lambda = \angle(c, m)$ is a function of the true anomaly θ of the point M . (By the angle between lines we mean the angle not exceeding $\pi/2$.) By continuity and periodicity, the function $\lambda(\theta)$ attains its maximum and minimum values. It is easy to prove that the minimum value is attained at vertices of the minor axis of the ellipse and equals $\arccos e$. (The maximum value $\pi/2$ is obviously attained at the apses for any e .)

Next, on the segment MN as a chord, let us construct a circle such that the tangents to it at the points M and N form an angle $\arccos e$ with the line c . In that way, two circles can be constructed. We only need their smaller arcs, which are shown in dashed lines in Fig. 1. Denote the midpoints of these arcs by K and L , and draw rays from these points toward M and N . As a result, the set $\mathcal{H}$ is defined by the infinite polygonal chains $AMLNC$ and $BMKND$ as in Fig. 1. One can prove that $\mathcal{E} \subset \mathcal{H}$ for any position of the line c .

Consider the basic properties of the set $\mathcal{H}$.

The set $\mathcal{H}$ and its boundary are pathwise connected sets. When c rotates about O , we obtain different sets $\mathcal{H}$. However, for a fixed e , they are all similar: only the distance between M and N changes. When $e = 0$, the rhombus $MLNK$

degenerates into a square inscribed in the circular orbit $\mathcal{E}$, while the orbit $\mathcal{E}$ itself can be considered as a result of the degeneration of the circular arcs (shown dashed in Fig. 1) into a union of two semicircles with common endpoints M and N .

Results and conclusions

The construction of the set $\mathcal{H}$ is discussed in more detail in [4]. It is also shown there that the distance $\rho(\mathcal{H}, \mathcal{H}') = C(e, e', I)\sigma$, where

$$C(e, e', I) = \sqrt{\frac{(1-e)(1-e')\sin^2 I}{(1-e)(1-e')\sin^2 I + 2(1+|\cos I|\sqrt{(1-e^2)(1-e'^2)} - ee')}} \quad (4)$$

and σ is defined by (1). Taking $\tau(\mathcal{E}, \mathcal{E}') = \rho(\mathcal{H}, \mathcal{H}')$, we finally conclude that the lower bound (3) holds, where the function $C(e, e', I)$ is given by (4). The function (4) is always positive and satisfies $C(e, e', I) = C(e', e, I)$.

The constructed estimate τ is most likely not sharp. For example, for circular perpendicular orbits, formulas (3) and (4) give $\tau = \sigma/\sqrt{3}$, whereas clearly $\rho = \sigma$ in this case. Combining (2) and (3), one obtains for the distance $\rho(\mathcal{E}, \mathcal{E}')$ an effective two-sided estimate

$$\tau(\mathcal{E}, \mathcal{E}') \leq \rho(\mathcal{E}, \mathcal{E}') \leq \sigma(\mathcal{E}, \mathcal{E}'), \quad (5)$$

where the functions τ , ρ , σ are either all zero ($\mathcal{E}$ and $\mathcal{E}'$ intersect) or all positive ($\mathcal{E}$ and $\mathcal{E}'$ do not intersect). Two-sided estimate (5) contains only simple and explicit functions of the osculating elements.

The practical efficiency of the estimate τ was studied in [4].

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Emergence of Deterministic Chaos in Mechanical Systems under Small Periodic External Nonautonomous and Parametric Self-Organizing Bifurcation Perturbations

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Abstract. Beginning with the classical Van der Pol equation, the methods of dynamical systems theory and synergetics have become widely used in the analysis of nonlinear phenomena. The primary objective is to identify stabilization mechanisms that govern the emergence of ordered regimes and self-sustained oscillations, which may either evolve into deterministic chaos or remain stable. These concepts are illustrated using models of forced and self-organizing combustion processes, systems of celestial mechanics, and nonlinear dynamical systems exhibiting finite-time blow-up solutions.

1. Emergence of Deterministic Chaos in Classical Mechanical Systems under Small Nonautonomous Periodic Perturbations

Consider the system

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y) + \mu F_{\text{ext}}, \quad 0 \leq \mu \leq 1, \quad (1)$$

which, when reduced to wave variables $\xi = x - ct$, leads to the situation when $u(x, t) = u(\xi)$, $v(x, t) = v(\xi)$. To which small periodic perturbations F_{ext} are then applied with a period T .

The Duffing Model [1].

The equation

$$x'' + kx' + \omega_0^2 x + ax^3 = F(\omega t), \quad (2)$$

where $k, \omega_0, a, F, \omega$ are constants, and $F \cos(\omega t)$ (i.e. $F_{\text{ext.}}(\omega t)$) denotes the external periodic forcing. The Duffing equation was first investigated by Duffing in 1918 and has since become one of the fundamental models in nonlinear dynamics. It continues to attract considerable attention because deterministic chaotic behavior

of its trajectories has been observed experimentally under appropriate conditions. Equation (2) is equivalent to the following system:

$$\frac{dv}{dt} = -[kv + \omega_0^2 x + ax^3] - F \cos(\omega t), \quad \frac{dx}{dt} = y, \quad (3)$$

where we assume $F \cos(\omega t) = \mu F_{\text{ext.}}(\omega t)$, $0 \leq \mu = 1$. For $\mu = 0$ system (3) on the phase plane $\{x, y\}$ has equilibrium states: $(x = 0, y = 0)$ – saddle, $(x = -\omega_0^2/a, y = 0)$ – center or focus. As well as a loop of separatrices 1 surrounding $(x = -\omega_0^2/a, y = 0)$ and passing through the saddle $(x = 0, y = 0)$. It gives a rough homoclinic structure $\{\Gamma_\mu\}$ in the region $G_{\varepsilon,1} : g_{\varepsilon,1} \otimes \{0 \leq t < \infty\}$, which provides chaotic behavior of trajectories.

The model of Ya. B. Zeldovich and D. A. Frank-Kamenetsky [2] describing the combustion process was studied by means of the wave variable $\xi = x - ct$, which leads to the equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - \delta u(u - \alpha)(u - 1), \quad (4)$$

where $u = u(t, x)$, $\lim_{x \rightarrow -\infty} u = 1$, $\lim_{x \rightarrow +\infty} u = 0$; $\alpha \in (0, 1)$, $\delta > 0$ are parameters, and in the absence of $F_{\text{ext.}}(u, x, t)$ in which the modes of self-oscillations are investigated. If we introduce $F_{\text{ext.}}(u, x, t)$ by the term $\mu \cdot F_{\text{ext.}}(u, x, t)$, $0 \leq \mu = 1$, then for $\mu = 0$, the system on the phase plane $\{u, v = u'_x\}$ has a separatrix contour 1 eye-type structure formed by separatrices $1_1, 1_2$ near the lying saddles $(u = 0, v = 0)$, $(u = 1, v = 0)$, surrounding the center $(u = \alpha, v = 0)$ at $\alpha\delta > 0$, or of the figure-eight type formed by the separatrices $1_1, 1_2$ of the saddle $(u = \alpha, v = 0)$, surrounding the centers on the phase plane $\{u, v\}$ at $\mu = 0$ $(u = 0, v = 0)$, $(u = 1, v = 0)$ when $\alpha\delta < 0$. For system (4) in the domain $G_{\varepsilon_1} : g_{\varepsilon_1} \otimes \{0 \leq t < \infty\}$, we have at $0 \neq |D| = 1$ a rough homoclinic structure $\{\Gamma_\mu\}$, which provides chaotic behavior of trajectories.

Theorem 1. *In the Duffing and Zel'dovich-Frank-Kamenetskii models, in the domain $G_{\varepsilon_1} : g_{\varepsilon_1} \otimes \{0 \leq t < \infty\}$ in the presence of a periodic external force $F_{\text{ext.}}$ ($0 \neq F_{\text{ext.}} = 1$) and a separatrix contour 1, there exists a rough homoclinic structure $\{\Gamma_\mu\}$ corresponding to the chaotic behavior of trajectories.*

2. Deterministic chaos in some models of mechanical systems with small periodic disturbances

Consider the *electric line model* [3] in the form:

$$\frac{\partial U}{\partial t} + v_0 \frac{\partial U}{\partial x} - \gamma \frac{\partial^2 U}{\partial x^2} = -\gamma (U - \beta U^3) + A_\omega \sin(x - ct), \quad c > 0, \quad (5)$$

reduced in the basic form to a system of ordinary differential equations:

$$\frac{du}{d\xi} = v, \quad \frac{dv}{d\xi} = \frac{1}{v} [-\gamma u + (v_0 - c)v + \gamma u^3] + A_\omega \sin \xi. \quad (6)$$

where $\xi = x - ct$, $|A_\omega| = 1$, $A_\omega = A_\omega|_{c=0} = 0$.

In the case $A = A_\omega$, system (6) under condition $v_0 = c$ has two saddles $(u_{1,2} = \pm 1, v = 0)$ in the phase plane $\{u, v\}$ and a loop of l -type separatrices surrounding the center $(0, 0)$. Then for $v_0 \neq c$, $0 \neq |A_\omega| = 1$ in the region $G_{\xi,1} : g_{\xi,1} \otimes \{0 \leq \xi < \infty\}$ we have the stochastic mode for the wave model (5).

Theorem 2. *In the power line model (5), if the condition is met $0 \neq |v_0 - c| < \varepsilon$, $0 \neq |A_\omega| = \varepsilon$, there is a rough homoclinic structure $\{G_\omega\}$ in $G_\omega : g_\omega \otimes \{0 \leq \xi < \infty\}$, corresponding to the chaotic behavior of trajectories.*

In [4], the motion of an electron in a one-dimensional system with a displacement phase transition was considered, and we show by qualitative analysis that under certain conditions it is characterized by stochastic wave solutions.

In this case, the equations of motion of the electron are a system of two coupled nonlinear oscillators, the displacement field of which is associated with the electron and has the form

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m_e} \psi_{xx} - g_1 \psi u_x = 0, \quad mu_t - mc_0^2 u_{xx} - |A|u + Bu^3 = g_2 e(|\psi|^2) x, \quad (7)$$

where $\psi(x, t)$ is the wave function of the electron; $u(t, x)$ is the displacement field of the atom; m_e is the mass of the electron; m is the mass of the atom; $0 < g = 1$ [7, § 4.4], where $g_1 = g_2 = g$.

By introducing [5] new variables

$$\xi = x - vt, \quad u = u(\xi), \quad \psi = \varphi(\xi) \left\{ \frac{1}{h} Et + \frac{1}{h} \left(m_e v_1 - \frac{1}{2} m_e v_2 t \right) \right\},$$

corresponding to solutions of the “traveling wave” type, i.e. solitons, we obtain a system second-order ordinary differential equations

$$\frac{\hbar^2}{2m} \frac{d^2 \varphi}{d\xi^2} = -E\varphi + g_1 \varphi \frac{du}{d\xi}, \quad (v^2 - c_0^2) \frac{d^2 u}{d\xi^2} - \frac{|A|}{m} u + \frac{B}{m} u^3 = g_2 \frac{e}{m} \frac{d(\varphi^2)}{d\xi},$$

which corresponds to the system

$$\frac{d\varphi}{d\xi} = \theta, \quad \frac{d\theta}{d\xi} = -M\varphi - g_1 \varphi v, \quad (8)$$

$$\frac{du}{d\xi} = v, \quad \frac{dv}{d\xi} = \frac{1}{m(v^2 - c_0^2)} \left\{ |A|u - Bu^3 + g_2 e \frac{d(\varphi^2)}{d\xi} \right\}, \quad (9)$$

with $\nu - c_0 < 0$, $M = 2mE/\hbar^2$, $\nu \neq c_0$, and a four-dimensional phase space $\{\varphi, \theta, u, v\}$.

Let us perform a qualitative analysis of system (9), assuming first $g_1 = g_2 = 0$, then $g_1 = 0$, $0 < g_2 \leq 1$, and finally $0 < g_{1,2} \leq 1$.

In the case $g_1 = g_2 = 0$, system (8) on the phase plane $\{\varphi, \theta\}$ has a center-type equilibrium state that defines solutions everywhere except $\varphi = \theta = 0$ periodic in ξ ones, which are parametrically expressed as

$$\varphi = r \cos \xi, \quad \theta = \frac{r}{\sqrt{M}} \sin \xi, \quad r > 0. \quad (10)$$

In the same case, the system (9) on the phase plane $\{u, w\}$ has: for $\nu - c_0 < 0$ a closed separatrix contour of the "eye" type, composed of rough saddles ($u = \pm 1, w = 0$) and their separatrices, enclosing the center ($u = 0, w = 0$) (this case for $m = mc_0^2 = |A| = 1$ is considered in [5, section 1.5], where parts $L \cap \{w > 0\}$, $L \cap \{w < 0\}$ of the separatrix L correspond to two types of solitons).

When $\nu - c_0 > 0$ is a closed separatrix contour of the "figure eight" type, composed of loops L_1, L_2 of separatrix L of the coarse saddle ($u = 0, w = 0$), surrounding the centers ($u = 1, w = 0$), ($u = -1, w = 0$) (each L_1 and L_2 has its own soliton solution).

In the case $g_1 = 0, 0 < g_2 \leq 1$, taking into account (10), we have $H = \frac{d(\phi^2)}{d\xi} = H \sin 2\xi$, where $H = \frac{e^2 h^2}{2mE}$, and the system (9) takes the form

$$\frac{du}{d\xi} = w, \quad \frac{dw}{d\xi} = \frac{1}{m(\nu^2 - c_0^2)} \{|A|u - Bu^3 + g_2 H \sin 2\xi\} \quad (11)$$

The system (11) is autonomous with small non-autonomous periodic perturbations in ξ . Applying the research methodology described in [3, 6, 7], the presence for $g_2 = 0$ of separatrix contours L or $L_{1,2}$ in (11) entails for $0 < g_2 \leq 1$ the presence in the extended phase space $\{u, w, \xi\}$ of a rough homoclinic structure. Such a structure is formed by the infinite intersection of two-dimensional separatrix manifolds induced by rough saddles of system (11) at $g_2 = 0$.

Theorem 3. *When $g_1 = 0, 0 < g_2 \leq 1$ in system (9), which in this case is equivalent to system (11), in a three-dimensional phase space $\{\phi, u, w\} \subset \{\phi, \theta, u, w\}$, where ϕ, θ are periodic in ξ , there is a rough homoclinic structure $\{\Gamma_{g_1}\}$ that defines chaotic wave solutions for system (7) at $g_1 = 0, 0 < g_2 \leq 1$.*

3. Chaotic modes and explosive blow-up solutions

As a result of changing the values of the system parameters, which is a manifestation of self-organization, in fact, we come to deterministic chaos. For such a process, the methods of "grafting" and synthesis are applicable [8, 9]. "Grafting" methods assume the presence of a family of periodic trajectories in the region of the system, which transfer (graft) periodicity to another system, which has regions with homoclinic structures with closed separatrix contours.

We proceed to the study of systems with explosive blow-up solutions [9], which are conditioned by sufficient conditions for the presence of a special type P_∞ of Poincaré circle, introduced in [10]. The generalized condition of a special type is given in [9], and, in particular, it is related to the presence of two-dimensional subsystems in the initial systems, which are reduced in a simplification of the condition from [9] to the solution of the equation

$$xP_m(x, y) - yQ_m(x, y) = 0, \quad (12)$$

where $P_m(x, y)$ and $Q_m(x, y)$ are the senior members in the initial system

$$\frac{dx}{dt} = \sum_{i=1}^m P_i(x, y), \quad \frac{dy}{dt} = \sum_{i=1}^m Q_i(x, y).$$

Let us further consider the departure of the trajectories of the system under study in P_∞ along directions in a finite time. In this case, there are cases: departure to the Poincare equator, or all of P_∞ , or on its arcs, or on special points P_∞ . Thus, the problem is reduced to finding the roots of the nonlinear equation (12) $(x_j, y_j, j = 1, n + 1)$. These roots determine the directions along which the trajectories of the system go in P_∞ in a finite time. Then the Poincare equator P_∞ is not a solution of the dynamical system under study. In the simplest cases, when the trajectories are directed to a point $y = 0$ or to $x = 0$, we find the roots of the equations $xP_m(x, y) = 0$, or $yQ_m(x, y) = 0$, respectively. In an even simpler case, finding roots can be limited to finding the roots of equations $P_m(x, y) = 0$, or $Q_m(x, y) = 0$, respectively. If in all these cases it is possible to identify divergence in directions, then we can assume that deterministic chaos is accompanied by explosive growth.

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Velocities-distance optimization of the rotation model for a Galactic ensemble of massive young stars with masers: effects of parallax reduction

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Abstract. The previously developed 4D method for modeling the kinematics of the homogeneous subsystem of the Galaxy is used to study in detail the effect of accounting for the uncertainty of the measured heliocentric distances (here, trigonometric parallaxes) on the results of this modeling. The study is based on an updated database of 213 HMSFR masers. It is shown that neglecting random errors in parallaxes (3D method) leads to an overestimation of the distance from the Sun to the center of the Galaxy, R_0 , by 0.2–0.5 kpc, which is significant compared to the statistical error $\pm\sim 0.1$ kpc. Also, the 3D method overestimates the velocity dispersion of masers in the Galactic plane by about 2 times, exaggerates the azimuthal and underestimates the radial residual velocities of the Sun, and also shows a tendency to underestimate the Oort constant A . Accounting for parallax errors leads to a decrease of the level of the rotation curve in the outer region of the Galaxy. It is demonstrated that these biases do not occur due to a change in the average distance scale, but rather as a result of the elimination of a fictitious expansion of the nominal distribution of objects in the Galactic plane relative to the Sun due to the measurement variance of distance estimates. Final estimates of universal (independent of the type of objects) Galactic characteristics are obtained from the updated data: $R_0 = 7.82 \pm 0.11$ kpc; angular and linear velocities of the Sun rotation around the center of the Galaxy $\omega_\odot = 30.44 \pm 0.18$ km/s/kpc, $\theta_\odot = 238.0 \pm 3.8$ km/s; the circular velocity of the Galaxy is $\theta_c = 233.24 \pm 3.8$ km/s.

Introduction

Previously, a 4D method was proposed for constructing a spatially-kinematic model of a homogeneous flat subsystem of the Galaxy, which optimizes the deviations of objects from the model according to three velocity characteristics (the

usual 3D approach) and according to parallaxes, thereby taking into account the statistical uncertainty of the latter [3, 4]. Since the more accurate 4D method is much more time-consuming and requires much longer calculations than the 3D method, this paper aims to show the advantages of the 4D method in terms of its impact on the results, and the systematic errors that can occur when using the simpler 3D approach.

1. Data

New measurements have been added to the database described in [4] for 7 masers in the high-mass star-forming regions (HMSFRs), including 3 new objects [1, 2]. The final catalog of HMSFRs used here contains 213 objects. These masers represent the most homogeneous subsystem of masers with trigonometric parallaxes.

2. Method

The 4D method is based on minimizing the likelihood function

$$L(\mathbf{a}) = \sum_{j=1}^N \left[\ln(\sigma_{V_r})_j + \ln(\sigma_{\mu'_l})_j + \ln(\sigma_{\mu_b})_j \right] + \frac{1}{2} \sum_{j=1}^N \min_{\varpi_{0,j}} \left\{ \frac{[V_{r,j} - V_{r,\text{mod}}(\varpi_{0,j})]^2}{(\sigma_{V_r})_j^2} + \frac{[\mu'_{l,j} - \mu'_{l,\text{mod}}(\varpi_{0,j})]^2}{(\sigma_{\mu'_l})_j^2} + \frac{[\mu_{b,j} - \mu_{b,\text{mod}}(\varpi_{0,j})]^2}{(\sigma_{\mu_b})_j^2} + \frac{(\varpi_j - \varpi_{0,j})^2}{\sigma_{\varpi,j}^2} \right\}. \quad (1)$$

Here $\mathbf{a}$ is the vector of parameters; $V_{r,j}$, $\mu'_{l,j}$, $\mu_{b,j}$, and ϖ_j are catalog (measured) values of heliocentric radial velocity V_r , longitude ($\mu'_l \equiv \mu_l \cos b$) and latitude (μ_b) proper motions, and parallax ϖ for the j -th object; $\varpi_{0,j}$ — the reduced parallax value of the j -th object; N is the sample size. The total variances of the velocity characteristics for the j -th object are the sums of two components: the measuring variance and contributions of natural dispersion $\sigma_j^{*2} = \sigma_j^{*2}(l_j, b_j, \varpi_{0,j}; \sigma_R, \sigma_\theta, \sigma_Z, R_0)$, i.e., the velocity ellipsoid $(\sigma_R, \sigma_\theta, \sigma_Z)$. Here l_j and b_j are the longitude and latitude of j -th object, correspondingly.

The complete iterative procedure of the 4D method includes simultaneous optimization of all parameters and the order of the rotation model, a flexible algorithm for eliminating outliers in the data to determine velocity variances (algorithm parameter $L' = 3$) and to evaluate the remaining parameters ($L' = 1$); for details, see [4]. In this work, objects that were spatially and kinematically isolated from the main data set were additionally excluded in order to reduce the influence of edge approximation effects. The 3D method used here for comparison includes the same procedure as the 4D method, except for the uncertainty in parallaxes—in expression (1), the last term in curly brackets is omitted, and $\varpi_{0,j}$ is replaced with ϖ_j everywhere.

3. Systematic effects caused by ignoring errors in parallaxes

The results obtained by 4D and 3D methods were compared for the same stages of the algorithms application, for the same samples (to check the effect of the sample), for popular restrictions on the sample (on the Galactocentric distance R , on the relative error of the parallax), and for the variant of the spherical ellipsoid of velocities with fixed components ($\sigma_R = \sigma_\theta = \sigma_Z = 5$ km/s; see [6]).

Systematic biases due to the neglect of errors in parallaxes in the 3D method, are most pronounced for the following pairs of correlated parameters: for R_0 the bias is $+(0.2 \div 0.5) \pm 0.1$ kpc ($+3 \div 6\%$), for the Oort parameter A is $(-0.8 \div 0.0) \pm (0.2 \div 0.4)$ km/s/kpc (Fig. 1); the dispersions σ_R and σ_θ are overestimated by ~ 2 times; the azimuthal residual velocity of the Sun is overestimated, and the radial velocity is underestimated by about one statistical uncertainty. The 4D method reduces (by decreasing the value of R_0) the level of the rotation curve by ~ 15 km/s in the outer region of the Galaxy ($R > 5$ kpc), which leads to a decrease in the mass of the halo, disk, and entire Galaxy, as well as to a decrease in the radial scale of the disk.

These systematic effects do not occur because of a change in the average distance scale (the 4D method leaves it unchanged), but as a result of the elimination of the nominal expansion relative to the Sun of the distribution of objects in the Galactic plane due to the dispersion of distance estimates: due to the geometric features of the contribution of rotation to the radial velocity, the reduction in the average distances to nominally distant objects reduces the value of R_0 to a greater extent than it increases due to the elongation of close distances.

Final estimates of universal characteristics: $R_0 = 7.817^{+0.109}_{-0.105}$ kpc; the angular and linear velocities of the Sun's rotation around the Galactic center $\omega_\odot = 30.44 \pm 0.18$ km/s/kpc, $\theta_\odot = 238.0 \pm 3.8$ km/s; the circular rotation speed of the Galaxy $\theta_c = 233.24 \pm 3.8$ km/s (at azimuth peculiar velocity $v_\odot^{\text{LSR}} = 4.76 \pm 0.49$ km/s [5]).

Conclusion

Ignoring parallax errors can lead to significant biases in the results of kinematic modeling of the Galaxy. The complexity of the 4D method is compensated by its systematic accuracy.

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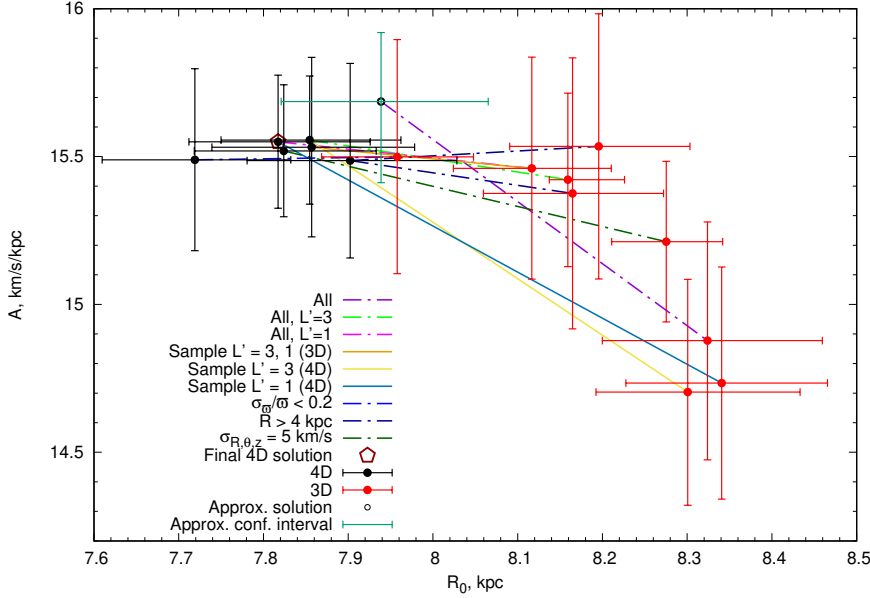


FIGURE 1. Effect of parallaxes' reduction (accounting for their statistical uncertainty) on estimates of the distance to the center of the Galaxy R_0 and the Oort constant A for different samples of HMSFR masers and fixed values of the velocity dispersion components ($\sigma_R = \sigma_\theta = \sigma_Z = 5$ km/s). Comparable estimates by the 4D method (black dots) and by the 3D method (red dots) are connected by straight lines: obtained independently by the full 4D- and 3D-algorithms under the same other conditions by dash-dotted lines, obtained by the same samples by solid lines.

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Sundiving solar sails for fast space exploration

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Abstract. Spacecraft with solar sails, if sent to the Sun’s vicinity, can achieve high heliocentric velocities—up to tens AU per year. This opens prospects for fast missions to various destinations in the Solar system and beyond: asteroids, distant planets, heliopause, solar gravitational lens’ focus, etc. This work reviews the concept of the so-called Sundivers with respect to its real capabilities, scientific potential, and implementation challenges.

Introduction

Solar sails have been investigated since the 20th century for both Solar System and interstellar exploration. To the best of the authors’ knowledge, the earliest scientific introduction of interstellar travel using solar sails appeared in the work of [1]. In the paper it was suggested to employ Earth-based lasers to propel a sailcraft up to 99% of the speed of light. Later, studies [2, 3, 4] put forward the idea of harnessing a natural radiation source—the Sun—to drive a sailcraft onto an interstellar path. One proposal was to send a sailcraft in a heliocentric orbit with a low perihelion and deploy the solar sail right at the perihelion in order to achieve substantial acceleration. This is the core idea of what is today known as *Sundiver*¹.

A Sundiver trajectory is a fast solar sailing trajectory in which a solar sailing spacecraft executes a close solar flyby to transition into a high-energy hyperbolic trajectory, exploiting both the Oberth effect and the strong solar radiation pressure. As interest in exploring the interstellar medium and the distant reaches of the Solar System grows, Sundiver trajectories are gaining increased relevance. It was demonstrated recently that realistic small spacecraft outfitted with solar sails have a variety of potential scientific uses, including those requiring Sundiver trajectories [5]. That paper highlighted a broad range of possibilities, spanning heliophysics and

¹The term can refer to a concept, or to a mission, or to a spacecraft, or to a trajectory, etc., depending on the context.

planetary science investigations, astrophysics applications, the solar gravitational lens focus mission, and more.

In this work we discuss: potential applications of Sunidver trajectories, development status and some engineering aspects, dynamical aspects—especially we pay attention to the maximum achievable velocities,—and current challenges.

1. Equations of motion and solar sail model

Throughout the work we consider a scenario when a solar sailing spacecraft (sailcraft) orbits the Sun, no other objects are involved. The motion of the spacecraft is modelled within the perturbed two-body framework:

$$\ddot{\mathbf{r}} = -\frac{\mu\mathbf{r}}{r^3} + \mathbf{a}_s$$

Here $\mathbf{r}$ is a radius vector from Sun to the spacecraft, μ is the Sun's gravitational parameter, and $\mathbf{a}_s$ is acceleration induced by the sail. To study orbital motion the sail is assumed to be flat specularly reflecting, with $\mathbf{a}_s$ given by

$$\mathbf{a}_s = a_c \left(\frac{1 AU}{r} \right)^2 \cos^2 \alpha_{\mathbf{n}} \mathbf{n}$$

Here a_c is the sailcraft's characteristic acceleration², $\mathbf{n}$ is a unit vector perpendicular to the sail's surface and directed away from the Sun (backside's unit normal vector), and $\alpha_{\mathbf{n}} = \alpha_{\mathbf{n}}(\mathbf{r}, \mathbf{n})$ is the angle between $\mathbf{r}$ and $\mathbf{n}$ called *the cone angle*. Vector $\mathbf{n}$ is considered as a control parameter when studying orbital motion, with only restriction $\alpha_{\mathbf{n}} \in [0, \pi/2]$ (so as to keep the backside in shadow).

When studying angular motion, the sailcraft is assumed to have the "Sunvane" architecture [5], i.e. a centrall truss hosting payload with six independent vanes attached to it (see Fig. 1). The vanes are assumed to have two limited degrees of freedom that will be used in practice to trim the sails and control angular motion. The sailcraft's mass is constituted by truss' mass that is constant, vanes' canvas mass that is proportional to the square of the vanes' size, and vanes support structure's mass that is a linear function of the vanes' size. Then, the area-to-mass ration of the sailcraft is

$$A/m = \frac{6 \cdot (R^2 - R_0^2) \sin 30^\circ}{m_{pl} + 6 \cdot [\sigma_{sq} (R^2 - R_0^2) \sin 30^\circ + \rho_{lin} R]}$$

where R is the size of a single vane, $R_0 \approx 0.5m$ is a size of a technical gap, σ_{sq} is the total surface density of everything that scales as R^2 , ρ_{lin} is the total linear density of everything that scales as R , m_{pl} is the payload mass, and $\sin 30^\circ$ appears from geometry. Based on [5], we assume $\sigma_{sq} \approx 3.55 g/m^2$ and $\rho_{lin} \approx 110.87 g/m^2$.

² $a_c := \eta \cdot P(1 AU) \cdot A/m$ where $\eta \in [1, 2]$ is the effectiveness coefficient, $P(1 AU)$ is SRP at one astronomical unit, A is the sail's area, and m is the total spacecraft's mass.



FIGURE 1. A scale model of a Sunvane sailcraft built at LGarde, Inc. Tustin, CA, USA shown on display at the Xplore, Inc. facility in Redmond, WA, in March 2023 [5].

2. Main results

Using a backwards-in-time propagation technique, described in [6], we determined hyperbolic excess velocities that a sailcraft can achieve depending on the flight distance and the characteristic acceleration (see Fig. 2). This data shows that the minimal required a_c for entering a hyperbolic trajectory is 0.8 mm/s^2 ($A/m \approx 95 \text{ m}^2/\text{kg}$) if r_π is limited to 50 solar radii and 0.32 mm/s^2 ($A/m \approx 38 \text{ m}^2/\text{kg}$) if r_π is limited to 20 solar radii.

Considering the "Sunvane" architecture, the numbers above correspond to the size of the vanes of $R \approx 57 \text{ m}$ and $R \approx 26.5 \text{ m}$ respectively, with payload weighting $m_{pl} = 30 \text{ kg}$. These sizes lead to inertia tensors that are of orders of $1 \cdot 10^5 \text{ kg m}^2$ and $6 \cdot 10^3 \text{ kg m}^2$ respectively. Knowing the inertia tensor, we can estimate the magnitude of control torques required to perform the Sundiver maneuver. As shown in [6], angular acceleration can be estimated as $\varepsilon = 20 \text{ marcsec/s}^2$, therefore the maximum required torque is $1 \cdot 10^5 \text{ kg m}^2 \times 20 \text{ marcsec/s}^2 \approx 0.56 \text{ N m}$. However, this estimate does not account for any disturbances that can be caused by vanes disalignment, meanwhile one 57-m vane creates a torque (relative to the connection point) of 0.32 N m at $r = 1 \text{ AU}$ and 37.4 N m at $r = 20$ solar radii. Clearly, no fly-wheel can compensate a 37.4 N m torque, therefore an adaptive and robust control algorithm will be required to orchestrate the sails so as to minimize misalignments and disturbance torques.

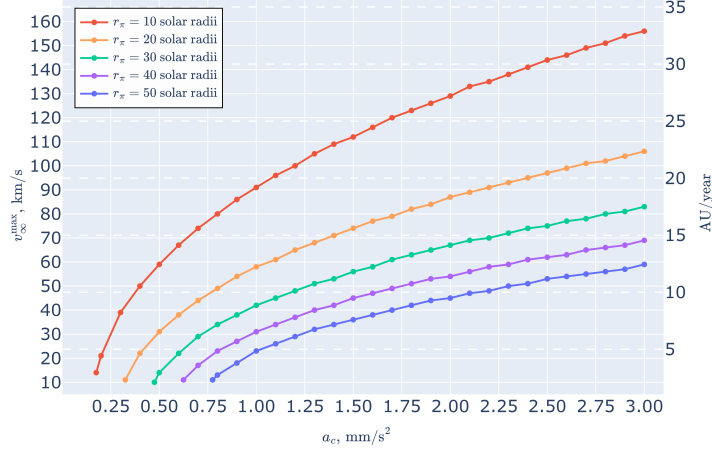


FIGURE 2. Maximum achievable hyperbolic excess velocity as a function of the solar sail’s characteristic acceleration for specific values of the hyperbola’s pericenter distance. The initial aphelion distance is limited to Earth’s orbit [6].

Conclusion

Sundiver concept provides great scientific prospects and poses great engineering challenges. To get v_∞ of at least 5 AU/year it requires characteristic acceleration of at least 1 mm/s^2 , with 0.8 mm/s^2 being the minimal characteristic acceleration required to enter a hyperbola ($r_\pi \approx 50$ solar radii). However, 0.8 mm/s^2 corresponds to the sail’s radius of approximately 57 m , which leads to enormous inertia tensor ($\sim 10^5 \text{ kg m}^2$) and external torques ($\sim 10^1 \text{ N m}$). Therefore, implementing such a mission means not only building a gigantic lightweight structure, but also developing a robust algorithm to precisely control the sails so as to ensure a required angular motion.

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Equilibrium configurations of rotating inhomogeneous liquid masses under gravity and shell elasticity

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Abstract. Equilibrium configurations of a rotating liquid mass subject to gravitational or elastic forces are considered. Routh reduction is used to eliminate the cyclic angular coordinate. By the Lyapunov–Poincaré theorem and within the Ritz approximation, the equilibrium and stability problem is reduced to minimizing a function of two dimensionless semiaxes of a triaxial ellipsoid. For a uniform density distribution, the proposed approach reproduces the known exact solutions: the families of Maclaurin spheroids and Jacobi ellipsoids. For a drop held together by surface tension, an approximate analytical solution of the equilibrium and stability problem is obtained in elementary functions and agrees with the results tabulated by Chandrasekhar. For an inhomogeneous mass distribution, a quadratic Roche model is used to obtain an analytical estimate of the Earth’s flattening that is close to the observed value.

Introduction

This study concerns the shape of a rotating self-gravitating liquid mass. The motion of such a system can be described by the Lagrange equations, with the Lagrangian equal to the difference between the kinetic and potential energies of the liquid. The angular variable is a cyclic coordinate. It can be eliminated by Routh reduction, after which the dynamics of the liquid mass is governed by the Routhian

$$R = T_2 + T_1 - U, \quad U = \Pi + \frac{K^2}{2J}, \quad K = J_0\Omega,$$

where T_2 is a positive-definite quadratic form and T_1 is a linear form in the generalized velocities. The function U is independent of the velocities. It is the effective potential energy and includes the rotational term determined by the moment of inertia.

The quantity K is the constant angular momentum of the ellipsoid. An equilibrium configuration is determined by the stationarity condition for U . If the equilibrium configuration is a local minimum of U , it is stable by the Lagrange–Dirichlet theorem and becomes asymptotically stable when dissipative forces are added. If the equilibrium configuration is not a minimum of U and the second variation can take negative values, then arbitrarily small resistance forces with complete dissipation make it unstable. The set of equilibrium configurations corresponding to minima of U is called the region of secular stability. This result is given in Appell’s monograph [1]. Appell referred to the formulation of the secular stability conditions for a rotating liquid mass in the presence of completely dissipative resistance forces as the Poincaré theorem.

We seek the minimum of the functional within a family of triaxial ellipsoids, which is the Ritz approximation. The problem is thereby reduced to minimizing a function of the ellipsoid semiaxes a_1 , a_2 , and a_3 .

For axisymmetric configurations, the problem reduces to minimizing a function of one variable, x , equal to the ratio of the ellipsoid semiaxes:

$$\begin{aligned}\tilde{U} &= \tilde{\Pi} + \frac{\tilde{K}^2}{2\tilde{J}}, & \tilde{K} &= \tilde{J}_0\tilde{\Omega}, & \tilde{\Omega} &= \frac{\Omega}{\sqrt{\pi G\rho}}, \\ \tilde{J} &= \frac{1}{2}x^{-2/3}, & \tilde{\Pi} &= -\frac{x^{1/3}\arccos x}{\sqrt{1-x^2}}, & x &= \frac{a_3}{a_1}.\end{aligned}$$

Here J , Π , Ω , and K denote the moment of inertia, potential energy, angular velocity, and angular momentum, respectively [2, 3]. Tildes denote dimensionless quantities; in particular, $\tilde{\Pi}$ is normalized by the magnitude of the gravitational energy of a sphere of radius $(a_1a_2a_3)^{1/3}$.

The necessary stationarity condition for $\tilde{U}$ yields the family of Maclaurin equilibrium configurations:

$$\tilde{\Omega}^2 = -\frac{6x^2}{1-x^2} + \frac{2x(1+2x^2)\arccos x}{(1-x^2)^{3/2}}.$$

The value $x_* = 0.582724$ corresponds to the eccentricity $e_* = 0.81267$. The Maclaurin formula then gives $\tilde{\Omega}_*^2 = 0.37423$. The point x_* is a bifurcation point from which the family of asymmetric Jacobi ellipsoids branches. Thus, the Maclaurin family is secularly stable for $x_* < x < 1$.

1. Earth’s flattening for a uniform density distribution

Consider an ellipsoid with semiaxes $a_1 = a_2 = a(1+\delta)$ and $a_3 = a(1+\delta)^{-2}$, where δ is a small deformation parameter. In terms of this parameter, the potential energy and moment of inertia, to leading order, are

$$\Pi = -\frac{16\pi^2 G\rho^2 a^5}{15} \left(1 - \frac{4}{5}\delta^2\right), \quad J = \frac{8}{15}\pi\rho a^5(1+2\delta).$$

The equilibrium equation for an ellipsoidal liquid mass rotating with angular velocity Ω is

$$\frac{d}{d\delta} \left(\frac{1}{2} J(\delta) \Omega^2 - \Pi(\delta) \right) = 0, \quad \delta = \frac{5\Omega^2}{16\pi G\rho}.$$

For the Earth, $\Omega = 7.29 \times 10^{-5} \text{ rad s}^{-1}$, $G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$, and $\rho = 5514 \text{ kg m}^{-3}$. Substitution gives $\delta = 0.00143$ and hence $a_1/a_3 = (1 + \delta)^3 \approx 1 + 3\delta = 1 + 1/232$. This result was already known to Newton and also follows from the Maclaurin formula. The observed ratio is approximately $a_1/a_3 = 1 + 1/298$. The discrepancy is explained by the nonuniform density distribution inside the Earth.

2. Earth's flattening for an inhomogeneous density distribution

Consider a model of the density distribution inside the planet in which the density depends quadratically on the distance r from the center: $\rho(r) = A - Br^2$, $0 \leq r \leq a$, where a is the mean radius of the planet. The constants A and B are determined from two global integral characteristics: the total mass M and the moment of inertia I . For the Earth, this gives

$$A \approx 10\,530 \text{ kg m}^{-3}, \quad B \approx 2.06 \times 10^{-10} \text{ kg m}^{-5}, \quad a = 6.37 \times 10^6 \text{ m}.$$

Figure 1 compares the density distribution in the Preliminary Reference Earth Model (PREM) with the quadratic approximation $\rho(r) = A - Br^2$.

According to PREM [7], the density at the Earth's center is about 13 g cm^{-3} , whereas the density of the crust near the surface is about 2.7 g cm^{-3} . The quadratic model slightly underestimates the density at both the center and the surface. To preserve the correct total mass and moment of inertia, it averages these values and produces a smoother profile. Nevertheless, this classical approximation is useful for analytical calculations in celestial mechanics and accurately reproduces integral quantities.

To leading order in δ , the moment of inertia $J(\delta)$ and the gravitational potential energy $\Pi(\delta)$ are

$$J(\delta) \approx J_0(1 + 2\delta), \quad J_0 = \frac{8\pi a^5}{15} \left(A - \frac{5}{7} B a^2 \right),$$

$$\Pi(\delta) \approx \Pi_0 \left(1 - \frac{4}{5} \delta^2 \right), \quad \Pi_0 = -\frac{16\pi^2 G}{15} a^5 \left(A^2 - \frac{10}{7} A B a^2 + \frac{25}{63} B^2 a^4 \right).$$

The equilibrium equation gives $\delta = -5J_0\Omega^2/(8\Pi_0)$. For the Earth, $J_0 = 8.0 \times 10^{37}$, $\Pi_0 = -2.47 \times 10^{32}$, and $\delta = 0.00108$. The corresponding flattening, $f = (a_1 - a_3)/a_1 \approx 3\delta = 0.00323$, is substantially closer to the observed value $1/298 \approx 0.00335$.

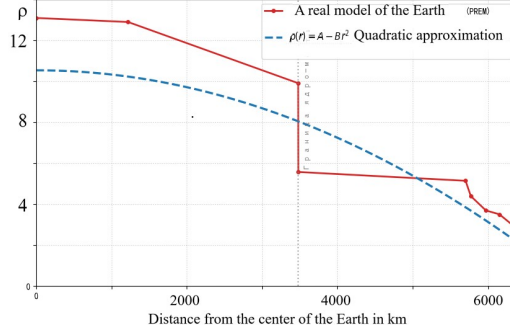


FIGURE 1. Comparison of the PREM density profile with the quadratic approximation for the Earth.

3. Effect of shell elasticity and other factors

Chandrasekhar [4] investigated the shape of a rotating liquid drop using the tensor virial method. The same approach can be extended to account for the elasticity of the Earth's shell by adding to Π the potential energy of the shell, which is proportional to its surface area [5]. V. A. Solonnikov and I. V. Denisova [7] studied the existence of axisymmetric equilibrium configurations of two-layer incompressible and compressible viscous liquids, taking into account surface tension and self-gravity.

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Set-theoretic distance and critical points of the surface distances on the example of asteroid 2102 Tantalus

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Abstract. In this paper, the evolution of the two-dimensional surface of the distances between the Keplerian orbits of asteroids and the Earth is investigated by long-term numerical integration of the equations of motion. The distance surface is considered as a smooth function on the torus of eccentric anomalies, the critical points of which determine local minima, maxima and saddles, including the value of the minimum distance between orbits (MOID). The model dynamics of MOID on the distance surface and the relevance of this status to different local minima during evolution over long time intervals are investigated.

Introduction

Determining the proximity of Keplerian trajectories is a fundamental task of celestial mechanics, which is of critical importance for assessing the risks of near-Earth asteroids (NEOs) or potentially dangerous objects (PHOs) colliding with our planet. From the point of view of mathematics, this problem is formulated as the search for the minimum distance between points of two one-dimensional sets in three-dimensional Euclidean space, that is, the search for the minimum distance between orbits (MOID). Historically, MOID search algorithms have evolved from pure analysis elliptical cases to more general configurations. In the classical work of K.V.Kholshevnikov and N.N.Vasiliev [2], the problem of determining critical points for two ellipses, it was reduced to finding the roots of a trigonometric polynomial of the 8th degree. Modern numerical methods make it possible to find all stationary points of the distance function by solving an algebraic equation of the 16th degree, ensuring high accuracy and reliability of calculations. The importance of finding all critical points (minima, maxima, and saddle points), not just the global minimum, is dictated by the topology of the distance function. For

two ellipses, this function is defined on a two-dimensional torus, and according to Morse theory [3], in the non-degenerate case it must have at least four critical points.

In cases with hyperbolic orbits, the domain of definition becomes non-compact (cylinder or plane), however, it remains there. the existence of several stationary points defining the geometry of convergence. Studies show that local minima and even saddle points may correspond to areas of possible close approaches during the evolution of the orbital elements.

Despite the availability of efficient algorithms for static computing MOID, the question of how these critical points move across the distance surface under the influence of secular perturbations remains less studied. Most existing methods focus on instantaneous distance values. In this paper, the evolution of the surface is investigated. distances over long time intervals. We analyze the trajectories of critical points along the surface of the distance, which allows us to better understand the mechanisms of changing the conditions of convergence of celestial bodies. Special attention is paid to the movement of points over significant time intervals, when a change in the elements of the orbits leads to topological rearrangements of the distance surface.

1. Set-theoretic distance and the critical points of the distance surface.

To find the distance between two points in elliptical orbits, we use the function given in the article by K.V.Kholshevnikov and N.N.Vasilyev [2]. In the same article, it is shown that the problem of finding the minimum distance between two points in elliptical orbits is reduced to finding the roots of a trigonometric polynomial of the 8th degree, which is equivalent to an algebraic polynomial of the 16th degree. We also note that our distance function, which depends on two eccentric anomalies, is defined on the torus. Hence, the number of critical points found according to Morse theory [3] satisfies the following equality:

$$N_{\text{saddle}} = N_{\text{min}} + N_{\text{max}},$$

where N_{saddle} is the number of saddle tokens,

N_{min} is the number of points of minimum distances,

N_{max} is the number of points of maximum distances.

This is a very important point in the context of MOID, because as will be shown later, it is not a specific point on the surface, but only a status that can be transferred between local minima. To assess the topological state of the interaction of two orbits, the value of the gearing coefficient is used, the calculation method of which is given in the textbook the manual [1]. According to him, this value for two elliptical orbits is calculated much more simply than the minimum distance between orbits (MOID).

2. The results are based on the example of asteroid 2102 Tantalus (1975 YA).

Let's consider the results of the study using the example of asteroid 2102 Tantalus from the Apollo group. During the simulation, the following phenomenon is observed over a time interval of 50,000 years: the inconstancy of the number of critical points. The observed change in their number ($6 \rightarrow 8 \rightarrow 6$) is interpreted as a result of bifurcations of the distance function that occur when the parameters of the orbits change.

As a result of the evolution of the distances between the orbits, a characteristic structure appears at critical points, superficially resembling a catastrophe of the type A "swallow tail" from the theory of catastrophes, which is formed as a result of the appearance of two critical points: a local maximum and a saddle point. One of the local maxima increases and becomes global, which is confirmed by comparing the values of the distance function at all critical points, but its predecessor in this status merges with the saddle point and disappears.

In all numerical experiments, as a result of evolution, a consistent behavior of the distances between orbits at critical points, perihelion and aphelion distances, eccentricity and inclination with a gearing coefficient is observed. The high peak of the gearing coefficient coincides with the peak of the global maximum, and the low peak coincides with the moment of the transition of the global status between the maxima in the "swallow's tail".

It is this apparent consistency in behavior that has become the motivation for research on the 500,000 year interval, where a long-period structure is observed. Similarly, the "highs of the maximum" occur at high peaks of the gearing coefficient, and the moments with the appearance of new critical points at low ones. At the same time, we note, that the positions of the "caps" of both types of peaks exceed 0 in values, which corresponds to the absence of engagement. Moreover, if the evolution of the semimajor axis, perihelion and aphelion are compared with the evolution of the gearing coefficient, it becomes clear that at the moments of "finding at low peaks of the gearing coefficient and the appearance of new critical points, the asteroid's orbit becomes external to the Earth, and at the moments of "being at high peaks" and "maximum value." On the contrary, the Earth's orbit is external to the asteroid. This corresponds to the transition between classification groups Cupid and Apollo.

During modeling for such long time intervals, the question arises about the stability and impact of errors in the initial data. The task is purely a model one and uses only Newtonian mechanics, but it is still interesting to find out how the influence of errors in the initial data will affect the results of the study. It turned out that the tube of trajectories when varying the Tantalum elements according to known errors, it has a small width for them. 50 experiments were conducted, and deviations were generated according to a normal distribution.

It has been numerically established that despite the change in the distances between the orbits at critical points, the general structure of evolution with a

"swallow tail" remains: somewhere it is formed earlier, somewhere later. This fact indicates the structural stability of the observed effect.

Conclusion

In the course of the work, studies were conducted on the evolution of the distances between the orbits of the asteroid 2102 Tantalus and the Earth. The equations of motion were integrated for different time intervals from 5 to 50,000 years, and graphs of the evolution of the Keplerian elements of the orbit and the distances between the orbits were studied in all variants. Based on the results of the study, the following conclusions can be drawn::

- changes in the orbital elements lead to bifurcations of the distance function, accompanied by the birth and disappearance of pairs of critical points;
- there is a consistent behavior between the evolution of a part of the elements the asteroid's orbit, the gearing coefficient, and the appearance or disappearance of critical points;
- it is shown that the minimum distance between orbits (MOID) is not associated with a fixed critical point of the distance function, but can switch between different local minima during the evolution of the orbital elements;
- it is shown that when the elements are varied in accordance with the errors in their definition, the pattern of changing orbits and the appearance/disappearance of critical points persists.

The results obtained indicate the need to take into account the full the structures of the critical points of the distance function in the analysis of the long-term dynamics of the approaches of celestial bodies.

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The history of the birth of cosmic dynamics at St. Petersburg State University

Elena Nikolaevna Polyakhova, Vladimir Stepanovich Korolev

Abstract.

The history of the birth on the basis of the Departments of Theoretical Mechanics, Celestial Mechanics and Control Theory of the new Department of Controlled Motion Mechanics at the new Faculty of Applied Mathematics and Control Processes (PM–PU) of Leningrad University (LSU), which jointly developed analytical and computational capabilities for solving problems of cosmic dynamics.

Introduction. Stages of birth.

An account of the events of the next stage of development and the completion of K.E. Tsiolkovsky's fantasies and first solutions to problems, F.A. Zander's scientific projects [1, 2], as well as S.P. Korolev's practical developments of the main tasks of Cosmonautics, which laid the foundation and created the foundation for space exploration and flights to other planets of the solar system based on analytical methods of space exploration speakers.

There were departments of theoretical mechanics, control theory and celestial mechanics at Leningrad University, whose heads were professors Nikolai Nikolaevich Polyakhov, Vladimir Ivanovich Zubov and Viktor Sergeevich Novoselov [3-6]. They proposed to form the Faculty of Applied Mathematics and Management Processes at Leningrad State University.

They turned to academics and other University professors for help and support. And they received a government decision – "So be it."

And we have a Department of Controlled Motion Mechanics. It became related to other departments, and together they created the Faculty of Applied Mathematics and Management Processes (PM-PU) and even the Institute of Computational Mathematics and Control Processes, where scientific laboratories were formed with the participation of employees and graduates of the departments of mathematics, mechanics, and astronomy of Leningrad State University.

They invited their students and graduate students. And the backbone or the basis for the life and work of the new faculty and various departments was formed. Some of them were remarkable, who became candidates and doctors of sciences:

Ageev P.A., Alferov G.V., Babajanyants L.K., Kirpichnikov S.N., Kovalenko A.N.,
Korolev V.S., Martynova N.F., Pupyshev Yu.A., Shmyrov A.S.

They were joined by other active employees of PM-PU who were interested in scientific papers that they could find in libraries and scientific journals. Everything that was useful for his scientific work [7-11].

We note their efforts and efforts in scientific research when working with students in the process of learning sciences and practical application of the acquired knowledge for the benefit of their native University, Russia and their beloved city, which later became St. Petersburg again [12-23]. They had a huge range of tasks, problems, hypotheses and theories that turned into new achievements, solution methods and algorithms. It turned out that when the possibilities of theoretical, analytical and celestial mechanics¹²⁹ were combined with control processes involving computer systems of machines and algorithms, a scientific direction appeared at the University: Cosmic Dynamics!

The mutual influence of astronomy, mathematics, and mechanics has been evident for a long time. Simple observations of the starry sky and theoretical justifications for the movement of celestial objects (such as the Moon, Sun and stars, planets and moons, comets and asteroids) led to the discoveries of Kepler's and Newton's laws, which make it possible to predict the motion and properties of orbits, determine the position at various points in time for the observed bodies or structures. The gravitational forces of all bodies, the pressure of the forces of the luminous flux, and many other effects were discovered.

The use of reactive forces for mechanical systems of variable composition in the process of their transformation makes it possible to describe and control the movement of artificial bodies (rockets, spacecraft, systems or structures). The influence of solar radiation forces [1, 2, 8, 11] on the surfaces of bodies led to the creation of a special photogravity field [24-27]. The appearance of a solar sail in spacecraft helped to organize management to create optimal orbits for interplanetary flights or to stabilize and steady trajectories.

Conclusion

The development of qualitative methods and applications of theory to computing systems and algorithms for solving complex problems allows us to obtain remarkable results in cosmic dynamics. The monograph "Analytical theory of optimization in gravitational fields" provides a general scheme for solving problems of optimizing the controlled motion of spacecraft [6]. The results were presented in a report at the XXIV Congress of the International Federation of Astronautics (1973).

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Analytical Problems of Sailing in Space for Solar System Research

Elena Polyakhova and Vladimir Titov

Abstract. We discussed shortly the development of analytical Solar Sailing theory applications to heliocentric transfer trajectories with the low thrust of light pressure from the Sun. Some sail projects, already realised successfully included, are demonstrated.

We discussed shortly the development of analytical Solar Sailing theory applications to heliocentric transfer trajectories with the low thrust of light pressure from the Sun. This part of general problems of Cosmonautics and Celestial Mechanics connects with the name of russian pioneer of rocket motion theory in Soviet Union in 1920–1930 Friederich Arturovich Zander (1887–1933). He interested and developed not only rocket engine and construction principle but also Solar Sailing in space for the destination of Solar System planets – the so called heliocentric Sailing with reflecting space mirrors known as Solar Sails.

To 2026 three Solar Sailing research centrum exist in Russia: in Moscow (Keldysh Institute of Applied Mathematics of Academy of Science and some others organizations), in Saint Petersburg (mainly in St. Petersburg State University) and in Samara University of S. P. Korolev.

Structures of theoretical research are as follows: analytical laws of solar sail orientation relatively to solar radiation direction, optimal inclination of sail normal for the heliocentric transfer, optimal thrust parameters of sail, effects of degradation degree of sail material under the aggressive influence of solar wind corpuscular radiation, especially when the transfer goes toward the Sun, comparison with electrical engine power spacecrafts, ecological aspects of solar sailing (illumination of Arctic regions, shadowing by sail-shield against global warming of the Earth), space robots transportation under the shadow of sail surface and so on.

The demonstration of some sail projects, already realised successfully included, is proposed.

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The Jacobi – Poisson Method of Building First integrals by Integral Manifolds for Generalized Hamiltonian Systems

Andrei Pranevich and Mikalai Skabelka

Abstract. In this paper we consider the generalized Hamiltonian differential systems (or the Arzanyh systems). The Jacobi – Poisson method of building first integrals in the form of Poisson brackets by known integral manifolds is developed. The results obtained in this paper can be used in the analytical theory of differential equations and in the analytical mechanics.

Introduction

One of the remarkable properties of the Hamiltonian differential systems

$$\frac{dq_i}{dt} = \partial_{p_i} H(t, q, p), \quad \frac{dp_i}{dt} = -\partial_{q_i} H(t, q, p), \quad i = 1, \dots, n, \quad (1)$$

is its invariance with respect to canonical (contact) transformations (see, for example [1, pp. 146 – 150] and [2, pp. 285 – 301]). This property of the Hamiltonian differential system (1) is very essential in the construction of its first integrals (or constants of motion), and, in particular, the Hamilton – Jacobi method of integrating the Hamiltonian differential system (1) follows directly from this property.

The most general systems of differential equations that are invariant with respect to the group of canonical transformations are systems of canonical equations of rank $m \geq 0$ or the generalized Hamiltonian systems (the Arzanyh systems) [3, 4]

$$\begin{aligned} \frac{dq_i}{dt} &= \partial_{p_i} H(t, q, p) + \sum_{j=1}^m K_j(t, q, p) \partial_{p_i} H_j(t, q, p), \\ \frac{dp_i}{dt} &= -\partial_{q_i} H(t, q, p) - \sum_{j=1}^m K_j(t, q, p) \partial_{q_i} H_j(t, q, p), \quad i = 1, \dots, n, \end{aligned} \quad (2)$$

where $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, respectively, $t \in \mathbb{R}$, the functions $H, H_j \in C^2(D), K_j \in C^1(D)$, $j = 1, \dots, m$, and the domain $D = T \times G$, $T \subset \mathbb{R}$, $G \subset \mathbb{R}^{2n}$.

Among the general methods of integration of the Hamiltonian system (1), the Jacobi – Poisson method is of particular importance. It gives the possibility to find the additional (third) first integral of the Hamiltonian system (1) by two known first integrals of the Hamiltonian system (1). And thus, in certain cases, to build an integral basis of the Hamiltonian system (1). Due to this property, the Poisson method is included in almost all monographs and textbooks on analytical mechanics (see, for example, [1, p. 100] and [5, p. 184]) and formulated as

Theorem (the Jacobi – Poisson theorem). *Suppose twice continuously differentiable functions $g_1: D' \rightarrow \mathbb{R}$ and $g_2: D' \rightarrow \mathbb{R}$ are first integrals on the domain $D' \subset D$ of the Hamiltonian system (1). Then the Poisson bracket*

$$g_{12}: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] \quad \forall (t, q, p) \in D'$$

of the functions g_1 and g_2 is also a first integral of the Hamiltonian system (1).

In his *Lectures on Dynamics* [6, pp. 240 – 241], C. Jacobi referred to the Jacobi – Poisson theorem as "one of the most remarkable theorems of the whole of integral calculus. In the particular case when $H = T - U$, it is the fundamental theorem of analytical mechanics. ... After I discovered this theorem I communicated it to the Academies of Berlin and Paris as an entirely new discovery. But I noticed soon after that this theorem had already been discovered and forgotten for 30 years, because one did not appreciate its real meaning, but had only used it as a lemma in a entirely different problem". Of course, the Jacobi – Poisson theorem does not always supply further first integrals. In some cases the result is trivial, the Poisson bracket being a constant. In other cases the first integral obtained is simply a function of the original integrals. If neither of these two possibilities occurs, however, then the Poisson bracket is a further first integral of system (1).

Our aim in this paper is to develop the Jacobi – Poisson method for integral manifolds of the generalized Hamiltonian system (2). This article continues the authors' research [7, 8, 9, 10, 11, 12] on the integral characteristics of ordinary and multidimensional Hamiltonian differential systems.

Main Results

To avoid ambiguity, we stipulate the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(D)$ is the function

$$[u, v]: (t, q, p) \rightarrow \sum_{i=1}^n (\partial_{q_i} u(t, q, p) \partial_{p_i} v(t, q, p) - \partial_{p_i} u(t, q, p) \partial_{q_i} v(t, q, p)) \quad \forall (t, q, p) \in D.$$

A function $g \in C^1(D')$ is called a *first integral on the domain $D' \subset D$* of the generalized Hamiltonian system (2) if

$$\mathfrak{G}g(t, q, p) = 0 \quad \forall (t, q, p) \in D',$$

where the linear differential operator

$$\begin{aligned} \mathfrak{G}(t, q, p) = & \partial_t + \sum_{i=1}^n \left(\left(\partial_{p_i} H(t, q, p) + \sum_{j=1}^m K_j(t, q, p) \partial_{p_i} H_j(t, q, p) \right) \partial_{q_i} - \right. \\ & \left. - \left(\partial_{q_i} H(t, q, p) + \sum_{j=1}^m K_j(t, q, p) \partial_{q_i} H_j(t, q, p) \right) \partial_{p_i} \right) \quad \forall (t, q, p) \in D. \end{aligned}$$

A smooth manifold $g(t, q, p) = 0$ is said to be an *integral manifold* of the generalized Hamiltonian system (2) if the derivative of the function $g \in C^1(D')$ by virtue of the generalized Hamiltonian system (2) is the identically zero on the manifold $g(t, q, p) = 0$, i.e.,

$$\mathfrak{G}g(t, q, p) = \Phi(t, q, p), \quad \Phi(t, q, p)|_{g(t, q, p)=0} = 0 \quad \forall (t, q, p) \in D'.$$

By $I_{D'}$ and $M_{D'}$, we denote the set of all first integrals and integral manifolds on the domain D' of the generalized Hamiltonian system (2), respectively. The phrase "the function g is an integral manifold with function Φ on the domain D' of the generalized Hamiltonian system (2)" is denoted by $(g, \Phi) \in M_{D'}$.

The main result of this article is the following statement.

Theorem 1. *Suppose $(g_k, \Phi_k) \in M_{D'}$, $g_k \in C^2(D')$, $k = 1, 2$. Then the function*

$$g: (t, q, p) \rightarrow [g_1(t, q, p), g_2(t, q, p)] - \int_{t_0}^t \varphi(\tau) d\tau \quad \forall (t, q, p) \in D', \quad t_0 \in T,$$

is a first integral of the generalized Hamiltonian differential system (2) if and only if the following identity holds

$$A(t, q, p) + [g_1(t, q, p), \Phi_2(t, q, p)] - [g_2(t, q, p), \Phi_1(t, q, p)] = \varphi(t) \quad \forall (t, q, p) \in D',$$

where $\varphi \in C(D')$ and the function

$$A(t, q, p) = \sum_{j=1}^m \left| \begin{array}{cc} [g_1(t, q, p), H_j(t, q, p)] & [g_2(t, q, p), H_j(t, q, p)] \\ [g_1(t, q, p), K_j(t, q, p)] & [g_2(t, q, p), K_j(t, q, p)] \end{array} \right| \quad \forall (t, q, p) \in D'.$$

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The empiric integral of motion in 3:2 resonance

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Here we report about the empiric quasi-integral of motion, bounded proper orbital elements (proper frequency g , eccentricity and semiajor axis) for a special case 3:2 resonance with Jupiter. The dependences between each pairs of orbital elements of known members of Hilda group are approximated. On base these approximations, the expression, contained a , e , and g is obtained. The approximate expression, connecting the proper elements a, e, g is:

$$I_1 = 2g - 109296\delta_a + 165.81/e \approx 502 \quad (1)$$

where $\delta_a = a - 3.9703$ is the distance to nominal resonance in AU. However, non-linear expression is better conserved:

$$I_2 = 2g - 109296\delta_a + 49/e^{3/2} \approx 242 \quad (2)$$

Also, we performed numerical integrations and showed, that test particles with arbitrary initial orbital elements in vicinity 3:2 resonance rapidly migrate to the described dependence. Therefore, we can conclude that only orbits, satisfying the obtained condition, are stable in the vicinity of 3:2 resonance. However, we have detected a group of 9 asteroids that that cannot be included in the studied dependence. All of them have small eccentricities $e < 0.07$. Perhaps these objects have recently experienced a collision or were moving under the influence of an extremely strong Yarkovsky effect.

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Three-axis Rigid Body under Action of Gyroscopic Forces for Classical Problems of Celestial Mechanics

Tatiana Salnikova and Eugene Kugushev

Abstract. To determine the stable stationary rotations of a triaxial satellite relative to its center of mass located at a libration point of the restricted circular three-body problem, within the satellite approximation model, the problem of the motion of a rigid body with a fixed point under the action of gyroscopic forces with a constant intensity vector is used.

Introduction

We discuss a generalization of the restricted circular three-body problem — instead of a small point mass, we consider the rotation of an extended, small, triaxial rigid body around its center of mass, located at the libration point. In an approximate satellite model for this problem, we determine the moments of all forces acting on the rigid body, relative to the center of mass, namely, the moments of gravitational, centrifugal, and Coriolis forces. We prove that in a linear approximation in the parameter of the vector of angular velocity of the orbital motion of the Primaries, only the moment of the Coriolis (gyroscopic) force exerts the dominant influence.

Thus, it becomes necessary to consider the problem of a triaxial rigid body with a fixed point, where the body is subject to gyroscopic forces of constant intensity. An arbitrary constant vector of the gyroscopic force intensity is in some sense analogous to the gravity acceleration vector in the classical problem for a heavy rigid body with a fixed point. In the restricted three-body problem, it corresponds to twice the angular velocity vector of the circular orbital motion of the two primary bodies.

1. Gyroscopic forces with constant intensity vector, equations of motion and first integrals

Let us consider the problem of the motion of a rigid body with a fixed point under the action of gyroscopic forces with a constant intensity vector [1]. A similar problem arises when analyzing the motion of a rigid body relative to its center of mass in the extended circular three-body problem.

A point mass dm with radius vector $\bar{r}$ is subject to a gyroscopic force $d\bar{F} = \bar{c} \times \bar{v} dm$. The constant vector in absolute space $\bar{c}$ is the intensity vector of the gyroscopic forces. We introduce a system $Ox_1x_2x_3$ of principal axes of inertia of a rigid body, and denote by $\bar{r} = (x_1, x_2, x_3)$ the radius vector of the rigid body's points in this system, and by $\bar{\omega} = (\omega_1, \omega_2, \omega_3)$ angular velocity of rigid body.

The total moment of gyroscopic forces acting on a rigid body

$$\bar{M}_O = \int_V \bar{r} \times d\bar{F} = \int_V \bar{r} \times (\bar{c} \times (\bar{\omega} \times \bar{r})) dm,$$

$$\bar{M}_O = -\bar{\omega} \times J^* \bar{c}, \quad J^* = \text{diag}\{I_1^*, I_2^*, I_3^*\} = \frac{(I_1 + I_2 + I_3)}{2} E - J.$$

The equations of motion of a rigid body in Euler-Poisson form:

$$J\dot{\bar{\omega}} + \bar{\omega} \times (J\bar{\omega} + J^* \bar{c}) = 0, \quad \dot{\bar{c}} + \bar{\omega} \times \bar{c} = 0, \quad (1)$$

where $J = \text{diag}\{I_1, I_2, I_3\}$ is the matrix of the rigid body's inertia tensor in the principal axes of inertia.

Applying to a rigid body with the center of mass at the libration point, we have gyroscopic force intensity vector $\bar{c} = (0, 0, -2\Omega)$, where Ω is the orbital angular velocity of the main bodies,

In the satellite approximation model, when calculating the moments of two gravitational forces, the centrifugal force and the Coriolis force relative to the libration point, it was found that in the linear approximation, the Coriolis force moment has a dominant effect — the moments of centrifugal and gravitational forces are of the order of $\approx \Omega^2$, and the moment of gyroscopic forces is of the order of $\approx \Omega$, both in the vicinity of triangular and collinear libration points.

So, the body is acted upon only by gyroscopic forces, their power is zero, so the energy integral exists: $H = (J\bar{\omega}, \bar{\omega})/2 = h = \text{const}$. As usual, the Euler-Poisson equations also admit the trivial integral $F = (\bar{c}, \bar{c}) = c_0^2 = \text{const}$. By direct differentiation, by virtue of equations (1), we can verify that the function $K = (J(\bar{\omega} - \bar{c}/2), \bar{c}) = k$ is a first integral. If we move to a system rotating with angular velocity $\bar{c}/2$, then K is the projection onto the vector $\bar{c}$ of the rigid body's angular momentum.

Note that if the body is dynamically symmetric, then the system also contains a fourth integral [1]. Since equations (1) admit an invariant measure with constant density, in this case they are integrated by quadratures.

2. Stability of the stationary rotations

Substituting the condition $\bar{\omega}(t) = \text{const}$ into (1), we obtain the conditions for the body's motion to be stationary: $\bar{\omega} \times (J\bar{\omega} + J^*\bar{c}) = 0$, and $\dot{\bar{c}} + \bar{\omega} \times \bar{c} = 0$. Differentiating the first equation with respect to time and substituting $\dot{\bar{c}}$ from the second, we obtain that the vector $\bar{c}$ is collinear with the vector $\bar{\omega}$, i.e. $\bar{c} = \mu\bar{\omega}$, where μ is some number. In this case, $\bar{\omega} \times \tilde{J}\bar{\omega} = 0$, where $\tilde{J} = J + \mu J^*$. Since J and $\tilde{J}$ are diagonal matrices and $\det J \neq 0$, then the eigenvectors of matrix J are also eigenvectors of matrix $\tilde{J}$. Consequently, rotations around an arbitrary principal axis of inertia with an arbitrary angular velocity are stationary rotations of a rigid body.

If μ is such that the matrix $\tilde{J}$ is non-singular (the non-critical case), then the set of its eigenvectors coincides with the set of eigenvectors of the matrix J . Then there are no other stationary motions of the rigid body.

Without loss of generality, we assume that the absolute coordinate system is chosen such that in it $\bar{c} = (c_0, 0, 0)$, $c_0 > 0$. Consider the stationary solution

$$\bar{\omega}(t) = (\omega_0, 0, 0), \quad \bar{c}(t) = (c_0, 0, 0), \quad c_0 > 0 \quad (2)$$

projected onto the principal axes of inertia of the body. Then $h = I_1\omega_0^2/2$, $k = I_1c_0\omega_0 - I_1c_0^2/2$

Assuming that the quantities $\omega_2, \omega_3, c_2, c_3$ are infinitesimal, we can express the quantities ω_1 and c_1 from the integrals H and F to third order. Substituting them into the integral K , we obtain to third order:

$$K = f_1(\omega_2, c_2) + f_2(\omega_3, c_3) + O(\omega_2^4 + \omega_3^4 + c_2^4 + c_3^4),$$

where

$$\begin{aligned} f_1(\omega_2, c_2) &= -\left(\frac{I_2c_0}{2\omega_0}\right)\omega_2^2 + I_2c_2\omega_2 + \left(-\frac{I_1\omega_0}{2c_0} + \frac{I_1 - I_2}{2}\right)c_2^2 \\ f_2(\omega_3, c_3) &= -\left(\frac{I_3c_0}{2\omega_0}\right)\omega_3^2 + I_3c_3\omega_3 + \left(-\frac{I_1\omega_0}{2c_0} + \frac{I_1 - I_3}{2}\right)c_3^2. \end{aligned}$$

According to Routh's theory, for the stationary motion (2) to be Lyapunov stable, it is sufficient that at the point $(\omega_2, \omega_3, c_2, c_3) = (0, 0, 0, 0)$ the quadratic part of K reaches a strict minimum or strict maximum. But the sign of the coefficient of ω_2^2 in the form f_1 is the same as the sign of the coefficient of ω_3^2 in the form f_2 . This means that the type of extremum of these forms (maximum or minimum) is the same. Therefore, a sufficient condition for stability is that both forms have definite signs. Using the fact that $I_2 > 0$ and $I_3 > 0$, we find that this condition consists of the simultaneous satisfaction of two inequalities:

$$(I_1 - I_2) \left(\frac{c_0}{\omega_0} - 1 \right) < 0, \quad (I_1 - I_3) \left(\frac{c_0}{\omega_0} - 1 \right) < 0 \quad (3)$$

Assuming that $I_2 \leq I_3$ is certain, we obtain three cases:

1. Stationary rotation around the axis with the smallest moment of inertia ($I_1 < I_2 \leq I_3$) is stable for $0 < \omega_0 < c_0$.

2. Stationary rotation about an axis with the intermediate moment of inertia ($I_2 < I_1 < I_3$) is unstable since the conditions (3) cannot be performed simultaneously.

3. Stationary rotation about an axis with the largest moment of inertia ($I_2 \leq I_3 < I_1$) is stable in two cases. If $\omega_0 > 0$, then it is sufficient that $c_0 < \omega_0 < +\infty$. If $\omega_0 < 0$, then it is sufficient that $-\infty < \omega_0 < 0$.

Unlike Euler's problem for a triaxial rigid body, in which stable rotations around the largest or smallest axis of the ellipsoid of inertia occur at any values of angular velocity, in this formulation, the intervals of admissible values of angular velocity for stationary states are not arbitrary.

Note that the obtained conditions for the stability of stationary rotation are sufficient conditions.

Conclusion

The above study has important application to the analysis of the evolution of a satellite with the center of mass at the libration point. Then gyroscopic force intensity vector $\bar{c} = (0, 0, -2\Omega)$ specifies one of the directions of the absolute coordinate system.

The results of an analytical study within the satellite approximation are compared with the results of numerical modeling for parameters of the Earth-Moon-Satellite system for various libration points (and also taking into account solar perturbation). Analytically obtained steady states are justified by numerical simulations.

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Modified Collocation Methods for Functional Differential Equations

Stepan Samulevich

Abstract. We modify classical collocation methods so that delayed and advanced arguments are evaluated by the same polynomial that represents the solution on a step. The derivative is interpolated at Lobatto nodes in Newton form, while shifted stage values are obtained by integrating this polynomial beyond the current step when necessary. The resulting implicit equations are solved by a Gauss–Seidel iteration initialized by extrapolation from the preceding step. The construction applies to first- and second-order functional differential equations and treats delay and advance terms uniformly.

Keywords: collocation methods, functional differential equations, delay differential equations, advanced arguments, Lobatto nodes.

Introduction

Functional differential equations contain values of the unknown function at shifted times. A representative first-order problem is

$$\dot{x}(t) = f(t, x(t), x(t \pm \tau)), \quad x(t_0) = x_0, \quad \tau > 0. \quad (1)$$

The minus sign describes a delay and the plus sign an advance. Standard delay solvers usually recover shifted values from a stored history, whereas an advanced value is not yet available. Nested integration can handle both cases, but requires repeatedly solving auxiliary initial-value problems [1, 2, 3].

The proposed modification is simple: shifted values are evaluated from the collocation polynomial itself. Delay and advance terms therefore enter the numerical scheme in exactly the same form. This is useful for problems that must be integrated both forward and backward in time.

1. Modified collocation scheme

Let

$$t_i = t_0 + hc_i, \quad 0 = c_1 < c_2 < \dots < c_s = 1,$$

where c_i are Lobatto nodes and h is the step size. Define

$$p(c) := \dot{u}(t_0 + hc), \quad (2)$$

where u approximates the solution on the step. The collocation conditions are

$$p(c_i) = f_i, \quad f_i = f(t_i, u(t_i), u(t_i \pm \tau)), \quad i = 1, \dots, s. \quad (3)$$

Introduce

$$\mathcal{I}_k p(c) := \frac{1}{(k-1)!} \int_0^c (c-\xi)^{k-1} p(\xi) d\xi, \quad k \geq 1. \quad (4)$$

For a first-order equation,

$$u(t_i) = x_0 + h\mathcal{I}_1 p(c_i), \quad u(t_i \pm \tau) = x_0 + h\mathcal{I}_1 p(c_i \pm \delta), \quad \delta = \frac{\tau}{h}. \quad (5)$$

The second formula is the essential modification. If $c_i \pm \delta$ lies outside $[0, 1]$, the same collocation polynomial is extrapolated rather than switching to a separate history or advance procedure.

For a second-order equation

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t), x(t \pm \tau), \dot{x}(t \pm \tau)), \quad (6)$$

set $p(c) = \ddot{u}(t_0 + hc)$. With $v = \dot{u}$,

$$\begin{aligned} u(t_0 + hc) &= x_0 + hx'_0 c + h^2 \mathcal{I}_2 p(c), \\ v(t_0 + hc) &= x'_0 + h\mathcal{I}_1 p(c), \\ u(t_0 + h(c \pm \delta)) &= x_0 + hx'_0(c \pm \delta) + h^2 \mathcal{I}_2 p(c \pm \delta), \\ v(t_0 + h(c \pm \delta)) &= x'_0 + h\mathcal{I}_1 p(c \pm \delta). \end{aligned} \quad (7)$$

Thus, the first- and second-order constructions differ only in the number of integrations applied to p .

1.1. Newton–Lobatto implementation

Write

$$p(c) = \sum_{j=1}^s a_j \prod_{m=1}^{j-1} (c - c_m), \quad a_j = [f_1, \dots, f_j]. \quad (8)$$

The integrated basis functions

$$\gamma_{j,k}(c) := \mathcal{I}_k \left[\prod_{m=1}^{j-1} (c - c_m) \right] \quad (9)$$

satisfy

$$\gamma_{j,k}(c) = (c - c_{j-1})\gamma_{j-1,k}(c) - k\gamma_{j-1,k+1}(c), \quad \gamma_{1,k}(c) = \frac{c^k}{k!}. \quad (10)$$

Hence, all stage values are linear combinations of a_j with precomputed weights evaluated at c_i or $c_i \pm \delta$.

Because f_i depends on the polynomial coefficients, the stage equations are implicit. They are solved by Gauss–Seidel iteration. The first step uses the initial condition, or a linear predictor for a second-order problem. Later steps use $p_{\text{prev}}(1+c_i)$ as the predictor. Since Lobatto nodes include both endpoints, the final derivative of one step is reused as the first derivative of the next step (FSAL).

The algorithm is:

1. predict the stage derivatives from the preceding polynomial;
2. integrate the Newton polynomial at c_i and $c_i \pm \tau/h$ to form all stage arguments;
3. update the stages and divided differences by Gauss–Seidel iteration;
4. evaluate the integrated polynomial at $c = 1$ to advance the solution.

2. Brief lunar-orbit example

As a short illustration, the method was applied to

$$\ddot{\mathbf{r}}(t) = -\frac{(\mu + \mu_{\oplus})\mathbf{r}(t)}{r^3} - \frac{3k_2\mu}{r^3} \left(1 + \frac{\mu}{\mu_{\oplus}}\right) \left(\frac{a_e}{r}\right)^5 \mathbf{r}(t - \tau), \quad r = \|\mathbf{r}(t)\|, \quad (11)$$

a simplified lunar tidal model [4]. The orbit was propagated for 40 years and then integrated backward. With $s = 11$, $h = 2.5$ days, and two Gauss–Seidel iterations per step, the collocation run used 116861 right-hand-side evaluations. The reference eleventh-order Adams–Bashforth–Moulton delayed calculation [3], with $h = 1/8$ day, used 243953 evaluations.

The example indicates that the modified scheme can use substantially larger steps. A systematic stability and convergence analysis with respect to τ/h remains necessary.

Conclusion

The main modification is the uniform evaluation of current, delayed, and advanced stage values from one integrated collocation polynomial. Lobatto nodes, Newton interpolation, precomputed integral weights, FSAL reuse, and a Gauss–Seidel stage iteration make the resulting implicit method practical. The same construction covers first- and second-order equations without a separate numerical mechanism for advanced arguments.

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Using Apparent Motion Parameters as Features for Machine Learning-based Tracklet Association: An Analysis of Feature Discriminative Power

T. N. Sannikova

Abstract. The association of short observational arcs (tracklets) becomes critically challenging in spacecraft fragmentation events, where multiple fragments move along closely spaced trajectories. Traditional orbit-determination-based methods generate numerous false associations in such conditions. This study explores geometric features derived directly from positional measurements $\alpha(t)$, $\delta(t)$ and apparent motion parameters (angular rate, position angle, small-circle parameters). We compute features for each pair, analyze their separability, and identify the most informative ones. The selected set includes six approximation residuals and three geometric criteria: the alignment angle α_{align} , the separation ratio R_{sep} , and the position angle difference $\Delta\psi$. Other parameters were discarded due to overlapping distributions or noise sensitivity. Three classifiers (Decision Tree, Random Forest, LightGBM) trained on these features demonstrate high performance. This approach provides reliable association without preliminary orbit determination, making it suitable for real-time applications.

Introduction

The association of short observational arcs (tracklets) is a fundamental problem in space surveillance, particularly critical in spacecraft fragmentation events where multiple debris fragments move along closely spaced trajectories. Classical methods based on preliminary orbit determination often generate numerous false associations due to orbital solution degeneracy, and require high computational cost. As an alternative, geometric characteristics of apparent motion offer a promising source of information. Modern CCD observations provide dense coordinate series $\alpha(t)$, $\delta(t)$, enabling the direct computation of apparent motion parameters (AMPs) without orbit propagation. In this paper, we analyze the separability of

various features derived from positional measurements for pairwise tracklet classification, select a compact nine-feature set, and validate its performance using machine learning models.

1. Features for a Pair of Tracklets

The dataset comprises 840 tracklets obtained during 2022–2023 at the Crimean Astrophysical Observatory. 363 tracklets belong to 168 distinct objects observed over one night (2–7 tracklets per object); the remaining tracklets from the same nights provide diverse false associations. Pairs were formed with two constraints: no temporal overlap, and a time difference not exceeding 10 hours, yielding 5315 pairs (238 true associations, 5077 false associations).

Following [1], for each tracklet containing n points $(t_j, \alpha_j, \delta_j)$, we compute AMPs for the mean epoch $t_0 = (t_1 + t_n)/2$: position vector $\mathbf{D}$, angular rate μ , position angle ψ , geodesic curvature κ , normal $\mathbf{P}$ and distance p of the osculating small circle, and tangent vector $\mathbf{T}$.

For each pair of tracklets A and B , the following features are computed:

1. **Joint Approximation Residuals:** Six RMS residuals $\sigma_{\text{line}}, \sigma_{\alpha}, \sigma_{\delta}, \sigma_{D1}, \sigma_{D2}, \sigma_{D3}$ from second-degree polynomial fits to $\delta(\alpha)$, $\alpha(t)$, $\delta(t)$, and the direction cosines $D_1 = \cos \delta \cos \alpha$, $D_2 = \cos \delta \sin \alpha$, $D_3 = \sin \delta$.
2. **Alignment Angle** between the displacement vector and the mean tangent:

$$\alpha_{\text{align}} = \frac{180}{\pi} \arccos \left(\frac{|(\mathbf{D}_B - \mathbf{D}_A) \cdot \mathbf{T}_{\text{mean}}|}{|\mathbf{D}_B - \mathbf{D}_A|} \right) \quad [\text{deg}], \quad \mathbf{T}_{\text{mean}} = \frac{\mathbf{T}_A + \mathbf{T}_B}{|\mathbf{T}_A + \mathbf{T}_B|}.$$

3. **Separation Ratio** of measured angular separation to expected separation:

$$R_{\text{sep}} = \frac{\rho_{AB}}{\frac{1}{2}(\mu_A + \mu_B) \Delta t_{AB}},$$

where Δt_{AB} is the time interval between mean epochs and ρ_{AB} is the angular separation between mean positions.

4. **Position Angle Difference** (reduced to $[0^\circ, 180^\circ]$):

$$\Delta\psi = \frac{180}{\pi} \min(|\psi_A - \psi_B|, 2\pi - |\psi_A - \psi_B|) \quad [\text{deg}].$$

5. **Consistency of planes of small circles:**

$$\theta_P = \arccos(|\mathbf{P}_A \cdot \mathbf{P}_B|) \quad [\text{deg}], \quad \Delta p = |p_A - p_B|.$$

6. **Difference in angular velocities** $\Delta\mu = |\mu_A - \mu_B|$ [angular sec/s].
7. **Difference in geodesic curvatures** $\Delta\kappa = |\kappa_A - \kappa_B|$.

In addition, the time interval Δt_{AB} and the angular separation ρ_{AB} were evaluated as auxiliary features. Their individual distributions inherently overlap strongly, which is expected for representative observational data; nevertheless, they were tested in combination with the primary features to explore whether they could provide complementary discriminative information.

2. Separability Analysis and Feature Selection

Analysis of the training set reveals that α_{align} exhibits the highest separability: 76% of true pairs have α_{align} below 0.1° (only 0.4% for false), and all true pairs satisfy $\alpha_{\text{align}} \leq 0.73^\circ$. R_{sep} also performs strongly, with 91% of true pairs lying within (0.90, 1.01), compared to only 5% of false pairs. The position angle difference $\Delta\psi$ typically localizes near 0° for true pairs. However, due to the topocentric effect (loops in the apparent trajectory), larger values are possible — a case of $\Delta\psi \approx 95^\circ$ was recorded in the training sample. Nevertheless, 71% of true pairs and only 24% of false pairs have $\Delta\psi < 5^\circ$.

The joint approximation residuals provide strong discriminative power. For a threshold of $30''$, σ_{line} is below the threshold for 96.2% of true pairs but only for 11.5% of false pairs. The remaining five residuals exhibit a similar pattern: the fraction of true pairs with residual $\leq 30''$ ranges from 71% to 91%, while for false pairs it never exceeds 12%. This confirms that the smoothness of the combined trajectory is a reliable indicator of a true association.

However, several investigated features were excluded from the final set: the angle between small-circle normals θ_P and the distance difference Δp proved too sensitive to measurement noise on short arcs, with strongly overlapping distributions. The differences in angular velocity $\Delta\mu$ and geodesic curvature $\Delta\kappa$ also showed significant overlap between classes. Furthermore, the inclusion of the time interval Δt_{AB} and angular separation ρ_{AB} into the feature set did not yield any improvement in the classification metrics; consequently, they were excluded.

Thus, the final feature set comprises six approximation residuals (σ_{line} , σ_α , σ_δ , σ_{D1} , σ_{D2} , σ_{D3}) and three geometric criteria (α_{align} , R_{sep} , $\Delta\psi$).

3. Machine Learning Models and Feature Importance

Three classifiers were trained on the selected features: Decision Tree (DT), Random Forest (RF), and LightGBM gradient boosting. Class imbalance was compensated by weighting true pairs inversely to their proportion. The full dataset was split into training and test subsets in a 75:25 ratio, and hyperparameters were tuned via cross-validation to maximize the area under the precision-recall curve (AUC-PR). Each trained classifier outputs the probability p_{AB} that a pair of tracklets belongs to the same physical object; a pair is considered a true association if $p_{AB} \geq 0.5$.

Feature importance analysis shows that α_{align} is dominant in all models. In DT, it accounts for 92.4% of total importance, with R_{sep} as the second most important feature at 7.3%. In RF, α_{align} decreases to 42.2%, while approximation residuals σ_{line} (19.6%), σ_{D2} (16.2%), and σ_{D3} (8.5%) become significant. In LightGBM, the contributions are more evenly distributed: α_{align} (21.9%), R_{sep} (15.7%), σ_{D2} (13.1%), and σ_α (11.8%). The use of multiple parameters allows ensemble methods (Random Forest and LightGBM) to achieve a better precision–recall balance compared to the decision tree, which relies almost entirely on α_{align} .

The high AUC-PR values (≥ 0.98) obtained for all three classifiers confirm strong discriminative power across all thresholds p_{AB} . At the 0.5 threshold, Random Forest missed 2 true pairs, while the other models achieved perfect recall (1.0). The difference in false positives is relatively small: Random Forest performed best with only 6 incorrectly linked pairs and a precision of 0.906; Decision Tree made 8 errors, and LightGBM produced 9 false positives, reducing its precision to 0.870.

Conclusion

The method was tested on 20 tracklets obtained during an 80-minute observation session of debris fragments from the ATLAS 5 CENTAUR R/B upper stage breakup in September 2024. The target set contains 7 objects with 2 to 5 tracklets each, plus 2 isolated tracklets, yielding 183 possible pairs (18 true, 165 false). On the target dataset, Decision Tree yielded the fewest false positives (19) while achieving 100% recall; Random Forest and LightGBM also achieved 100% recall with 20 and 37 false positives, respectively.

After applying maximum weight matching in a bipartite graph to form consistent chains of tracklets, all three classifiers yield the correct partition on the target data. On the full dataset (168 chains (each with 2 to 7 tracklets) and 477 single tracklets), Random Forest and LightGBM correctly reconstructed 166 complete chains (98.8%), while Decision Tree reconstructed 165 (98.2%). Random Forest produced the fewest false merges (4) compared to 6 (LightGBM) and 11 (Decision Tree), but fragmented 2 chains; the other two models did not fragment any chains.

This study shows that approximation residuals that characterize trajectory smoothness and geometric criteria based on apparent motion parameters provide effective discrimination for tracklet association. The proposed approach requires no preliminary orbit determination, making it computationally efficient and suitable for real-time applications in space surveillance and fragmentation monitoring.

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Comparison of D -criteria and metrics in the space of Keplerian orbits

Gubina M.D. and Shaifulin V.Sh.

Abstract. This work is devoted to the analysis of the properties of the classical D -criteria of orbital dissimilarity, the g -metrics of Kholoshevnikov, and the g -metric based on physical integrals of motion. The behavior of these metrics is analyzed as the Keplerian orbital elements change. This analysis shows how well the available tools distinguish between different orbits. It also allows for the selection of a criterion or metric that exhibits better sensitivity under specific conditions.

Introduction

The assessment of the proximity of celestial bodies' orbits is a fundamental task of celestial mechanics, necessary for the search of asteroid families, meteoroid streams, and the identification of bodies of common origin. The paper provides a comparative analysis of existing criteria and metrics, and studies their local extremes and asymptotics, which allows for the selection of the optimal tool for specific astronomical tasks.

Definitions of D -criteria and metrics

Let us introduce the notations: a is the semi-major axis, e is the eccentricity, i is the inclination, Ω is the longitude of the ascending node, ω is the argument of pericenter, $q = a(1 - e)$, $p = a(1 + e)$. The indices 1 and 2 refer to the compared orbits.

The main formulas are summarized in table 1.

Here, I is the mutual inclination, P is the angle between the Laplace–Runge–Lenz vectors, Π and Π' are the differences in the longitudes of the pericenters, $E = -\mu/(2a)$, $L = \sqrt{\mu p}$, and q is the unit orientation quaternion.

TABLE 1. Main criteria and metrics

Name	Formula	Features
D_1 (Southworth–Hawkins)	$D_1^2 = \frac{(q_1 - q_2)^2}{L^2} + 4 \sin^2 \frac{I}{2} + (e_1 - e_2)^2 + (e_1 + e_2)^2 \sin^2 \frac{\Pi}{2}$	Not defined for circular orbits [1]
D_2 (Southworth–Hawkins)	$D_2^2 = \frac{(q_1 - q_2)^2}{L^2} + 4 \sin^2 \frac{I}{2} + (e_1 - e_2)^2 + (e_1 + e_2)^2 \sin^2 \frac{\Pi'}{2}$	Depends on the choice of the main plane [1]
D_3 (Drummond)	$D_3^2 = \left(\frac{q_1 - q_2}{q_1 + q_2} \right)^2 + \left(\frac{e_1 - e_2}{e_1 + e_2} \right)^2 + I^2 + \left(\frac{e_1 + e_2}{2} \right)^2 P^2$	Not defined when $e_1 = 0, e_2 > 0$ [2]
D_J (Jopek)	$D_J^2 = \left(\frac{q_1 - q_2}{q_1 + q_2} \right)^2 + (e_1 - e_2)^2 + 4 \sin^2 \frac{I}{2} + (e_1 + e_2)^2 \sin^2 \frac{\Pi}{2}$	Combination of D_1 and D_3 [3]
ρ_1	$\rho_1^2 = \frac{1}{L} (p_1 + p_2 - 2\sqrt{p_1 p_2} \cos I) + (e_1^2 + e_2^2 - 2e_1 e_2 \cos P)$	Contains the arbitrary parameter L [4]
ρ_2	$\rho_2^2 = (1 + e_1^2)p_1 + (1 + e_2^2)p_2 - 2\sqrt{p_1 p_2} (\cos I + e_1 e_2 \cos P)$	Basic Kholshevnikov metric [5]
ρ_3	$\rho_3^2 = \min_{\Omega_1, \Omega_2} \rho_2^2$	Factor space over Ω [5]
ρ_4	$\rho_4^2 = \min_{\omega_1, \omega_2} \rho_2^2$	Factor space over ω [5]
ρ_5	$\rho_5^2 = \min_{\Omega_1, \Omega_2, \omega_1, \omega_2} \rho_2^2$	Factor space over Ω and ω [5]
g (new)	$g^2 = \left(\frac{E_1 - E_2}{E_{\text{scale}}} \right)^2 + \left(\frac{L_1 - L_2}{L_{\text{scale}}} \right)^2 + d_q^2(q_1, q_2)$	Based on integrals of motion; quaternion representation [6]

Analysis results

Variations of the major semi-axis a_1

The extremum conditions and asymptotics are presented in table 2.

TABLE 2. Properties when varying a_1

Metric	Extremum condition	Asymptotics as $a_1 \rightarrow \infty$
D_1, D_2	$q_1 = q_2$	$\sim a_1^2$ (quadratic growth)
D_3, D_J	$q_1 = q_2$	$\rightarrow \text{const}$ (loss of sensitivity)
ρ_1	$\sqrt{p_2/p_1} \cos I = 1$	$\sim a_1$ (linear growth)
$\rho_2 - \rho_5$	$1 + e_1^2 = \sqrt{p_2/p_1} \mathcal{F}$	$\sim a_1$
g	polynomial of the 6th degree	$\sim a_1$

Conclusion: D_3 and D_J are unsuitable for distant objects, while the Kholshevnikov and g metrics provide a balanced sensitivity.

Variations of eccentricity e_1

Complexity of extremum equations: D_1, D_2 — linear; D_J — polynomial of 4th degree; D_3 — 7th; ρ_1 – ρ_5 — irrational, reduced to polynomial of 8th degree; g — $e_1 = 0$ or $L_1 = L_2$ (simplest).

Limits:

- When $e_1 \rightarrow 0^+$: for D_3 , the term $\left(\frac{e_1 - e_2}{e_1 + e_2}\right)^2 \rightarrow 1$ is non-physical; the others have finite limits.
- When $e_1 \rightarrow 1^-$: ρ has a derivative $\sim 1/\sqrt{1 - e_1}$ (strong divergence); g has the same, but with a smaller coefficient, which gives better stability.

Variations of the inclination i_1

Analytical conditions for extremum are obtained for D_2 ($i_1 = \arctan(\tan i_2 \cos(\Omega_1 - \Omega_2))$), ρ_4 (the same without πk) and ρ_5 ($i_1 = i_2$). The rest are transcendental equations.

When $i_1 \rightarrow 0^+$ and $i_1 \rightarrow \pi^-$, all metrics have finite limits. The metric g stands out because its derivative does not depend on the eccentricities, which is important for highly eccentric orbits.

Variations of the node Ω_1

All functions are 2π -periodic. The derivatives at $\Omega_1 = 0$ and 2π have opposite signs. The metrics ρ_3 and ρ_5 do not depend on Ω_1 . For ρ_4 , the extremum condition is $\sin i_1 \sin i_2 \sin(\Omega_1 - \Omega_2) = 0$, which implies $\Omega_1 = \Omega_2$ or $\Omega_1 = \Omega_2 + \pi$.

Variations of the pericenter argument ω_1

For D_1, D_2, D_J , the extremum condition is $\Pi = \pi n$ or $\Pi' = \pi n$, which is the simplest. For D_3 , it is transcendental. The metrics ρ_4 and ρ_5 do not depend on ω_1 . The derivatives ρ_1 – ρ_3 and g are proportional to $e_1 e_2$, so they become zero when $e_1 = 0$ or $e_2 = 0$.

Recommendations for selection

- **For general application** the metric g is preferred: physically justified, free from singularities, and has simple conditions for e and L .
- **For circular orbits** use only ρ or g (classical D is not applicable).
- **For highly eccentric orbits**, g is better than ρ due to lower divergence.
- **For coplanar orbits**, g gives an advantage (the derivative does not depend on e).
- **For fast calculations** D_1, D_2, D_J give simple conditions for ω_1 ; ρ_4, ρ_5 — for i_1 .
- **For quotient spaces** ρ_3, ρ_5 are independent of Ω ; ρ_4, ρ_5 are independent of ω .

Conclusion

The conducted comparative analysis shows that the classical D -criteria, despite their historical value, have serious mathematical drawbacks. The Kholshevnikov metrics are strict metrics, but they give complex extremum equations and strong divergence in the parabolic limit. The g metric, based on physical integrals, combines strictness, absence of singularities, and simplicity of the mathematical structure, which makes it the most universal tool for orbit identification problems.

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The Batrakov Effect

Ivan I. Shevchenko

Abstract. The report is dedicated to the centenary of the birth of the outstanding Russian scientist and celestial mechanician Yuri Vasilievich Batrakov. The report focuses on two of his classic works, “Periodic Motions of a Particle in the Gravitational Field of a Rotating Triaxial Ellipsoid” and “On the Distribution of Mean Motions of Asteroids Near Commensurabilities,” whose publication in 1957 and 1958 coincided with the beginning of the space age.

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On classical and modern approaches to the inverse problem of celestial mechanics

Sergey Shorokhov

Abstract. The inverse problem of celestial mechanics seeks to determine the potential force field that produces a given family of orbits. We address both classical and modern approaches to this problem. We present a classical analytical approach based on a system of partial differential equations (PDEs) for the force and energy functions. We also explore a neural network approach using neural ordinary differential equations (neural ODEs), outlining the forward- and reverse-time ODE systems required to train the neural network to learn the force function. This synthesis of approaches highlights the complementary nature of classical analytical methods and modern computational techniques.

Introduction

The inverse problem of celestial mechanics is a foundational challenge in dynamical astronomy and theoretical mechanics: given a prescribed family of orbits or a dataset of trajectory observations, determine the underlying potential force field that governs a dynamical system's motion [1].

The systematic study of the inverse problem gained significant momentum in the mid-20th century, driven by observations of satellites and other space objects. V. Szebehely [2] formalized the condition (first-order PDE) under which a given family of curves can be realized as orbits of a dynamical system, particularly in the context of satellite observations and Earth's rotation. G. Bozis [3] investigated the inverse problem for families of orbits in two and three dimensions, highlighting the role of energy function and integrability conditions. The exact solvability of partial differential equation (PDE) systems for force and energy functions was established in [4], where specific classes of potentials admitting prescribed orbital families were derived. Recent papers by T. Kotoulas [5] have explored the algebraic and geometric structures underlying these PDE systems, offering new approaches to their resolution.

Recently, the inverse problem has evolved beyond purely analytical methods. Massive astrometric datasets from missions such as Gaia, combined with advanced computational methods, demand new approaches. Meanwhile, machine learning techniques have proven effective at inferring functional dependencies directly from noisy, discrete observations. Our goal is to bridge these approaches by presenting both traditional PDE formulations and their contemporary neural network counterparts.

1. Partial differential equations approach

We study the inverse problem of constructing a force function $U(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3)$ for a material point in three-dimensional space in a Cartesian coordinate system, with equations of motion

$$\ddot{\mathbf{x}} = \nabla_{\mathbf{x}} U(\mathbf{x}), \quad (1)$$

where $\nabla_{\mathbf{x}}$ denotes the operation of computing the gradient with respect to $\mathbf{x}$, admitting a family of trajectories defined in domain $\Omega \subset \mathbb{R}^3$ by relations of the form

$$\omega_1(\mathbf{x}) = c_1, \omega_2(\mathbf{x}) = c_2, c_1, c_2 \in \mathbb{R} \quad (2)$$

The Jacobian matrix $\frac{\partial(\omega_1, \omega_2)}{\partial(x_1, x_2, x_3)}$ is assumed to have maximum rank.

When a material point moves in a potential force field, the energy integral is conserved:

$$\frac{1}{2} \|\dot{\mathbf{x}}\|^2 - U(\mathbf{x}) = h, \|\dot{\mathbf{x}}\|^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2.$$

Each curve within the family (2) corresponds to a specific energy level. Thus, the energy function $h(\mathbf{x})$ remains constant along any given trajectory:

$$h(\mathbf{x}) = h(\omega_1(\mathbf{x}), \omega_2(\mathbf{x})).$$

As was demonstrated in [4], solving the inverse problem of celestial mechanics requires investigating the compatibility and finding a solution to the following system of first-order partial differential equations for the unknown functions $U(\mathbf{x})$ and $h(\mathbf{x})$:

$$\begin{cases} \langle \nabla_{\mathbf{x}} \omega_1, \nabla_{\mathbf{x}} U - (U + h) \mathbf{p} \rangle = 0, \\ \langle \nabla_{\mathbf{x}} \omega_2, \nabla_{\mathbf{x}} U - (U + h) \mathbf{p} \rangle = 0, \\ \langle E \mathbf{s}, \nabla_{\mathbf{x}} U - (U + h) \mathbf{p} \rangle = -2(U + h) \operatorname{div}(E \mathbf{s}) - \langle \mathbf{q}, \nabla_{\mathbf{x}} h \rangle, \\ \langle \mathbf{s}, \nabla_{\mathbf{x}} h \rangle = 0, \end{cases} \quad (3)$$

where $\mathbf{s} = \nabla_{\mathbf{x}} \omega_1 \times \nabla_{\mathbf{x}} \omega_2$, $E = \frac{1}{\|\mathbf{s}\|^2} \langle \mathbf{s}, \operatorname{rots} \rangle$, $\mathbf{p} = \frac{1}{\|\mathbf{s}\|^2} (\nabla_{\mathbf{x}}^T \mathbf{s}) \mathbf{s}$, $\mathbf{q} = \mathbf{p} \times \mathbf{s}$. Here, the angle brackets denote the scalar product in $\mathbb{R}^3$, and the operation $\times$ denotes the vector product in $\mathbb{R}^3$.

If the functions $\omega_1(\mathbf{x})$ and $\omega_2(\mathbf{x})$ in (2) are known, the PDE system (3) is a linear first-order system of four PDEs in the two unknowns $U(\mathbf{x})$ and $h(\mathbf{x})$. But often the functions $\omega_1(\mathbf{x})$ and $\omega_2(\mathbf{x})$ in (2) are not given in advance, and PDE

system (3) is a nonlinear system of four PDEs in the four unknowns $\omega_1(\mathbf{x})$, $\omega_2(\mathbf{x})$, $U(\mathbf{x})$, and $h(\mathbf{x})$, where three PDEs in (3) are first-order and one is third-order.

Because solving the inverse problem of celestial mechanics involves a system of PDEs for several unknown functions, this suggests the possible application of either Riquier theory [6] or the Gröbner basis method [7]. In practice, finding exact solutions to PDE system (3) is highly challenging. To address this complexity, various simplifying assumptions are introduced regarding the form of the unknown functions [5]. This yields simpler differential equations for which exact analytical solutions can sometimes be obtained.

2. Neural network approach

In the neural network approach, the unknown force function U is assumed to be the output of a neural network $\mathcal{F}$ with input $\mathbf{x} \in \mathbb{R}^3$ and parameters (weights) $\boldsymbol{\theta} \in \mathbb{R}^p$.

So we analyze a second-order ODE system

$$\ddot{\mathbf{x}} = \nabla_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}) \quad (4)$$

where the gradient of the force function $U(\mathbf{x}, \boldsymbol{\theta})$ with respect to the variables $\mathbf{x}$ in (4) is computed using the exact automatic differentiation implemented in modern deep learning frameworks.

We train a neural network $\mathcal{F}$ on the existing set of data points

$$\mathbf{D} = \left\{ \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, x_1'^{(k)}, x_2'^{(k)}, x_3'^{(k)}, t_k \right), k = \overline{1, N} \right\},$$

where the vector $\mathbf{x}_k = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)} \right)$ represents the observed position and the vector $\mathbf{x}'_k = \left(x_1'^{(k)}, x_2'^{(k)}, x_3'^{(k)} \right)$ represents the observed velocity of the material point at time t_k , so that the trajectories $\mathbf{x}(t; \boldsymbol{\theta})$ of system (4) match the observations in $\mathbf{D}$ as closely as possible (i.e., the trajectories pass through the points in $\mathbf{D}$ or are in their vicinity).

We replace the system (4) of three second-order equations with the following system of six first-order equations in the variables $(\mathbf{x}, \mathbf{x}') = (x_1, x_2, x_3, x'_1, x'_2, x'_3)$:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{x}', \\ \dot{\mathbf{x}}' = \nabla_{\mathbf{x}} U(\mathbf{x}, \boldsymbol{\theta}), \end{cases} \quad (5)$$

By integrating the system (5) using a numerical algorithm, we can obtain the set of trajectory points $(\mathbf{x}(t; \boldsymbol{\theta}), \mathbf{x}'(t; \boldsymbol{\theta}))$ at times $t_k, k = \overline{1, N}$: $\mathbf{x}_k = \mathbf{x}(t_k; \boldsymbol{\theta})$, $\mathbf{x}'_k = \mathbf{x}'(t_k; \boldsymbol{\theta})$.

We assume that the loss functional $\mathcal{L}$ for training the neural network $\mathcal{F}$ takes the form of the sum of the estimates of the deviations of the trajectory $\mathbf{x}(t; \boldsymbol{\theta})$

from the points in the observation set $\mathbf{D}$, measured in both position and velocity:

$$\mathcal{L}[\mathbf{x}(t; \boldsymbol{\theta}), \mathbf{x}'(t; \boldsymbol{\theta})] = \sum_{k=1}^N \rho(\mathbf{x}(t_k; \boldsymbol{\theta}), \mathbf{x}'(t_k; \boldsymbol{\theta}), \mathbf{x}^{(k)}, \mathbf{x}'^{(k)}), \quad (6)$$

where $\rho(\mathbf{x}(t_k; \boldsymbol{\theta}), \mathbf{x}'(t_k; \boldsymbol{\theta}), \mathbf{x}^{(k)}, \mathbf{x}'^{(k)}) = \|\mathbf{x}(t_k; \boldsymbol{\theta}) - \mathbf{x}^{(k)}\|^2 + \|\mathbf{x}'(t_k; \boldsymbol{\theta}) - \mathbf{x}'^{(k)}\|^2$. To minimize the loss $\mathcal{L}$ using gradient descent in the parameter space $\boldsymbol{\theta}$ of the neural network $\mathcal{F}$, we must compute the gradient of $\mathcal{L}$ with respect to $\boldsymbol{\theta}$.

Following [8], we introduce the adjoint variables $\boldsymbol{\chi}(t) = \nabla_{\mathbf{x}} \mathcal{L}$, $\boldsymbol{\chi}'(t) = \nabla_{\mathbf{x}'} \mathcal{L}$, and $\boldsymbol{\vartheta}(t) = \nabla_{\boldsymbol{\theta}} \mathcal{L}$, which represent the gradients of the loss functional $\mathcal{L}$ with respect to the states $\mathbf{x}$, the velocities $\mathbf{x}'$, and the parameters $\boldsymbol{\theta}$ at time $t \in [t_0, T_N]$.

The dynamics of the variables (gradients) $\boldsymbol{\chi}(t)$, $\boldsymbol{\chi}'(t)$, and $\boldsymbol{\vartheta}(t)$ are governed by systems of first- and second-order ODEs in reverse time, defined on the intervals $[t_k, t_{k+1}]$ for $k = N-1, N-2, \dots, 0$:

$$\begin{cases} \ddot{\mathbf{x}} = \nabla_{\mathbf{x}} U, & \mathbf{x}(t_{k+1}) = \mathbf{x}_{k+1}, \\ & \dot{\mathbf{x}}(t_{k+1}) = \mathbf{x}'_{k+1}, \\ \ddot{\boldsymbol{\chi}}' = \nabla_{\mathbf{x}'}^2 U \boldsymbol{\chi}', & \boldsymbol{\chi}'(t_{k+1}) = \nabla_{\mathbf{x}'} \rho(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1}, \mathbf{x}^{(k+1)}, \mathbf{x}'^{(k+1)}), \\ & \dot{\boldsymbol{\chi}}'(t_{k+1}) = -\nabla_{\mathbf{x}} \rho(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1}, \mathbf{x}^{(k+1)}, \mathbf{x}'^{(k+1)}), \\ \dot{\boldsymbol{\vartheta}} = -\nabla_{\boldsymbol{\theta}} (\nabla_{\mathbf{x}}^T U \boldsymbol{\chi}'), & \boldsymbol{\vartheta}(t_{k+1}) = \mathbf{0}. \end{cases} \quad (7)$$

The system of first- and second-order ODEs (7) in the variables $\mathbf{x}(t)$, $\boldsymbol{\chi}'(t)$, and $\boldsymbol{\vartheta}(t)$ must be solved backward in time from t_N to t_0 .

Since the functional $\mathcal{L}$ depends on the position and velocity at the final time t_N and on the positions and velocities at intermediate times $t_1, t_2, \dots, t_{N-1}$, we can solve the system of equations (7) in reverse time on each interval $[t_{N-1}, t_N]$, $[t_{N-2}, t_{N-1}]$, etc. (in reverse order), and obtain the resulting gradients $\boldsymbol{\vartheta}(t_k)$ of the loss functional $\mathcal{L}$ at times $t_0, t_1, \dots, t_{N-1}$.

Thus, the gradient of the loss functional $\mathcal{L}[\mathbf{x}(t; \boldsymbol{\theta})]$ with respect to $\boldsymbol{\theta}$ is obtained as the sum of the gradients $\boldsymbol{\vartheta}(t_k)$ computed by solving the systems (7) on the time intervals $[t_{N-1}, t_N]$, $[t_{N-2}, t_{N-1}]$, ..., $[t_0, t_1]$:

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}[\mathbf{x}(t; \boldsymbol{\theta})] = \sum_{k=0}^{N-1} \boldsymbol{\vartheta}(t_k; \boldsymbol{\theta}). \quad (8)$$

Training the neural network for the force function U on families of curves obtained in [4] demonstrates the convergence of the learning process and the closeness of the trajectories of the ODE system (4) to the curves of the target family of orbits.

Conclusion

The inverse problem of celestial mechanics has evolved from a purely theoretical mathematical challenge into a dynamic, data-driven field of research. The classical approach, based on partial differential equations, provides rigorous necessary and sufficient conditions for the existence of potentials that produce prescribed orbital families.

However, the limitations of classical PDE methods in handling noisy, high-dimensional, or incomplete observational data have paved the way for modern machine learning techniques. Among these, the neural ODE framework offers a powerful alternative to traditional PDE-based formulations. By casting the learning process as a coupled system of forward- and reverse-time ODEs, this method efficiently computes gradients and learns complex potentials directly from data while preserving the continuous-time dynamical structure of the underlying physics.

Ultimately, the future of this field lies in the synergy between both paradigms. Classical PDE theory can contribute physical constraints, symmetries, and regularization terms to guide neural network training, ensuring that the learned potentials are both data-consistent and analytically sound. As astrometric datasets grow in volume and precision, this hybrid framework will become indispensable for uncovering the hidden gravitational landscapes of the universe.

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Dynamics of entering and leaving the quasi-satellite regime among the known Earth co-orbitals

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Abstract. A quasi-satellite is a small body that orbits the Sun while remaining near a planet for an extended period without being gravitationally bound to it. Quasi-satellites are an important part of the population of co-orbitals of planets in the Solar System. For Earth, quasi-satellites can be described as a subclass of co-orbitals that is better identified than other subclasses. The first quasi-satellite of Earth was revealed in 2004 and subsequently received the name Cardea. Currently, seven asteroids are more or less unanimously classified by specialists as Earth's quasi-satellites. Since the quasi-satellite motion of all known objects is temporary, understanding the patterns of its transformation is of great importance. To study these patterns, we propose a semi-analytical model that describes the secular evolution of asteroid motion in the 1:1 mean-motion resonance with Earth. As an example, we consider the repeated transitions of asteroid (164207) Cardea into and out of the quasi-satellite regime.

Introduction

For near-Earth asteroids, the 1:1 mean-motion resonance (MMR) manifests itself through the oscillatory behaviour of the resonant phase $\varphi = \lambda - \lambda_3$, where λ is the asteroid's mean longitude and λ_3 is Earth's mean longitude. In general, this resonance admits several distinct regimes: the quasi-satellite (QS), Trojan (T), and horseshoe (HS) regimes, as well as various compound states (e.g., QS+HS). These regimes differ in the value of the resonant phase around which they librate and in the libration amplitude [1]. In the QS regime, in particular, the resonant phase φ librates around a value close to 0° .

Studying a system with several coexisting resonant regimes requires analysing not only the transitions from non-resonant to resonant motion and vice versa, but also the transitions between different resonant modes. Such mutual transitions of

near-Earth asteroids between the QS regime and other regimes allowed by the 1:1 MMR have been examined, for instance, in [2, 3].

To study the secular evolution of quasi-satellite motions and to identify possible transformation scenarios, [4] and [5] used an analysis of approximate integrals of motion - adiabatic invariants (AI). The systematic use of this approach for investigating resonant dynamics in Celestial Mechanics was pioneered in [6]. How this method should be adapted to situations with coexisting resonant regimes is discussed in [7].

The studies [4] and [5] were mainly methodological in scope, focusing on the general features of quasi-satellite motion. Given the considerable number of known quasi-satellite asteroids, it is relevant to apply the adiabatic approximation systematically - to better understand the dynamics of both individual asteroids and the population as a whole. As a first step, we chose to examine the population of Earth's quasi-satellites.

Our preliminary estimations have shown that a realistic dynamical model cannot rely solely on Earth's gravitational attraction to describe the secular evolution of the heliocentric motion of its co-orbitals. Since [4] and [5] were based on the restricted three-body problem, we had to extend their model to include perturbations from five planets: Venus, Earth, Mars, Jupiter, and Saturn.

1. Semi-analytical model for the transformation of motion in the 1:1 MMR

In our semi-analytical model, we treated the planets as point masses moving on circular coplanar orbits. The equations of motion were initially written in Delaunay variables. We then applied a canonical transformation that made the resonant phase φ one of the phase variables.

The Hamiltonian of the problem was averaged over the fast orbital motions of the planets and the asteroid. The resulting intermediate system was split into a subsystem describing the evolution of φ and its conjugate momentum Φ , and a subsystem describing the secular evolution of the eccentricity e and the argument of perihelion ω . This separation was justified by the different characteristic rates of change of these variables:

$$\dot{\varphi} \sim \dot{\Phi} \gg \dot{e} \sim \dot{\omega}.$$

For fixed slow variables e and ω , the subsystem for the resonant phase is a Hamiltonian system with one degree of freedom. The action variable of this 1DOF Hamiltonian is an adiabatic invariant of the problem.

A further averaging, now performed along the solutions of the subsystem for the resonant phase, leads to evolutionary equations for the slow variables. Then, on the (e, ω) plane, one can construct regions such that when the solutions of the evolutionary equations reach the boundaries of these regions, transitions between resonant regimes occur (either in a deterministic or a quasi-probabilistic manner).

2. Transitions between resonant regimes for asteroid (164207) Cardea

Figure 1 shows the behaviour of the phase trajectories of the evolutionary equations for parameter values corresponding to the case of asteroid (164207) Cardea. The red lines show the evolution of the osculating elements e and ω during the secular evolution of the asteroid's orbit, obtained by numerical integration. The left half of the figure presents the results of the integration with initial data taken from the NASA/JPL Small Body Database (SBDB). The right half shows the evolution of a clone of the asteroid, which differs in the initial value of the mean anomaly.

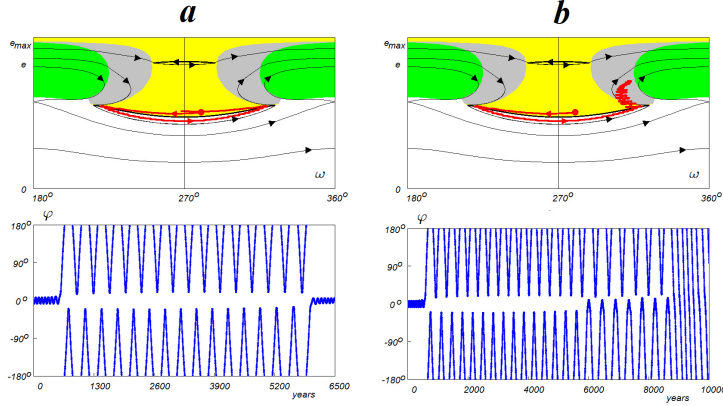


FIGURE 1. Presumed transitions between resonant regimes for asteroid (164207) Cardea. The upper part shows the phase portrait in the plane of slow variables (yellow region: coexistence of QS and HS regimes; grey region: compound QS+HS regime; white region: HS regime; green region: non-resonant (passing) regime). Panel (a) corresponds to the nominal values of the osculating variables; panel (b) corresponds to a clone differing in the mean anomaly (with a difference of $1.3 \cdot 10^{-4}$ degrees).

From the diagram, it can be seen that the asteroid is currently located in the region of coexistence of the QS and HS regimes (highlighted in yellow). The phase point characterising the dynamical state of the asteroid will then shift to the left until it reaches the left boundary of this region. An analysis of the behaviour of the trajectories of the evolutionary equations shows that the vicinity of the boundary can only be left in the HS regime. Thus, the first regime transition for Cardea should be deterministic. The evolution in the HS regime will bring the phase point to the right boundary of the coexistence region. Here, the transition is of a quasi-probabilistic nature: the asteroid may either return to the QS regime (Fig. 1a) or

enter the compound QS+HS regime (Fig. 1b). The transition probabilities can be calculated using the formulas given in [7]:

$$P_{HS \rightarrow QS} \approx 0.94461, \quad P_{HS \rightarrow QS+HS} \approx 0.05539.$$

Conclusion

The main result of our study is that a number of Earth's quasi-satellites (Cardea, Kamo'oalewa) and some of its other co-orbitals (2015SO₂, 2016CO₂₄₆, 2020PN₁) can be regarded as belonging to the population of regularly transient asteroids. During their secular evolution, these asteroids alternate between the QS and HS regimes, occasionally interrupted by transitions into the compound QS+HS regime. With the transition patterns established, we plan to obtain estimates of the distribution of regularly transient asteroids among the different regimes.

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Professor M. L. Lidov: Per Aspera ad Astra (to the 100th Anniversary)

Vladislav Sidorenko

Abstract. The year 2026 marks the 100th anniversary of the birth of Professor Mikhail Lvovich Lidov, an outstanding mechanician. M. L. Lidov made a great contribution to the success of many projects that are the pride of Russian cosmonautics: obtaining the first images of the far side of the Moon, the first lunar landing, the first artificial satellite of the Moon, and the first broadcasting satellites that covered the entire territory of the USSR. One of the fundamental effects of celestial mechanics—the von Zeipel–Lidov–Kozai effect—is named after him. My talk outlines the main milestones in Lidov’s scientific career and provides a brief overview of his research.

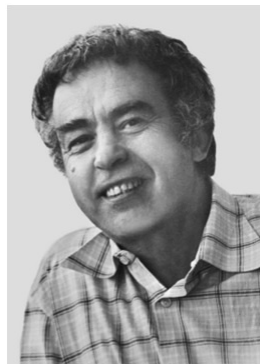


FIGURE 1. Mikhail Lvovich Lidov (04.10.1926 — 30.12.1993)

Introduction

Mikhail Lvovich Lidov was an leading Soviet scientist in the fields of celestial mechanics, space flight mechanics, and the study of the planets of the Solar System.

He was awarded the Lenin Prize and received numerous other state awards. He was equally at home in abstract theory and in practical engineering.

M. L. Lidov belongs to the generation of outstanding scientists and engineers whose talent made possible the emergence and rapid development of the space era. In the year of the centenary of his birth, it is worth recalling the main stages of his life and noting his unique scientific achievements, many of which remain relevant in modern celestial mechanics.

Early Years

Mikhail Lvovich Lidov was born on October 4, 1926, in Cherkassy. His childhood and youth fell on a dramatic period in Soviet history. His father, Lev Iosifovich Lidov, was convicted in the 1930s on charges of counter-revolutionary activity, spent time in camps and exile, and was fully rehabilitated in 1955. In 1940, Lidov's elder sister Vera was also arrested and convicted.

In 1944, Mikhail Lvovich began military service as an aircraft armament mechanic. After the war, while still in service, he completed his secondary education and in 1949 entered the Faculty of Mechanics and Mathematics of Moscow State University. After his demobilization in 1951, he continued his studies as a full-time student at the same faculty.

Lidov's first scientific investigations were carried out under the supervision of Academician L. I. Sedov. In 1954, his paper on gas motion taking into account Newtonian gravitation was published in the *Doklady of the Academy of Sciences of the USSR*. In the same year, he graduated with honours from Moscow University in Mechanics.

Because his father had been convicted of counter-revolutionary activity, the young graduate could not find a job for almost a year. In 1955, he was employed by the Interdepartmental Commission for Interplanetary Communications. In December 1957, with the support of V. A. Egorov and D. E. Okhotsimsky, M. L. Lidov joined the Department of Applied Mathematics (OPM — the Russian abbreviation for *Otdelenie Prikladnoi Matematiki*) of the Steklov Mathematical Institute of the Academy of Sciences of the USSR, later reorganized as the Institute of Applied Mathematics.

Scientific Achievements

One of Lidov's first studies at the Division of Applied Mathematics concerned the atmospheric drag experienced by the earliest artificial Earth satellites. From observations he determined the density of the upper atmosphere and identified regularities in its variation, including the relationship between the intensity of orbital decay and the migration of the orbital perigee between the daytime and nighttime sectors of the Earth's atmosphere.

An important stage of his activity was his participation in the preparation of the flight of the spacecraft Luna-3. Lidov and his colleagues from OPM needed to find a trajectory that would allow photographing the far side of the Moon and ensure the transmission of images to Earth using ground stations in the USSR. They proposed a trajectory for Luna-3 that involved a gravity assist at the Moon, which ensured the spacecraft's return to the radio visibility zone from the Soviet Union.

In October 1959, Luna-3 transmitted the first images of the far side of the Moon to Earth. For his contribution to the success of this mission, M. L. Lidov received the Lenin Prize in 1960.

In 1961, Lidov published his fundamental work on the evolution of planetary satellite orbits under gravitational perturbations from external bodies. It showed that the attraction of a distant celestial body can cause coupled periodic changes in the eccentricity of the orbit and its orientation in space. Under certain conditions, the eccentricity can increase so significantly that the satellite approaches the surface of the central body or collides with it.

As an illustration, M. L. Lidov considered the motion of a hypothetical "polar Moon", whose orbital plane is perpendicular to the plane of the Earth's motion around the Sun. His approximate formulas predicted the collision of such a satellite with the Earth after 52 revolutions. Subsequent numerical integration of the equations of motion gave a collision after 55 revolutions, confirming the high accuracy of the analytical result.

Similar regularities were investigated at the beginning of the 20th century by the Swedish astronomer Hugo von Zeipel, and in 1962 by the Japanese scientist Yoshida Kozai. Currently, the corresponding phenomenon is commonly called the von Zeipel–Lidov–Kozai effect. Its manifestations are found in a wide range of problems: from the motion of artificial and natural satellites of planets to the dynamics of stellar systems and the evolution of compact astrophysical objects.

Another major area of Lidov's research was the theory of guaranteed estimation. In 1964, his pioneering paper appeared, in which he studied the problem of determining the parameters of a system without using probabilistic assumptions about measurement errors. It was assumed only that the absolute values of the errors did not exceed specified bounds. Lidov showed that finding a guaranteed estimate, valid for all admissible error realizations, could be reduced to a linear programming problem.

M. L. Lidov also developed methods for estimating motion in the presence of unmodeled accelerations. Such problems were interpreted as differential games, in which the estimation algorithm opposes the most unfavorable admissible perturbation. The theoretical results were developed into computational algorithms and applied to specific problems of spacecraft navigation and control.

A significant part of Lidov's research in the 1970s–1980s concerned the dynamics of telecommunication satellites. Together with A. A. Soloviev, he established important regularities in the orbital evolution of Molniya satellites, which provided television broadcasting to Siberia and the Far East.

Together with M. A. Vashkovyak and A. P. Markeev, M. L. Lidov studied the motion of spacecraft in the vicinity of the unstable libration points of the Earth–Moon system. The possibility of placing a relay satellite in such a region to provide communication with a lander on the far side of the Moon was considered. The practical effectiveness of such a scheme was confirmed decades later.

In the last years of his life, Lidov worked on a scheme for a fly-by of several asteroids during a single mission, analyzed the use of the Moon’s gravitational field to insert a spacecraft into a halo orbit in the Sun–Earth system, and studied quasi-satellite orbits.

Conclusion

Lidov’s scientific achievements received high recognition. In addition to the Lenin Prize, he was awarded the Order of the Red Banner of Labour. His PhD dissertation, prepared in 1964, was recognized as a work deserving the degree of Doctor of Physical and Mathematical Sciences.

The scientific legacy of Lidov combines fundamental studies in celestial mechanics with the solution of specific problems arising in space missions. His methods for investigating orbital evolution, guaranteed estimation, and trajectory correction retain both scientific and practical significance. His name is preserved in the von Zeipel-Lidov-Kozai effect and was also assigned to asteroid (4236) Lidov.

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A numerical survey of von Zeipel–Lidov–Kozai resonances and their coupling with mean-motion resonances in the trans-Neptunian region and the inner Solar System

Evgeny Smirnov and Ivana Milić Žitnik

Abstract. We present two catalog-wide numerical surveys of the von Zeipel–Lidov–Kozai (ZLK) resonance and its coupling with mean-motion resonances (MMRs), one in the trans-Neptunian region ($a > 30$ au) and one inside Jupiter’s orbit ($a < 4.9$ au). Beyond Neptune every one of the 81 ZLK-resonant objects is also trapped in a two-body MMR with Neptune, while inside Jupiter’s orbit only 11 of the 201 ZLK-resonant asteroids are in an MMR. The coupling is thus strong and universal beyond Neptune, but weak inside Jupiter’s orbit, where the ZLK resonance is an overwhelmingly planet-crossing phenomenon that extends below the classical critical inclination.

Introduction

The von Zeipel–Lidov–Kozai (ZLK) resonance, first found by von Zeipel (1910) and later developed by Lidov (1962) and Kozai (1962), produces a libration of the argument of pericenter ω with coupled anti-correlated oscillations of the eccentricity e and inclination i . In the quadrupole test-particle limit the libration is possible above a critical inclination $\arccos \sqrt{3/5} \approx 39.2^\circ$. This value holds only in the $\alpha \rightarrow 0$ limit ($\alpha = a/a_p$, the ratio of the two semi-major axes) and grows with α (Kozai 1962; Vashkoviak 1981). Following Morbidelli (2002); Shevchenko (2017); Sidorenko (2018) we treat the ZLK effect as a nonlinear resonance with resonant angle $\sigma \equiv \omega$; a full account of the theory is given by Shevchenko (2017).

In the outer Solar System the ZLK resonance is closely tied to MMRs. Gallardo (2006) showed that the locking of the semi-major axis inside an MMR allows ZLK libration below the non-resonant critical inclination, and Lei et al. (2022) showed that the libration centers can then shift away from 90° and 270° . Inside Jupiter’s orbit the picture is more fragmented: the libration centers shift to 0° and 180° in the low-inclination, planet-crossing regime (Michel and Thomas 1996;

Gronchi and Milani 1999), where Carruba et al. (2026) recently identified ZLK-resonant co-orbitals of the terrestrial planets. A catalog-wide census of the ZLK resonance had not been carried out in either region. We provide such a census for both and test whether the tight MMR–ZLK coupling seen for trans-Neptunian objects (TNOs) also holds inside Jupiter’s orbit. The full surveys are presented in Smirnov and Milić Žitnik (2026a) and Smirnov and Milić Žitnik (2026b).

1 Methods

Both surveys use the **resonances** package (Smirnov 2023), which wraps **rebound** (Rein and Liu 2012), in a full N-body model with the Sun, all planets, and Pluto. For each object the angle $\sigma \equiv \omega$ was classified as resonant (a clean libration over at least one full cycle), possible (temporarily captured or near the separatrix), or non-resonant, and inside Jupiter’s orbit the librators were further split into regular and irregular ones. The two integration setups differ because the ZLK timescales do. For the 1037 TNOs with $a > 30$ au we integrated for 10 Myr and re-integrated the ambiguous objects in stages up to 300 Myr. For the 863 471 asteroids with $a < 4.9$ au we used a 100 kyr automated phase and a 500 kyr manual phase; because many of these objects are highly eccentric, highly inclined, or planet-crossing, only the **ias15** integrator (Rein and Spiegel 2015) gave stable results for the whole sample. Every resonant object was then checked for concurrent membership in two-body and three-body MMRs.

2 Results

From the 1037 TNOs, 81 (7.8%) are in the ZLK resonance and 23 are transient. The main result is that all 81 ZLK-resonant TNOs are simultaneously trapped in a two-body MMR with Neptune, a direct empirical confirmation of the predicted coupling in this region. The most populated resonances are 2N-3 (58 objects), 4N-7 (15), 1N-2 (9), and 3N-5 (7), which together hold about 86% of the ZLK-resonant TNOs. The libration centers cluster near 90° and 270° , except in 1N-2, where they are shifted to asymmetric values, as expected for 1:N resonances. Among individual objects, the retrograde TNO (585899) 2020 HM98 is temporarily in the resonance 2N+9, and the most distant, (652920) 2014 GR53 at $a = 202.8$ au, is in 1N-18.

From the 863 471 asteroids, 201 are confirmed ZLK-resonant (163 regular and 38 irregular), less than 0.02% of the sample. The population is overwhelmingly planet-crossing: 144 are near-Earth asteroids ($q < 1.3$ au) and 28 are Mars-crossers, while the main belt is almost empty. In sharp contrast with the TNOs, only 11 of the 201 (about 5%) are also in a two-body or three-body MMR, and the surviving pairs concentrate in the 2J-1 resonance with Jupiter, so the two resonances appear decoupled here. About one-third of the librators (62 objects) have inclinations below 39.2° , and 24 lie above the quadrupole threshold $c_1 = (1 - e^2) \cos^2 i = 3/5$, which the classical test-particle theory forbids. This is consistent with the shifted equilibria inside MMRs (Gallardo 2006; Lei et al. 2022) and with the planet-crossing centers at $\omega = 0^\circ$ and 180° (Michel and Thomas 1996; Gronchi and Milani

1999; Carruba et al. 2026), and part of it with the higher α -corrected threshold. The four regular 2J-1 librators fall on the highly inclined island of the two-island structure of Chrenko et al. (2015).

Conclusions

Beyond Neptune the ZLK resonance and the MMR network are tightly coupled: every ZLK-resonant TNO is locked in a two-body MMR with Neptune, and the locking of the semi-major axis creates the conditions for the ZLK libration. Inside Jupiter’s orbit this coupling is almost absent. Fewer than 6% of the ZLK-resonant asteroids are in an MMR, the surviving pairs concentrate in the 2J-1 resonance, and the ZLK resonance is instead a planet-crossing phenomenon that reaches below the classical critical inclination. We attribute the contrast to MMR overlap and chaos in the main belt, to eccentricity pumping onto planet-crossing orbits, and to competition with strong secular resonances.

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Artificial Periodic Orbits of a Solar Sail Spacecraft in Moon Vicinity

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Abstract. In the framework of future projects for the exploration and development of the Moon, relay satellites for communication between the Earth and the Moon—especially between the Earth and the far side and polar regions of the Moon, where permanent bases are planned to be built—are of significant interest. This article develops a methodology for determining resonant orbits near the L2 libration point in the Earth-Moon system based on the model of the circular restricted three-body problem. In addition, the influence of changes in the initial position of the Sun, the magnitude of nominal acceleration from light pressure, and the installation angles of the solar sail on such artificial resonant orbits is considered. As a result of the study, new types of periodic orbits for placing relay satellites in the Earth-Moon system, and the dependencies of the shape and size of artificial resonant orbits on design parameters and control parameters of spacecraft with a solar sail, were obtained.

Keywords. Resonant orbit, solar sail, L2 libration point, Earth-Moon system, multiple shooting method, initial position of the Sun.

Introduction

In the near future, the exploration and use of the Moon will become the main task of space exploration. The far side and polar regions of the Moon are of particular interest for development. The functioning of objects on the Lunar surface in these areas will require the creation of satellite systems to ensure continuous communication between the Earth and the Moon. Usually, such satellites are placed in natural periodic orbits (for example, Lyapunov orbits and halo orbits [1, 2]) near the L2 point. However, for some tasks, it is required to simultaneously meet the requirements for the period, the shape, and the position of the orbit.

Solar sails (SS) represent a relatively new technology for creating control action in space due to light pressure [3]. The key advantage of an SS is the absence

of fuel consumption. Bookless and McInnes [4] investigated a method for forming artificial Lagrange points in the Sun-Earth system created by shifting natural libration points due to solar light pressure. Changing the parameters and control laws of the SS allows obtaining various artificial Lagrange points or forming artificial periodic orbits in accordance with specific requirements.

We have developed a new methodology for determining the resonant orbits of an SS near the L2 point in the Earth-Moon system, corresponding to given sail installation angles, the magnitude of the nominal SS acceleration, and the initial position of the Sun.

1. Mathematical Models and Methods

If we neglect all perturbations, the mass of the spacecraft, and consider circular trajectories of the Earth and the Moon around the Sun and the Earth respectively, then the motion of the spacecraft in the coordinate system $\mathbf{oxyz}$ can be considered within the framework of the circular restricted three-body problem model. In this case, the equation of spacecraft motion in the rotating coordinate system has the form:

$$\ddot{\mathbf{r}} + 2\omega_M \times \dot{\mathbf{r}} + \nabla U(\mathbf{r}) = \mathbf{a}_s, \quad (1)$$

where $\mathbf{r}$ is the spacecraft position vector; ω_M is the angular velocity of the Moon around the Earth; $\mathbf{a}_s$ is the acceleration from light pressure forces; $U(\mathbf{r})$ is the pseudo-potential function.

If we assume that the inclination of the Moon's orbit $i_M \approx 0$ is small, then subsequent analysis is significantly simplified. The rotating coordinate system the direction vector of sunlight $\mathbf{s}$ has the coordinates

$$\mathbf{s} = (\cos(\omega_s t + \theta_0), -\sin(\omega_s t + \theta_0), 0)^T, \quad (2)$$

where θ_0 is the initial position of the Sun; ω_s is the angular velocity of the Sun's rotation around the Earth-Moon system in the rotating coordinate system.

Assuming that the SS has a perfectly reflecting surface, the acceleration vector acting on the spacecraft $\mathbf{a}_s$ can be expressed as:

$$\mathbf{a}_s = \kappa(\mathbf{s} \cdot \mathbf{n})^2 \mathbf{n}, \quad (3)$$

where κ is the maximum (nominal) acceleration from light pressure at a given distance from the Sun; $\mathbf{n}$ is the unit normal vector to the SS. Let the vector $\mathbf{n}$ in the coordinate system be defined by the orientation angles α and δ :

$$\mathbf{n} = (\cos \alpha \cos \delta, \sin \alpha \cos \delta, \sin \delta)^T, \quad (4)$$

where α is the angle between the vector $\mathbf{n}$ and the plane; δ is the angle of rotation of the vector $\mathbf{n}$ around the axis.

The nominal acceleration from the SS κ and the angle δ are constant and the acceleration vector $\mathbf{a}_s$ from the light pressure forces is equal to

$$\mathbf{a}_s = \kappa \cos^2 \alpha_0 \cos^2 \delta \mathbf{n}. \quad (5)$$

Analysis of equation (5) shows that the influence of the angle δ on the acceleration vector $\mathbf{a}_s$ can be considered equivalent to a change in the magnitude κ and the angle α .

In this work, to determine the resonant orbit corresponding to fixed installation angles, nominal SS acceleration, and the initial position of the Sun, a two-point boundary value problem is solved, in which the initial vector of the spacecraft's phase coordinates is determined, ensuring the closure of the spacecraft's flight trajectory over a period T . Natural orbits with a period equal to T are used as an initial approximation.

2. Simulation results

Let us consider the influence of the initial position of the Sun θ_0 , the magnitude of nominal acceleration κ and the SS installation angle α on these orbits.

1. For the entire possible range of the Sun's initial position, only two configurations of resonant orbits can be obtained. They correspond to $\theta_0 = 0^\circ$ or 180° , and $\theta_0 = 90^\circ$ or 270° . Resonant orbits of the first configuration ($\theta_0 = 0^\circ$ or 180°) have two identical orbital loops, symmetrical relative to y . Resonant orbits of the second configuration ($\theta_0 = 90^\circ$ or 270°) have one large loop and one small loop, symmetrical to themselves relative to x .

2. The magnitude of nominal acceleration κ determines the displacement of the resonant orbits relative to the natural periodic orbits. Figure 1 presents the dependencies of the maximum displacements of the positions of resonant orbits Δy in the direction of the axis y relative to the natural periodic orbits on the magnitudes of the nominal acceleration κ . The upper line corresponds to the maximum displacement of the outer loop of the orbits, and the lower line corresponds to the maximum displacement of the inner loop of the orbits.

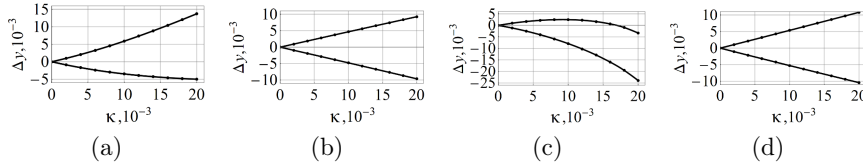


FIGURE 1. Dependencies of the maximum position displacements of resonant orbits Δz in the direction relative to natural periodic orbits on the inclination angle α .

3. Figure 2 presents the dependencies of the maximum position displacements of the resonant orbits relative to the natural periodic orbits on the sail installation angle α . The red lines represent the displacements of the highest points of the orbits, and the blue lines represent the displacement of the lowest points of the orbits.

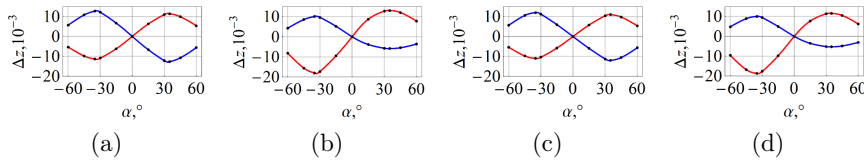


FIGURE 2. Dependencies of the maximum position displacements of resonant orbits Δz in the direction relative to natural periodic orbits on the inclination angle α .

Conclusion

In this work, a methodology for determining resonant periodic orbits relative to the L2 libration point of the Earth-Moon system is proposed, and it is determined how their configuration and displacement change compared to natural halo orbits when the initial position of the Sun, the magnitude of the nominal acceleration, and the SS installation angle change. By combining different values of the control variables, it is possible to create diverse orbits adaptable to various mission needs.

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Regularized figure-eight orbit

Vladimir Titov

Аннотация. The figure-eight orbit in the regularized shape space of the three-body problem is considered. The Lemaitre regularization is used: the shape sphere (or the corresponding extended complex plane) is mapped onto the regularized shape sphere. This mapping is four-sheeted. The mapping is constructed and its properties are studied.

1. Introduction

The shape space is the quotient space of $\mathbb{R}^n$ by translations and rotations. The figure-eight orbit is well known, it is obtained numerically with sufficient accuracy; it is stable and far from collisions. Nevertheless, its study in the regularized space allows one to look at this orbit from a different perspective. Since the masses of the bodies are comparable (equal), it is natural to use a global regularization, namely the Lemaitre regularization:

$$\zeta = \frac{z^3 + \sqrt{8}}{1 - \sqrt{8}z^3},$$

To represent the orbit, we will use its Fourier series expansion (with period 2π) as a solution in the plane $\mathbb{R}^2$:

$$\begin{aligned}x_1(t) &= 1.0958785 \sin t & -0.0252775 \sin 5t - 0.0058497 \sin 7t + \dots \\y_1(t) &= 0.3372826 \sin 2t & +0.0557118 \sin 4t - 0.0029908 \sin 8t + \dots \\x_2(t) &= x_1(t + 2\pi/3) \\y_2(t) &= y_1(t + 2\pi/3) \\x_3(t) &= x_1(t + 4\pi/3) \\y_3(t) &= y_1(t + 4\pi/3)\end{aligned}$$

Using the scaling symmetry, this solution can be easily transformed into a figure-eight solution with any period or any size.

2. The Figure-Eight Orbit in Shape Space

Factoring out translations and rotations, we pass to the shape space Ξ . The corresponding expansions are easily obtained:

$$\begin{aligned}
 & \text{Сдвиг на } 2\pi/3 \\
 \xi_1 = & 1.7446017 \cos(2t) + 0.2542500 \cos(4t) - 0.0137548 \cos(8t) \\
 & -0.0034641 \cos(10t) + 0.0002985 \cos(14t) + 0.0000878 \cos(16t) \\
 & -0.0000089 \cos(20t) - 0.0000029 \cos(22t) \\
 \xi_2 = & 1.7446017 \sin(2t) - 0.2542500 \sin(4t) - 0.0137548 \sin(8t) \\
 & +0.0034641 \sin(10t) + 0.0002985 \sin(14t) - 0.0000878 \sin(16t) \\
 & -0.0000089 \sin(20t) + 0.0000029 \sin(22t) \\
 \xi_3 = & -1.2672068 \sin(3t) + 0.0137536 \sin(9t) - 0.0003240 \sin(15t) \\
 & +0.0000102 \sin(21t)
 \end{aligned}$$

The expression for the moment of inertia (or the radius-vector (ξ_1, ξ_2, ξ_3)):

$$\begin{aligned}
 \rho = & 1.9777440 + 0.0043046 \cos(6t) - 0.0000416 \cos(12t) \\
 & + 0.0000007 \cos(18t) - 0.0000005 \cos(24t).
 \end{aligned}$$

The space Ξ is the cone $S^2 \times \mathbb{R}^+$. The singularities of double collisions are the points $(-1, 0)$, $(1/2, \sqrt{3}/2)$ and $(1/2, -\sqrt{3}/2)$. The preimage of the first point is $(\sqrt{2}-\sqrt{6})/2$ and $(\sqrt{2}+\sqrt{6})/2$; the preimages of the remaining points are obtained by rotations of 120° and 240° .

All non-singular points have four preimages. For any non-singular point ζ , they are easily obtained as the roots of the 4th-degree polynomial:

$$z^4 + \sqrt{8}\zeta z^3 + \sqrt{8}z - \zeta = 0$$

3. The Figure-Eight Orbit on the Regularized Shape Sphere

On the shape sphere S^2 or $\mathbb{Z}_\infty$, the figure-eight trajectory is a closed continuously differentiable curve $\zeta(t)$. We need to determine the corresponding continuous curves $z_i(t)$ on the regularized shape sphere ($i = 1, \dots, 4$).

Each curve $z_i(t)$ can be identified by its initial point $z_i(0)$. For all subsequent times t , from the four roots for a given curve z_k , one must choose the nearest one. Theoretically, there could be bifurcation points where the polynomial has multiple roots, but this is not the case for the figure-eight.

The figure shows the four preimages of the figure-eight orbit. The red, blue, and brown trajectories are rotated relative to each other by 120° ($e^{2\pi i/3}$). The cyan trajectory (drawn with a thick line) is close to the unit circle. All preimages pass through 6 Euler points, with a total of 12 such points, 3 on each sheet. Only on the cyan trajectory are the points evenly spaced, but if projected onto the sphere, they are all close to great circles; the cyan trajectory is close to the equator, while the planes of the others are inclined to the equator by approximately 71° .

It is not difficult to obtain expansions, for example, for the cyan curve.

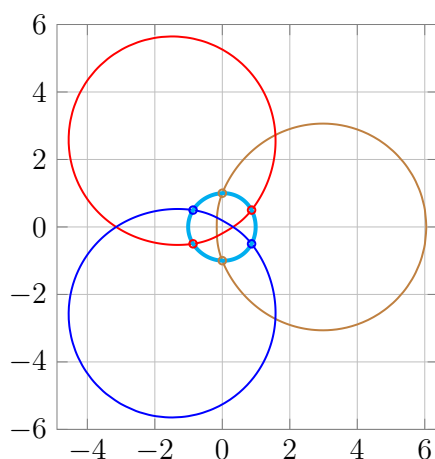


Рис. 1. Figure-eight on the regularized extended complex plane

4. Conclusion

The regularized figure-eight orbit appears simpler than the orbit in shape space. By choosing the preimage close to the unit circle (or to the equator on the shape sphere), we obtain a simple expression that can be used for further analysis.

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Neural network simulating the Lomb-Scargle periodogram

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Abstract. A convolutional neural network is developed for detecting a sinusoidal signal in unevenly sampled time series with periodic gaps and Gaussian white noise. The model, trained with binary cross-entropy, approximates the Bayesian posterior probability in the limit of infinite sample size. On test data, the network’s root-mean-square deviation from the theoretical detection probability is about 0.07, while its processing speed exceeds numerical Bayesian modelling by 5–6 orders of magnitude. A critical sensitivity to initial weights is revealed, and empirical distributions for their initialization are proposed.

Introduction

The search for periodic signals is an important problem in astronomy (exoplanet detection, variable stars) and other fields. This work investigates the potential of artificial neural networks for detecting a sinusoidal signal in noise. The main goal is to assess the capabilities of neural networks, identify “pitfalls”, and outline directions for further development. As a test case, we consider the detection of a harmonic signal in an uneven time series with gaps and Gaussian white noise. The neural network effectively emulates the Lomb–Scargle periodogram [1], [2].

Bayesian interpretation of the neural network classifier

When a neural network is trained with the binary cross-entropy (BCE) loss function, minimisation in the limit of an infinite sample leads to the network output approximating the posterior probability of signal presence, $g_1(x) = P(\omega_1|x)$. This generalises the known result for the L_2 mere [3]. The derivation is based on variational calculus: the BCE functional attains its global minimum at $F^*(x) = g_1(x)$. Second variations show that BCE is more sensitive to errors in regions where the posterior probability is close to 0 or 1, i.e. near the boundaries.

Uneven time series with gaps

The time series may be uneven (a model with periodic gaps; the parameters λ and μ are chosen randomly: $\lambda \sim \mathcal{B}(1, 2)$ and $\mu \sim \mathcal{B}(1, 1)$), and may contain an arbitrary offset and noise variance. To remove the influence of offset and variance, each series is centred and normalised. The input data are supplemented with information about the observation times through the spectral window. For each frequency f_k , both $\Omega(f_k)$ and $\Omega(2f_k)$ are included in the feature vector (the latter is related to the generalised Lomb–Scargle periodogram [4]). A switch to a convolutional architecture is made: the first layer is a convolutional kernel of size $1 \times 6 \times m$ (where 6 is the number of input channels: real and imaginary parts of the DFT, $\Omega(f_k)$, $\Omega(2f_k)$, the second layer is a convolution $1 \times m \times 1$, followed by a fully connected layer and the output. This reduces computational cost and allows scaling.

The problem of initial weights

Training with random weights practically never converged to a good minimum due to “dead” neurons and subtraction of nearly equal numbers. An empirical initialisation for the weights of the first layer was proposed, based on modelling parameter distributions to avoid pathologies. For each frequency block, the quantities $H_i = \sqrt{C_i^2 + S_i^2}$, $R_i = -B_i/H_i$, $\alpha_i = \text{atan2}(S_i, C_i)$ were introduced, where C_i , S_i are the weights for the real and imaginary parts of the DFT, and B_i is the bias. It was found that H_i and R_i are close to normal, α_i to uniform, and the output layer weights D_i to half-normal. Additionally, it was empirically discovered that the weights handling $\Omega(2f)$ must be related to the DFT weights by $\frac{S''}{C''} = \frac{SC}{C^2 - S^2}$, $S''^2 + C''^2 \approx 0.5(S^2 + C^2)$. Setting such initial approximations significantly improved convergence. The weights for $\Omega(f)$ turned out to be not very informative and had little effect on accuracy. This starting point allowed reaching a meaningful minimum. The optimal block size is 16 or 32 neurons.

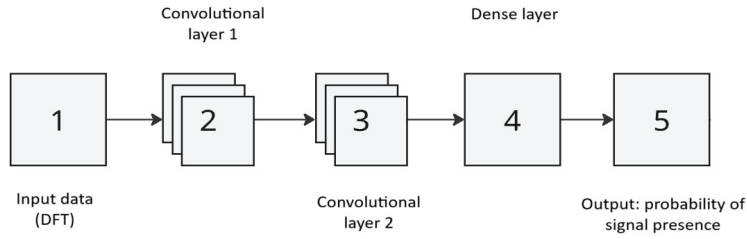


FIGURE 1. Neural network structure.

Results

For a test sample of 40,000 series, the Bayesian posterior detection probability of 0.3851. The convolutional network with $m = 16$ or 32 achieved BCE of about 0.4048 and 0.4047, respectively, and a root-mean-square error of about 0.07. A comparison on a two-dimensional histogram showed good agreement with the theoretical posterior probability, with the majority of points concentrated in the corner regions (probabilities near 0 or 1), where BCE is most sensitive [5].

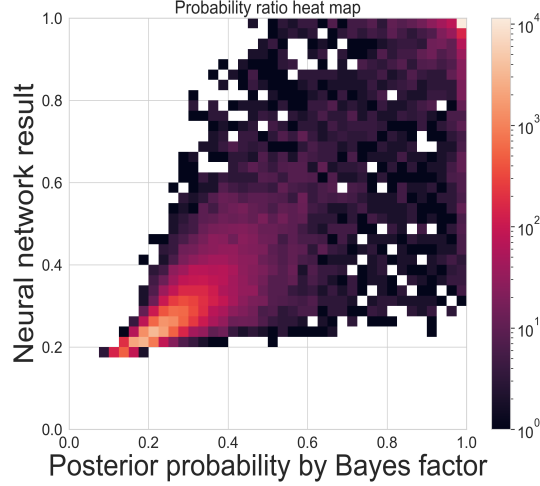


FIGURE 2. Comparison of the final neural network model output with the Bayesian posterior probability of signal presence. A two-dimensional histogram is shown, where colour indicates the density of test points (40,000 points in total from the validation sample).

Discussion and prospects

The main advantage of the neural network model is its processing speed: running through 10^6 series takes about 1 minute on a GPU, while computing the Bayesian probability for 40,000 series on a CPU took about a week (a gap of 5–6 orders of magnitude). This makes the method suitable for processing large astronomical catalogues. However, training is extremely sensitive to initial weights – without an empirical “hint”, convergence to the global minimum is unlikely. Future plans include:

- increasing the number of observations K and the spectral range;
- adapting the model for non-sinusoidal signals (e.g., highly eccentric orbits);
- accounting for correlated noise (non-uniform spectrum).

Thus, the work demonstrates that neural networks can effectively approximate Bayesian inference in periodic signal detection, but they require careful tuning of initial conditions. The proposed approach can serve as a foundation for more complex applications.

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Secular Perturbations of Orbital Elements of Asteroids in the Thule Group

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Abstract. Secular perturbations of orbital elements of asteroids in the Thule group were studied. Asteroids of this group move in mean motion resonance 4:3 with Jupiter. The study was conducted using numerical integration of equations of motion for real asteroids. A simple dynamic model was used, taking into account perturbations from Jupiter, moving in a fixed elliptical orbit. After identifying classical secular perturbations, their influence on orbital elements was eliminated. This allowed us to identify the Lidov-Kozai mechanism in this resonant region. As a result, it was found that the maximum eccentricity, and correspondingly the minimum orbital inclination, are observed not at the usual values of the argument of perihelion $\omega = 90^\circ, 270^\circ$, but at the points $\omega = 0^\circ, 180^\circ$.

Introduction

Secular perturbations of asteroids, both classical and Lidov-Kozai mechanism (LKM), are treated in the analytical theory by assuming that no mean-motion commensurabilities take place [1, 2, 3]. However, there are very strong mean motion resonances in the asteroid belt, where analytical methods cannot be applied.

The LKM in the region of resonances was studied by Y. Kozai [4] using a semi-analytical method. It was shown that in resonant regions the LKM manifests itself differently than in non-resonant ones. For the 4:3 resonance, i.e. in the Thule group, it was found that eccentricities e take maximum and inclinations i take minimum at $2\omega = 0^\circ$ and vice versa at $2\omega = 180^\circ$.

Also, T. Vinogradova [5] used another method to study the LKM in Hildas and Trojans. The method is based on numerical integration of asteroid's equations of motion and elimination of classical secular perturbations. This method is used in the present work.

1. Method

The method used long-term numerical integration. An orbit evolution of real asteroids in the Thule group was investigated. As a source of initial osculating elements, the IAU Minor Planet Center (MPC) database of orbital elements version Nov. 2024 was employed. Elements of the Jupiter's orbit were fixed at the moment corresponding to the epoch of asteroid orbital elements.

At first, classical secular perturbations were identified. Then these perturbations were excluded from osculating elements. An exclusion of classical perturbations can be made very easy using a coordinate transformation formula. After excluding classical secular perturbations, LKM i and e oscillations can become visible in the distributions of proper elements.

2. Classical secular perturbations

In the case of classical secular perturbations, i variations depends on the longitude of the ascending node Ω , and e variations – on the longitude of perihelion ϖ .

As an example, the motion of (185290) 2006UB219 from the Thule group was considered. The results of numerical integration in Figure 1 show well classical secular perturbations in the orbital elements of this asteroid.

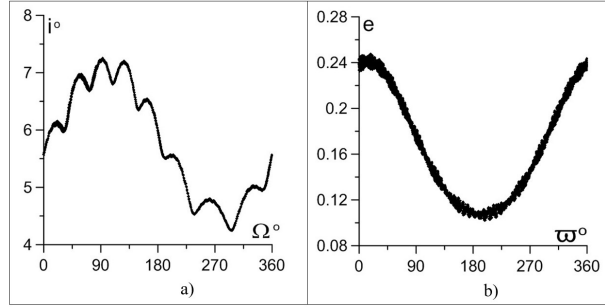


FIGURE 1. The results of numerical integration show variations of orbital elements of the asteroid (185290)2006 UB219 under the influence of classical secular perturbations: (i, Ω) and (e, ϖ) .

3. Lidov-Kozai perturbations

The LKM is a secular perturbation which causes long-period oscillations in the inclination i and eccentricity e depending on a perihelion argument ω . Under the influence of LKM, i and e change in concert.

After excluding classical secular perturbations, LKM i and e oscillations can become visible in distributions of proper elements (i_p, ω_p) and (e_p, ω_p) . These oscillations can be seen in Figure 2. One can see that usual positions of the maximum e and, accordingly, minimum i ($\omega = 90^\circ, 270^\circ$) are shifted by 90° and is achieved with $\omega = 0^\circ, 180^\circ$.

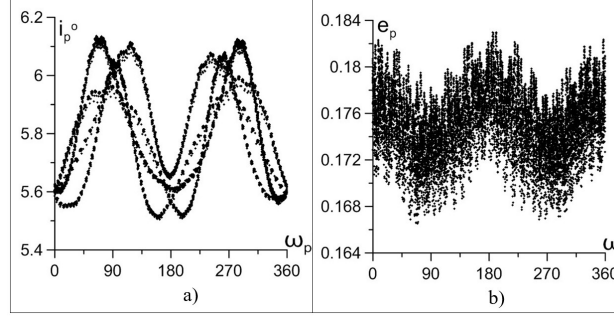


FIGURE 2. LKM oscillations of the orbital elements i and e of the asteroid (185290)2006 UB219 in the Thule group.

Conclusion

The results by Y. Kozai [4], concerning LKM in resonance 4:3, are confirmed in this work using a different method.

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Hamiltonian structures and normal forms of dynamic systems with periodic perturbations

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Abstract. Conditions are proposed for the reducibility of a linear periodic system of differential equations whose characteristic polynomials are recurrent to an equivalent Hamiltonian system. The constructions are based on a transition from the linear part of the system to the normal forms of the corresponding Hamiltonian systems, followed by a transformation of the resulting system. The proposed scheme does not require complex and cumbersome transformations of the original system.

The presence of a Hamiltonian structure and, consequently, the existence of first integrals and various types of symmetries in a dynamical system allows for significant advances in the study of the system's dynamics. The existence of a Hamiltonian structure for certain types of differential equations and, accordingly, the construction of an equivalent Hamiltonian system for such equations in various settings have been discussed in a number of papers (see, for example, [1]). These papers focus on the study of Hamiltonian structure problems for autonomous differential equations, while nonautonomous (and, in particular, periodic) systems have been studied significantly less. The study of such problems and, in particular, the description of classes of differential equations with a Hamiltonian structure, the development of efficient algorithms for constructing the corresponding transformations, appears to be an important and interesting area of research.

The article develops the results previously obtained by the authors (see [2]).

We consider a dynamical system dependent on a small parameter ε , described by the differential equation

$$z' = [B_0 + \varepsilon S(t)]z, \quad z \in R^{2n}, \quad (1)$$

in which B_0 is a constant matrix, and $S(t)$ is a continuous and T -periodic matrix.

The following conditions are assumed to be satisfied:

- V1) There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the characteristic polynomial of the monodromy matrix $V(\varepsilon)$ of the system (1) is recurrent.
- V2) All eigenvalues of the matrix B_0 are prime.

V3) the matrix B_0 has no eigenvalues comparable in absolute value $\frac{2\pi i}{T}$, i.e. for any two distinct eigenvalues λ_1 and λ_2 , the relation

$$\lambda_1 - \lambda_2 \neq \frac{2\pi ki}{T} \quad \text{for } k = 0, \pm 1, \pm 2, \dots$$

The following assertion is key.

Theorem 1. *Let conditions V1-V3 be satisfied. Then there exists a non-degenerate, continuously differentiable, T -periodic in t , and analytic for small $|\varepsilon|$ substitution $z = K(t, \varepsilon)u$, transforming the periodic system (1) to Hamiltonian form.*

$$\frac{du}{dt} = JA(\varepsilon)u, \quad (2)$$

in which $JA(\varepsilon)$ is an autonomous Hamiltonian matrix in normal form, analytically depending on ε .

The transition to the Hamiltonian system (2) can be implemented using the following scheme. According to the Floquet-Lyapunov theory (see, e.g., [3]), the fundamental solution matrix $Z(t, \varepsilon)$ of the system (1) can be represented as

$$Z(t, \varepsilon) = P(t, \varepsilon)e^{B(\varepsilon)t}, \quad (3)$$

where $P(t, \varepsilon)$ is a continuous and T -periodic in t matrix such that $P(0, \varepsilon) = P(T, \varepsilon) = I$, $B(\varepsilon)$ is a constant (independent of t) matrix. Moreover, by virtue of the condition V3, the matrices $P(t, \varepsilon)$ and $B(\varepsilon)$ can be chosen to be analytic with respect to the variable ε in some neighborhood of the zero value $\varepsilon = 0$ so that $P(t, 0) = I$ and $B(0) = B_0$. The monodromy matrix $V(\varepsilon) = Z(T, \varepsilon)$ of the system (1) satisfies the equality

$$V(\varepsilon) = e^{B(\varepsilon)T}. \quad (4)$$

By condition V1 and equality (4), the properties of the eigenvalues of the matrix $B(\varepsilon)$ are similar to those of Hamiltonian matrices. Therefore, the characteristic polynomial $L(p, \varepsilon)$ of the matrix $B(\varepsilon)$ with a given set of roots can be associated with one or more corresponding Hamiltonian matrices $JA(\varepsilon)$ in normal form with the same set of eigenvalues.

The Hamiltonian matrix $JA(\varepsilon)$ is analytic:

$$JA(\varepsilon) = JA_0 + \varepsilon JA_1 + \varepsilon^2 JA_2 + \dots$$

To construct the Hamiltonian matrices JA_m ($m = 0, 1, 2, \dots$) it is sufficient to construct (in the appropriate approximation) the eigenvalues of the matrix $B(\varepsilon)$ involved in the equality (3). The eigenvalues $\lambda(\varepsilon)$ of the matrix $B(\varepsilon)$ are related to the eigenvalues $\mu(\varepsilon)$ of the matrix $V(\varepsilon)$ by the equality $\mu(\varepsilon) = e^{T\lambda(\varepsilon)}$. Therefore, to construct $\lambda(\varepsilon)$, it suffices to construct the eigenvalues $\mu(\varepsilon)$ of the monodromy matrix $V(\varepsilon)$ of the system (1). Here, methods of perturbation theory for linear operators can be used in the problem of approximately constructing multipliers of periodic systems.

As an application, problems of constructing normal forms for differential equations of the plane restricted elliptic three-body problem are considered (see, for example, [4]).

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